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NONNEGATIVE SOLUTIONS TO SOME SINGULAR SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. We prove the existence of a nonnegative weak solution $0 \not\equiv u \in H_0^1(\Omega)$ to the singular semilinear elliptic problem $-\Delta u = \chi_{\{u>0\}} a u^{-\alpha} + f(., u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^n$, $0 < \alpha < 3$, $a \in L^\infty(\Omega)$, $0 \not\equiv a \geq 0$, and $f : \Omega \times [0, \infty) \to \mathbb{R}$ is a Carathéodory function which satisfies some suitable hypothesis. We also obtain results about the problem with a parameter $-\Delta u = \chi_{\{u>0\}} a u^{-\alpha} + \lambda f(., u)$ in Ω, $u \geq 0$ in Ω, $u = 0$ on $\partial\Omega$.

Keywords. Singular elliptic problem; Variational technique; Nonnegative solution; Bifurcation problem.

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1. Introduction

Let us consider the singular semilinear elliptic problem:

$$
\begin{cases}
-\Delta u = \chi_{\{u>0\}} a u^{-\alpha} + f(x, u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial\Omega, \\
u \geq 0 \text{ in } \Omega, \; u \not\equiv 0 \text{ in } \Omega.
\end{cases}
\tag{1.1}
$$

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and the related problem with a parameter λ:

$$
\begin{cases} -\Delta u = \chi_{\{u>0\}} a u^{-\alpha} + \lambda f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega,\ u \not\equiv 0 \text{ in } \Omega. \end{cases} \tag{1.2}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, $0<\alpha<3$, $\lambda\in\mathbb{R}$, a, f are functions defined on Ω and $\Omega\times[0,\infty)$ respectively; and where $\chi_{\{u>0\}}au^{-\alpha}$ stands for the function defined by $\chi_{\{u>0\}}au^{-\alpha}(x) := a(x)u(x)^{-\alpha}$ if $u(x)\neq 0$, and $\chi_{\{u>0\}}au^{-\alpha}(x) := 0$ if $u(x)=0$.

These problems have received considerable interest in the literature and appear in applications to chemical catalysts process, non-Newtonian fluids, and in models for the temperature of electrical conductors (see e.g., [6], [4], [12], [15] and the references therein). The existence of positive solutions (i.e. such that $u(x)>0$ for all $x\in\Omega$) to problem (1.1) was proved, for the case $f=0$, and under various assumptions on a, in [15], [12], [7], [21], [10] and [3]. Existence theorems for positive classical solutions to problem (1.2) were obtained by Shi and Yao in [24], when Ω and a are regular enough, with a non necessarily nonnegative, $f(x,s)=s^p$ and $0<\alpha,p<1$. The free boundary singular elliptic bifurcation problem $-\Delta u = \chi_{\{u>0\}}(-u^{-\alpha}+\lambda f(.,u))$ in Ω, $u=0$ on $\partial\Omega$, $u\geq 0$ in Ω, $u\not\equiv 0$ (that is: $|\{x\in\Omega : u(x)>0\}|>0$) was studied by Dávila and Montenegro in [9], under the assumptions that $0<\alpha<1$, $\lambda>0$, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a nonnegative Carathéodory function, $f(x,s)$ is nondecreasing and concave in s, and $\lim_{s\to\infty} f(x,s)=0$ uniformly on $x\in\Omega$.

Bifurcation problems of the form $-\Delta u = g(x,u)+f(x,\lambda u)$ in Ω, $u=0$ on $\partial\Omega$, $u>0$ in Ω, were studied by Coclite and Palmieri [5]. It was proved there that, if $g(x,u)=au^{-\alpha}$, $a\in C^1\left(\overline{\Omega}\right)$, $a>0$ in $\overline{\Omega}$, and $f\in C^1\left(\overline{\Omega}\times[0,\infty)\right)$, then there exists $\lambda^*>0$ such that, for any $\lambda\in[0,\lambda^*)$, (1.2) has a positive classical solution $u\in C^2(\Omega)\cap C\left(\overline{\Omega}\right)$. Furthermore; for any $\lambda\geq 0$, a positive classical solution exists if, in addition, $\overline{\lim}_{s\to\infty}\frac{f(x,s)}{s}\leq 0$ uniformly on $x\in\overline{\Omega}$ (see [5], Theorem 1).

Multi-parameter singular bifurcation problems of the form $-\Delta u = g(u)+\lambda|\nabla u|^p+\mu f(.,u)$ in Ω, $u=0$ on $\partial\Omega$, $u>0$ in Ω were studied, by Ghergu and Rădulescu in [18]. Dupaigne, Ghergu and Rădulescu [14] obtained existence and nonexistence theorems for Lane–Emden–Fowler equations with convection and singular potential. Rădulescu [23] stated existence, nonexistence, and uniqueness theorems for blow-up boundary solutions of logistic equations, and for Lane-Emden-Fowler equations, with singular nonlinearities and subquadratic convection term. Existence and nonexistence results for positive solutions to the inequality $Lu \geq K(x)u^p$ on the punctured ball $\Omega = B_r(0) \setminus \{0\}$ were obtained by Ghergu, Liskevich and Sobol [16] for second order linear elliptic operators L without zero order term, and $K \in L^{\infty}_{loc}(\Omega)$ such that $0 < ess\inf K$. A Liouville comparison principle for entire weak solutions of quasilinear singular parabolic second-order partial differential inequalities was obtained in [20] by Kurta and existence and uniqueness results were obtained by Bougherara and Giacomoni [1] for mild solutions to singular initial value parabolic problems involving the p-Laplacian. Singularly perturbed elliptic problems on an annulus whose solutions concentrate in a circle were studied by Manna and Srikanth [22].

The following problem

$$\begin{cases} -\Delta u = ag(u) + \lambda f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega. \end{cases} \tag{1.3}$$

was considered by Cîrstea, Ghergu and Rădulescu [8] under the following assumptions: Ω is a regular enough bounded domain in $\mathbb{R}^n$, $0 \leq a \in C^{\beta}\left(\overline{\Omega}\right)$, $0 < f \in C^{0,\beta}[0,\infty)$ for some $\beta \in (0,1)$, f is nondecreasing on $[0,\infty)$, $f(s)/s$ is nonincreasing for $s > 0$, g is nonincreasing on $(0,\infty)$, $\lim_{s\to 0^+} g(s) = +\infty$; and there exist $\alpha \in (0,1)$, $\sigma_0 > 0$, and $c > 0$, such that $g(s) \leq cs^{-\alpha}$ for $s \in (0,\sigma_0)$. Under these hypothesis, and defining $\mu := \lim_{s\to\infty} f(s)/s$, $\lambda^* := \lambda_1/\mu$ (where λ_1 stands for the first Dirichlet eigenvalue of $-\Delta$ in Ω), and $\mathscr{E} := \left\{u \in C^2(\Omega) \cap C^{1,1-\alpha}\left(\overline{\Omega}\right) : \Delta u \in L^1(\Omega)\right\}$, the following results were proved:

([8], Theorem 1): If $\mu = 0$ and $\min_{\overline{\Omega}} a > 0$ (respectively $\min_{\overline{\Omega}} a = 0$), then, for all $\lambda \in \mathbb{R}$ (resp. $\lambda \geq 0$), problem (1.3) has a unique solution $u_\lambda \in \mathscr{E}$, the map $\lambda \to u_\lambda$ is strictly increasing, and each u_λ satisfies $c_1 d_\Omega \leq u_\lambda \leq c_2 d_\Omega$ for some positive constants c_1 and c_2, where $d_\Omega := dist(.,\partial\Omega)$

([8], Theorem 2): If $\mu > 0$ and $\lambda \geq \lambda^*$, then (1.3) has no solutions in $\mathscr{E}$. Furthermore, if $\mu > 0$ and $\min_{\overline{\Omega}} a > 0$ (respectively $\min_{\overline{\Omega}} a = 0$), then (1.3) has a unique solution $u_\lambda \in \mathscr{E}$ for any $\lambda < \lambda^*$ (resp. $0 \leq \lambda < \lambda^*$) and, again, the map $\lambda \to u_\lambda$ is strictly increasing; and each u_λ satisfies $c_1 d_\Omega \leq u_\lambda \leq c_2 d_\Omega$ for some positive constants c_1 and c_2. Moreover, $\lim_{\lambda\to(\lambda^*)^-} u_\lambda = +\infty$ uniformly on compact subsets of Ω.

Finally, let us mention that in [19], the authors proved the existence of nonnegative solutions for a restricted version of problem (1.1), namely when *H1)* holds, $0 < \alpha < 1$, and $f(.,u) = -bu^p$, with $0 < p < \frac{n+2}{n-2}$, and $0 \leq b \in L^r(\Omega)$ for suitable values of r.

Additional references, and a comprehensive treatment of the subject, can be found in [17], [23], see also [11].

The aim of this work is to prove, under suitable hypothesis on a and f, existence results for nonnegative weak solutions to problems (1.1) and (1.2). By a weak solution we mean a solution in the sense of the following.

Definition 1.1. We say that $u : \Omega \to \mathbb{R}$ is a weak solution of problem (1.1) if $u \in H_0^1(\Omega)$, $u \geq 0$, $\chi_{\{u>0\}} a u^{-\alpha} \varphi \in L^1(\Omega)$ and

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \chi_{\{u>0\}} a u^{-\alpha} \varphi + \int_\Omega f(x,u)\varphi. \tag{1.4}$$

for all φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

For $b \in L^\infty(\Omega)$ such that $b^+ \not\equiv 0$, we will write $\lambda_1(b)$ for the positive principal eigenvalue for $-\Delta$ on Ω, with homogeneous Dirichlet boundary condition and weight function b. With this notation, our first result reads as follows.

Theorem 1.2. *Let $\alpha \in (0,3)$ and assume the following conditions:*

H1) $a \in L^\infty(\Omega)$, $a \geq 0$, *and* $a \not\equiv 0$,

H2) $f : \Omega \times [0,\infty) \to \mathbb{R}$ *is a Carathéodory function on* $\Omega \times [0,\infty)$, *i.e.,* $f(.,s)$ *is measurable for any* $s \in [0,\infty)$, *and* $f(x,.)$ *is continuous a.e.* $x \in \Omega$,

H3) $\sup_{0 \leq s \leq M} |f(.,s)| \in L^1(\Omega)$ *for any* $M > 0$,

H4) One of the two following conditions holds:

H4') $\sup_{s>0} \frac{f(.,s)}{s} \leq b$ *for some* $b \in L^\infty(\Omega)$ *such that* $b^+ \not\equiv 0$, *and* $\lambda_1(b) > m$ *for some integer* $m \geq \max\{2, 1+\alpha\}$,

H4") $f \in L^\infty(\Omega \times (0,\sigma))$ *for all* $\sigma > 0$, *and* $\overline{\lim}_{s\to\infty} \frac{f(.,s)}{s} \leq 0$ *uniformly on* Ω, *i.e., for any* $\varepsilon > 0$ *there exists* $s_0 > 0$ *such that* $\sup_{s \geq s_0} \frac{f(.,s)}{s} \leq \varepsilon$, *a.e. in* Ω,

H5) $f(.,0) \geq 0$.

Under these hypothesis, (1.1) has a weak solution u (in the sense of Definition 1.1), that belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$; *and satisfies:*

i) $u > 0$ *a.e. in* $\{a > 0\}$. *In particular,* $\chi_{\{u>0\}} a u^{-\alpha} \not\equiv 0$ *and, if* $a > 0$ *a.e. in* Ω, *then* $u > 0$ *a.e. in* Ω.

ii) If $f(.,0) > 0$ a.e. in Ω, then $u > 0$ a.e. in Ω.

Note that, if $f \geq 0$ in $\Omega \times [0,\infty)$ then, by the maximum principle, the solutions to problem (1.1) that satisfy $\chi_{\{u>0\}} a u^{-\alpha} \not\equiv 0$ are positive *a.e.* in Ω. Example 3.7 in [19] shows that conditions like the ones stated above are needed in order to ensure the existence of a strictly positive weak solution.

Concerning problem (1.2) our results are the following.

Theorem 1.3. *Let $\alpha \in (0,3)$, and assume that H1)-H3), H4") and H5) hold. Then, for all $\lambda \geq 0$, (1.2) has a weak solution u_λ (in the sense of Definition 1.1); this solution u_λ is in $H_0^1(\Omega) \cap L^\infty(\Omega)$, satisfies $\chi_{\{u>0\}} a u_\lambda^{-\alpha} \not\equiv 0$ and $u_\lambda > 0$ a.e. in $\{a > 0\}$. These results remain valid for any negative λ if, in addition, $f(.,0) = 0$ and $\lim_{s\to\infty} \frac{f(.,s)}{s} = 0$ uniformly on Ω.*

Moreover, for $\lambda \geq 0$, if $f(.,0) > 0$ a.e. in Ω, then $u_\lambda > 0$ a.e. in Ω .

Theorem 1.4. *Let $\alpha \in (0,3)$; assume H1)-H3), H5), and that one of the two following conditions holds:*

H6) $\underset{(x,s)\in\Omega\times(0,\infty)}{ess\sup} \frac{f(x,s)}{s} < \infty$,

H7) $f \in L^\infty(\Omega \times (0,\sigma))$ *for all* $\sigma > 0$.

Then there exists $\lambda^* > 0$ such that, for any nonnegative $\lambda < \lambda^*$, (1.2) has a weak solution (in the sense of Definition 1.1) $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ that satisfies $\chi_{\{u_\lambda>0\}} a u_\lambda^{-\alpha} \not\equiv 0$ and $u_\lambda > 0$ a.e. in $\{a > 0\}$.

Theorems 1.3 and 1.4, can be viewed as partial generalizations of the already mentioned existence results contained in [8]. Let us briefly compare those results with ours: On the one hand, our assumptions on a and f are weaker than those imposed in [8]: we allow that $|\{x \in \Omega : a(x) = 0\}| > 0$; and we do not require $f > 0$. Notice also that we allow f to depend on (x,u); and that we do not require monotonicity, either on f, or on $s \to f(s)/s$. Our range of values of α is wider than the range allowed in [8]. On the other hand, we cannot guarantee that the solutions that we found are strictly positive in Ω. Moreover, we obtain neither the uniqueness, nor the monotonicity obtained in [8]. Finally, our λ^* does not have the optimality property of its counterpart in ([8], Theorem 2).

Our approach to study problem (1.1) is variational, and adapted from the one followed in [19]. Note that problem (1.1) has additional challenges with respect to the one considered in [19]: not only the nonlinearity is more general, but a further obstacle is posed by the fact that, when $\alpha \geq 1$, the domain of the corresponding energy functional J is not an open subset of $H_0^1(\Omega)$. In order to circumvent this obstacle we will consider, for any $M > 0$, the functional J on the set D_M^α formed by the nonnegative functions $u \in H_0^1(\Omega)$ that are bounded by M, and such that $J(u)$ is well defined and finite. In Section 2 we prove that, on D_M^α, J has a nonnegative minimizer $u_M \not\equiv 0$; and that $\|u_M\|_\infty \leq \mathscr{M}$, with $\mathscr{M}$ constant and independent of M. From these facts, and some auxiliary lemmas, Theorem 1.2 is proved in Section 3 by showing that, for M large enough, u_M is a weak solution of (1.1) (in spite of the possible lack of differentiability of J at u_M). Finally, at the end of Section 3, we use Theorem 1.2 to obtain Theorems 1.3 and 1.4.

2. Preliminaries

Let us recall that $\lambda \in \mathbb{R}$ is called a principal eigenvalue for $-\Delta$ in Ω, with homogeneous Dirichlet boundary condition and weight function b, if the problem $-\Delta u = \lambda b u$ in Ω, $u = 0$ on $\partial\Omega$ has a solution ϕ such that $\phi > 0$ in Ω.

Remark 2.1. The following facts are well known (see e.g., [13]). If Ω is a $C^{1,1}$ domain in $\mathbb{R}^n$, $b \in L^\infty(\Omega)$ and $b^+ \not\equiv 0$ then:

i) There exists a unique positive principal eigenvalue $\lambda_1(b)$, its eigenspace is one dimensional, and is included in $C^1\left(\overline{\Omega}\right)$. Moreover, for each positive eigenfunction ϕ, there are positive constants c_1, c_2 such that $c_1 d_\Omega \leq \phi \leq c_2 d_\Omega$ in Ω. In particular, for $\gamma \in \mathbb{R}$, ϕ^γ is integrable if, and only if, $\gamma > -1$.

ii) If $0 < \lambda < \lambda_1(b)$ and $h \in L^\infty(\Omega)$, the problem $-\Delta u = \lambda b u + h$ in Ω, $u = 0$ on $\partial\Omega$, has a unique solution $u \in \cap_{1 \leq p < \infty} W^{2,p}(\Omega)$, and the corresponding solution operator $(-\Delta - \lambda b)^{-1} : L^\infty(\Omega) \to C_0^1\left(\overline{\Omega}\right)$ is bounded and strongly positive, i.e., if $h \in L^\infty(\Omega)$ and $0 \leq h \not\equiv 0$ then u belongs to the interior of the positive cone of $C_0^1\left(\overline{\Omega}\right)$ where $C_0^1\left(\overline{\Omega}\right) := \left\{v \in C^1\left(\overline{\Omega}\right) : v = 0 \text{ on } \partial\Omega\right\}$.

iii) If $b^* \in L^\infty(\Omega)$ and $b \leq b^*$ then $\lambda_1(b^*) \leq \lambda_1(b)$.

For $M > 0$ and $0 < \alpha < 3$, let $D_M^\alpha \subset H_0^1(\Omega)$ be defined by

$D_M^\alpha := \left\{u \in H_0^1(\Omega) : 0 \leq u \leq M\right\}$ if $0 < \alpha < 1$,

$D_M^\alpha := \left\{u \in H_0^1(\Omega) : 0 \leq u \leq M \text{ and } \int_{\{a>0\}} a\,|\ln u| < \infty\right\}$ if $\alpha = 1$,

$D_M^\alpha := \left\{u \in H_0^1(\Omega) : 0 \leq u \leq M \text{ and } \int_{\{a>0\}} a u^{1-\alpha} < \infty\right\}$ if $1 < \alpha < 3$.

Lemma 2.2. *Assume H1). Then $D_M^\alpha \neq \varnothing$ for any $M > 0$ and $\alpha \in (0,3)$.*

Proof. The lemma is immediate when $0<\alpha<1$. For $1<\alpha<3$, we can proceed as follows: Let ϕ be a positive principal eigenfunction for $-\Delta$ in Ω with homogeneous Dirichlet boundary condition, with weight function 1, and normalized such that $\|\phi\|_\infty = M^{\frac{1+\alpha}{2}}$. Note that $\left|\nabla\left(\phi^{\frac{2}{1+\alpha}}\right)\right|^2 = \left(\frac{2}{1+\alpha}\right)^2 \phi^{\frac{2(1-\alpha)}{1+\alpha}} |\nabla\phi|^2$ and that, since $\alpha<3$, we have $\frac{2(1-\alpha)}{1+\alpha} > -1$. Thus $\left|\nabla\left(\phi^{\frac{2}{1+\alpha}}\right)\right| \in L^2(\Omega)$. Clearly $\phi^{\frac{2}{1+\alpha}} \in L^2(\Omega)$ and $a\phi^{\frac{2(1-\alpha)}{1+\alpha}} \in L^1(\Omega)$, and then $\phi^{\frac{2}{1+\alpha}} \in D_M^\alpha$.

Consider now the case $\alpha=1$. Let $\delta\in(0,1)$ and let $\beta=1+\delta$. Thus $1<\beta<3$. Since $\lim_{s\to0^+} s^\delta |\ln(s)| = 0$, and $|\ln(s)|<s$ for $s>1$, there is a positive constant c such that $|\ln(s)| \le c\left(s^{-\delta}+s\right)$ for any $s>0$. Let ϕ be a principal eigenfunction as above, but normalized now by $\|\phi\|_\infty = M^{\frac{1+\beta}{2}}$. As before, we have $\phi^{\frac{2}{1+\beta}} \in H_0^1(\Omega)$. Also, $\left|\ln\left(\phi^{\frac{2}{1+\beta}}\right)\right| \le c\left(\phi^{-\frac{2\delta}{1+\beta}} + \phi^{\frac{2}{1+\beta}}\right) = c\left(\phi^{\frac{2(1-\beta)}{1+\beta}} + \phi^{\frac{2}{1+\beta}}\right)$. Since $\phi^{\frac{2(1-\beta)}{1+\beta}}$ and $\phi^{\frac{2}{1+\beta}}$ belong to $L^1(\Omega)$, it follows that $\int_\Omega a\left|\ln\left(\phi^{\frac{2}{1+\beta}}\right)\right| < \infty$, and so $\phi^{\frac{2}{1+\beta}} \in D_M^1$.

For $0<\alpha<3$, let $J:\cup_{M>0} D_M^\alpha \to \mathbb{R}$ be defined by

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{1-\alpha}\int_{\{a>0\}} a u^{1-\alpha} - \int_\Omega F(.,u) \text{ if } \alpha\neq 1, \tag{2.1}$$

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\{a>0\}} a\ln(u) - \int_\Omega F(.,u) \text{ if } \alpha = 1,$$

where $F(x,s) := \int_0^s f(x,\sigma)\,d\sigma$.

Lemma 2.3. *i) Assume H2) and H3). Let $M>0$, and let $\{u_j\}_{j\in\mathbf{N}}$ be a sequence of measurable functions on Ω such that $0\le u_j\le M$ for all $j\in\mathbb{N}$, and $\lim_{j\to\infty} u_j = u$ a.e. in Ω for some $u:\Omega\to\mathbb{R}$. Then $\lim_{j\to\infty}\int_\Omega F(.,u_j) = \int_\Omega F(.,u)$.*

ii) If H2) and H3) hold, and if u,v are nonnegative functions in $L^\infty(\Omega)$, then

$$\lim_{t\to0^+} \frac{1}{t}\int_\Omega \left(F(.,u+tv) - F(.,u)\right) = \int_\Omega v f(.,u) \text{ and} \tag{2.2}$$

$$\lim_{t\to0^+} \int_\Omega \left(F(.,u+tv) - F(.,u)\right) = 0. \tag{2.3}$$

If, in addition, $u-\varepsilon_0 v\ge 0$ for some $\varepsilon_0>0$. then

$$\lim_{t\to0^-} \frac{1}{t}\int_\Omega \left(F(.,u+tv) - F(.,u)\right) = \int_\Omega v f(.,u) \text{ and} \tag{2.4}$$

$$\lim_{t\to0^-} \int_\Omega \left(F(.,u+tv) - F(.,u)\right) = 0. \tag{2.5}$$

Proof. i) follows easily from *H2)* and *H3)* applying Lebesgue's dominated convergence theorem.

To see ii) note that, for $0<t<1$, by the mean value theorem,

$$F(.,u+tv)-F(.,u)=tvf(.,u+\eta_t) \tag{2.6}$$

in $\{v>0\}$; where $\eta_t:\{v>0\}\to\mathbb{R}$ depends on u,v,t, and satisfies $0\leq\eta_t\leq t\|v\|_\infty$. We define $\eta_t=0$ in $\{v=0\}$, so that (2.6) holds in Ω. Now,

$$\begin{aligned}&\left|\frac{1}{t}\int_\Omega(F(.,u+tv)-F(.,u))-\int_\Omega vf(.,u)\right|\\&=\left|\int_\Omega v(f(.,u+\eta_t)-f(.,u))\right|\leq\int_\Omega v|f(.,u+\eta_t)-f(.,u)|.\end{aligned}$$

By *H2)*, $\lim_{t\to0^+}v|f(.,u+\eta_t)-f(.,u)|=0$ *a.e.* in Ω; and, by *H3)*,

$$v|f(.,u+\eta_t)-f(.,u)|\leq 2M\sup_{0\leq s\leq 2M}|f(.,s)|\in L^1(\Omega),$$

where $M:=\|u\|_\infty+\|v\|_\infty$. Then, by Lebesgue's dominated convergence theorem,

$$\lim_{t\to0^+}\int_\Omega v|f(.,u+\eta_t)-f(.,u)|=0.$$

Thus (2.2) (and so also (2.3)) holds. The proofs of (2.4) and (2.5) are similar.

Lemma 2.4. *Assume H1)-H3), and let $M>0$, $\alpha\in(0,3)$. Then*

i) J is coercive on D_M^α with respect to the topology of $H_0^1(\Omega)$; i.e., $J(u)\to\infty$ when $u\in D_M^\alpha$, and $\|\nabla u\|_2\to\infty$.

ii) $\inf_{u\in D_M^\alpha}J(u)$ is achieved at some $u\in D_M^\alpha$.

Proof. For $u\in D_M^\alpha$ we have $|\int_\Omega F(.,u)|\leq MB_M$, where $B_M:=\int_\Omega\sup_{0\leq s\leq M}|f(.,s)|$. Note that, by *H3)*, $B_M<\infty$.

If $1<\alpha<3$, we have $-\frac{1}{1-\alpha}\int_\Omega au^{1-\alpha}\geq0$, then $J(u)\geq\frac{1}{2}\int_\Omega|\nabla u|^2-MB_M$, which implies i).

If $0<\alpha<1$, from Hölder's and Poincaré's inequalities we get $\frac{1}{1-\alpha}\int_\Omega au^{1-\alpha}\leq c\|\nabla u\|_2^{1-\alpha}$ for some positive constant c independent of u. Thus $J(u)\geq\frac{1}{2}\|\nabla u\|_2^2-c\|\nabla u\|_2^{1-\alpha}-MB_M$; therefore i) holds also in this case.

If $\alpha=1$, using Poincaré's inequality, and that $\ln s\leq s$ for $s>0$, for some positive constant c independent of u we get

$$-\int_{\{a>0\}}a\ln u\geq-\int_{\{a>0\}\cap\{u\geq1\}}a\ln u\geq-\int_{\{a>0\}\cap\{u\geq1\}}au\geq-\int_\Omega au\geq-c\|\nabla u\|_2$$

and then $J(u)\geq\frac{1}{2}\|\nabla u\|_2^2-c\|\nabla u\|_2-B_M$; consequently i) holds when $\alpha=1$.

To prove ii), let $\beta := \inf_{u\in D_M^\alpha} J(u)$. Since $D_M^\alpha \neq \varnothing$, we have $\beta < \infty$. Consider a sequence $\{u_j\}_{j\in\mathbf{N}} \subset D_M^\alpha$ such that $\lim_{j\to\infty} J(u_j) = \beta$; it follows from i) that $\{u_j\}_{j\in\mathbf{N}}$ is bounded in $H_0^1(\Omega)$. Since the inclusion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, there exist $u \in H_0^1(\Omega)$, and a subsequence $\{u_{j_k}\}_{k\in\mathbf{N}}$ such that $\{u_{j_k}\}_{k\in\mathbf{N}}$ converges strongly in $L^2(\Omega)$, and such that $\{\nabla u_{j_k}\}_{k\in\mathbb{N}}$ converges weakly to ∇u in $L^2(\Omega, \mathbf{R}^n)$. Taking a subsequence if necessary, we can assume that $\{u_{j_k}\}_{k\in\mathbf{N}}$ converges to u *a.e.* in Ω. Thus

$$\|\nabla u\|_2 \leq \underline{\lim}_{k\to\infty} \|\nabla u_{j_k}\|_2 . \tag{2.7}$$

Note that $u \in D_M^\alpha$. Indeed, since $0 \leq u_{j_k} \leq M$ for all k, we have $0 \leq u \leq M$, and so $u \in D_M^\alpha$ when $0 < \alpha < 1$. If $1 < \alpha < 3$, by Fatou's lemma,

$$\begin{aligned} -\frac{1}{1-\alpha}\int_\Omega au^{1-\alpha} &\leq \underline{\lim}_{k\to\infty} \int_\Omega \frac{-1}{1-\alpha} au_{j_k}^{1-\alpha} \\ &= \underline{\lim}_{k\to\infty} \left(J(u_{j_k}) - \frac{1}{2}\int_\Omega |\nabla u_{j_k}|^2 + \int_\Omega F(., u_{j_k}) \right) \\ &\leq \underline{\lim}_{k\to\infty} J(u_{j_k}) + MB_M < \infty \end{aligned}$$

and then $u \in D_M^\alpha$ when $1 < \alpha < 3$. If $\alpha = 1$, again by Fatou's Lemma,

$$\begin{aligned} \int_{\{a>0\}} a|\ln u| &= \int_{\{a>0\}} \underline{\lim}_{k\to\infty} a|\ln u_{j_k}| \\ &\leq \underline{\lim}_{k\to\infty} \left(-\int_{\{a>0\}\cap\{u_{j_k}\leq 1\}} a\ln u_{j_k} + \int_{\{a>0\}\cap\{u_{j_k}>1\}} a\ln u_{j_k} \right) \\ &= \underline{\lim}_{k\to\infty} \left(-\int_{\{a>0\}} a\ln u_{j_k} + 2\int_{\{a>0\}\cap\{u_{j_k}>1\}} a\ln u_{j_k} \right) \\ &\leq \underline{\lim}_{k\to\infty} \left(-\int_{\{a>0\}} a\ln u_{j_k} + 2\int_{\{a>0\}} au_{j_k} \right) \end{aligned}$$

and, since $\{au_{j_k}\}_{k\in\mathbb{N}}$ converges to au in the $L^1(\Omega)$ norm,

$$\begin{aligned} &\underline{\lim}_{k\to\infty} \left(-\int_{\{a>0\}} a\ln u_{j_k} + 2\int_{\{a>0\}} au_{j_k} \right) \\ &= \underline{\lim}_{k\to\infty} \left(-\int_{\{a>0\}} a\ln u_{j_k} \right) + 2\int_{\{a>0\}} au \\ &= \underline{\lim}_{k\to\infty} \left(J(u_{j_k}) - \frac{1}{2}\int_\Omega |\nabla u_{j_k}|^2 + \int_\Omega F(., u_{j_k}) \right) + 2\int_{\{a>0\}} au \\ &\leq \underline{\lim}_{k\to\infty} J(u_{j_k}) + MB_M + 2\int_{\{a>0\}} au < \infty. \end{aligned}$$

Then $u \in D_M^\alpha$ also when $\alpha = 1$. Since $u \in D_M^\alpha$, we have $J(u) \geq \beta$; therefore, to prove ii), it remains to show that $J(u) \leq \beta$. To do this observe that, by Lemma 2.3 i),

$$\lim_{k\to\infty} \int_\Omega F\left(.,u_{j_k}\right) = \int_\Omega F(.,u)\,. \tag{2.8}$$

If $1 < \alpha < 3$, from (2.7), (2.8) and Fatou's lemma, we get

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{1-\alpha}\int_{\{a>0\}} a u^{1-\alpha} - \int_\Omega F(.,u) \\ &\leq \underline{\lim}_{k\to\infty}\left(\frac{1}{2}\int_\Omega \left|\nabla u_{j_k}\right|^2\right) + \underline{\lim}_{k\to\infty}\left(-\frac{1}{1-\alpha}\int_{\{a>0\}} a u_{j_k}^{1-\alpha}\right) \\ &+ \lim_{k\to\infty}\int_\Omega F\left(.,u_{j_k}\right) \\ &\leq \underline{\lim}_{k\to\infty}\left(\frac{1}{2}\int_\Omega \left|\nabla u_{j_k}\right|^2 - \frac{1}{1-\alpha}\int_{\{a>0\}} a u^{1-\alpha} - \int_\Omega F\left(.,u_{j_k}\right)\right) \\ &= \underline{\lim}_{k\to\infty} J\left(u_{j_k}\right) = \beta. \end{aligned} \tag{2.9}$$

Then $J(u) \leq \beta$ when $1 < \alpha < 3$. Consider now the case $0 < \alpha < 1$: since $0 \leq u_{j_k} \leq M$ for all k, Lebesgue's dominated convergence theorem gives $\lim_{k\to\infty} \frac{1}{1-\alpha}\int_{\{a>0\}} a u_{j_k}^{1-\alpha} = \frac{1}{1-\alpha}\int_{\{a>0\}} a u^{1-\alpha}$, and then, as in (2.9), we get $J(u) \leq \beta$.

Finally suppose $\alpha = 1$: Since $u_{j_k} \in D_M^\alpha$ we have $a \ln\left(M/u_{j_k}\right) \geq 0$, and then Fatou's lemma gives

$$\begin{aligned} -\int_{\{a>0\}} a \ln u &= \int_{\{a>0\}} a \ln\left(\frac{M}{u}\right) - \int_{\{a>0\}} a \ln M \\ &= \int_{\{a>0\}} \underline{\lim}_{k\to\infty} a \ln\left(\frac{M}{u_{j_k}}\right) - \int_{\{a>0\}} a \ln M \\ &\leq \underline{\lim}_{k\to\infty} \int_{\{a>0\}} a \ln\left(\frac{M}{u_{j_k}}\right) - \int_{\{a>0\}} a \ln M \\ &= \underline{\lim}_{k\to\infty}\left(\int_{\{a>0\}} a \ln M - \int_{\{a>0\}} a \ln u_{j_k}\right) - \int_{\{a>0\}} a \ln M \\ &= \underline{\lim}_{k\to\infty}\left(-\int_{\{a>0\}} a \ln u_{j_k}\right). \end{aligned}$$

Now, we proceed as in (2.9), replacing there $-\frac{1}{1-\alpha}\int_{\{a>0\}} a u_{j_k}^{1-\alpha}$ by $-\int_{\{a>0\}} a \ln u_{j_k}$, and $-\frac{1}{1-\alpha}\int_{\{a>0\}} a u^{1-\alpha}$ by $-\int_{\{a>0\}} a \ln u$, to conclude that $J(u) \leq \beta$ also for $\alpha = 1$.

Lemma 2.5. *Assume H1)-H3), and let $M > 0$, $\alpha \in (0,3)$. Then*

$$\int_\Omega \langle \nabla u, \nabla (u\varphi)\rangle \leq \int_\Omega \chi_{\{u>0\}} a u^{1-\alpha}\varphi + \int_\Omega f(.,u)\, u\varphi \tag{2.10}$$

for any minimizer u for J on D_M^α, and for any nonnegative $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Let u be a minimizer for J on D_M^α, $\tau \in (-1,0)$; and let φ be a nonnegative function in $H^1(\Omega) \cap L^\infty(\Omega)$ that, in addition, satisfies $\|\varphi\|_\infty \leq \frac{1}{2}$.

Note that $u+\tau u\varphi \in D_M^\alpha$. Indeed, $0 \leq u+\tau u\varphi \leq M$ and (since $u \in L^\infty(\Omega)$) $u+\tau u\varphi \in H_0^1(\Omega)$. In particular, this gives $u+\tau u\varphi \in D_M^\alpha$ when $0<\alpha<1$.

If $1<\alpha<3$ we have also $\left|a(u+\tau u\varphi)^{1-\alpha}\right| \leq \frac{1}{2^{1-\alpha}} a u^{1-\alpha} \in L^1(\{a>0\})$, and so $u+\tau u\varphi \in D_M^\alpha$.

If $\alpha=1$ then $|a\ln(u+\tau u\varphi)| = a|\ln u + \ln(1+\tau\varphi)| \leq a|\ln u| + a|\ln(1+\tau\varphi)| \in L^1(\{a>0\})$ and so, again in this case, $u+\tau u\varphi \in D_M^\alpha$.

To prove (2.10) we consider first the case where $\alpha \neq 1$. Since $J(u) \leq J(u+\tau u\varphi)$, a computation gives

$$\begin{aligned} &\tau \int_\Omega \langle \nabla u, \nabla(u\varphi)\rangle \\ &\geq \frac{1}{1-\alpha} \int_{\{a>0\}} a u^{1-\alpha}\left((1+\tau\varphi)^{1-\alpha}-1\right) + \int_\Omega (F(.,u+\tau\varphi u) - F(.,u)) \\ &- \frac{\tau^2}{2}\int_\Omega u^2|\nabla\varphi|^2 - \frac{\tau^2}{2}\int_\Omega \varphi^2|\nabla u|^2 - \tau^2\int_\Omega u\varphi\langle\nabla u,\nabla\varphi\rangle, \end{aligned} \tag{2.11}$$

and a Taylor expansion gives

$$(1+\tau\varphi)^{1-\alpha} - 1 = (1-\alpha)\tau\varphi + \frac{\tau^2}{2}(1-\alpha)\alpha(1+\zeta)^{-\alpha-1}\varphi^2$$

for some measurable function ζ such that $-\frac{1}{2} \leq \tau\varphi \leq \zeta \leq 0$. Since $au^{1-\alpha} \in L^1(\{a>0\})$, and $1+\zeta \geq \frac{1}{2}$, we have $\left|\int_{\{a>0\}} a u^{1-\alpha}(1+\zeta)^{-\alpha-1}\varphi^2\right| \leq c$ where c is a positive constant independent of τ; and so,

$$\lim_{\tau\to 0^-} \frac{1}{(1-\alpha)\tau}\int_{\{a>0\}} a u^{1-\alpha}\left((1+\tau\varphi)^{1-\alpha}-1\right) = \int_{\{a>0\}} a u^{1-\alpha}\varphi. \tag{2.12}$$

Also, by Lemma 2.3 ii) we have

$$\lim_{\tau\to 0^-} \frac{1}{\tau}\int_\Omega (F(.,u+\tau\varphi u) - F(.,u)) = \int_\Omega \varphi u f(.,u). \tag{2.13}$$

Dividing by τ the inequality (2.11), letting $\tau \to 0^-$, and using (2.12) and (2.13), we get

$$\int_\Omega \langle \nabla u, \nabla(u\varphi)\rangle \leq \int_{\{a>0\}} a u^{1-\alpha}\varphi + \int_\Omega f(.,u)u\varphi. \tag{2.14}$$

Note that $au^{1-\alpha}\varphi = \chi_{\{u>0\}} a u^{1-\alpha}\varphi$ (this clearly holds when $0<\alpha<1$; and when $1 \leq \alpha < 3$ the equality follows from the fact that $u>0$ *a.e.* in $\{a>0\}$). Thus (2.14) gives (2.10) and, since both sides in (2.10) are linear on φ, our additional assumption $\|\varphi\|_\infty \leq \frac{1}{2}$ can be removed. Thus the lemma holds when $\alpha \neq 1$.

If $\alpha = 1$ we have, as before, (2.11), with the term $\frac{1}{1-\alpha}\int_{\{a>0\}} au^{1-\alpha}\left((1+\tau\varphi)^{1-\alpha}-1\right)$ replaced by $\int_{\{a>0\}} a\left(\ln\left(u\left(1+\tau\varphi\right)\right)-\ln u\right) = \int_{\{a>0\}} a\ln\left(1+\tau\varphi\right)$; and a Taylor expansion gives $\ln\left(1+\tau\varphi\right) = \tau\varphi - \left(1+\zeta_\tau\right)^{-2}\tau^2\varphi^2$ for some measurable function $\zeta_\tau : \Omega \to \mathbb{R}$ satisfying $-\frac{1}{2} \leq \tau\varphi \leq \zeta_\tau \leq 0$. Then

$$\lim_{\tau\to 0^-}\frac{1}{\tau}\int_{\{a>0\}} a\left(\ln\left(u\left(1+\tau\varphi\right)\right)-\ln u\right) = \int_{\{a>0\}} au\varphi$$

and so, proceeding as in the previous case, we conclude that (2.10) holds when $\|\varphi\|_\infty \leq \frac{1}{2}$; and, as before, this additional assumption on φ can be removed.

Lemma 2.6. *Assume H1)-H3). Let* $M > 0$, $\alpha \in (0,3)$. *Let* m *be an integer such that* $m \geq \max\{2, 1+\alpha\}$, *and let* u *be a minimizer for* J *on* D_M^α. *Then, for any nonnegative* $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \left\langle \nabla\left(u^m\right), \nabla\left(\varphi\right)\right\rangle \leq m\int_\Omega \left(au^{m-1-\alpha} + u^{m-1}f\left(.,u\right)\right)\varphi$$

Proof. u is bounded, therefore $u^m \in H_0^1(\Omega)$ and $\nabla\left(u^m\right) = mu^{m-1}\nabla u$. Also $u^{m-2}\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for any nonnegative $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then

$$\begin{aligned} &\int_\Omega \left\langle \nabla\left(u^m\right), \nabla\varphi\right\rangle \\ &= m\int_\Omega u^{m-1}\left\langle \nabla u, \nabla\varphi\right\rangle \\ &= m\int_\Omega \left\langle \nabla u, \nabla\left(u^{m-1}\varphi\right)\right\rangle - m(m-1)\int_\Omega u^{m-2}\varphi\left|\nabla u\right|^2 \\ &\leq m\int_\Omega \left(\chi_{\{u>0\}}au^{m-1-\alpha} + u^{m-1}f\left(.,u\right)\right)\varphi, \end{aligned} \tag{2.15}$$

the last inequality by Lemma 2.5. Since $\chi_{\{u>0\}}au^{m-1-\alpha} = au^{m-1-\alpha}$, the lemma follows.

Remark 2.7. Let $u \in L^1_{loc}(\Omega)$ such that $\nabla u \in L^2(\Omega)$, and let $w \in L^\infty(\Omega)$. If $\int_\Omega \langle \nabla u, \nabla\varphi\rangle \leq \int_\Omega w\varphi$ (respectively $\int_\Omega \langle \nabla u, \nabla\varphi\rangle \geq \int_\Omega w\varphi$) for every nonnegative $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ then the corresponding inequality holds for all nonnegative $\varphi \in H_0^1(\Omega)$ (by using a density argument with the truncations $\varphi_j(x) := \min\{\varphi(x), j\}$, $j \in \mathbb{N}$).

Lemma 2.8. *Assume H1)-H4), and* $\alpha \in (0,3)$. *Then there exists a positive number* $\mathscr{M}$ *such that, for any* $M > 0$, *and any minimizer* u *for* J *on* D_M^α, $\|u\|_\infty \leq \mathscr{M}$.

Proof. Let $M>0$, and let u be a minimizer for J on D_M^α. Assume first that *H4')* holds, and let m and b be as there. By Lemma 2.6, for $0\leq\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega \langle\nabla(u^m),\nabla(\varphi)\rangle \leq m\int_\Omega\left(\chi_{\{u>0\}}au^{m-1-\alpha}+u^{m-1}f(.,u)\right)\varphi$$
$$\leq m\int_\Omega\left(\chi_{\{u>0\}}au^{m-1-\alpha}+bu^m\right)\varphi$$

and so, by Remark 2.7, the same inequalities hold for all nonnegative $\varphi\in H_0^1(\Omega)$, i.e.,

$$(-\Delta-mb)(u^m)\leq m\chi_{\{u>0\}}au^{m-1-\alpha}=mau^{m-1-\alpha} \text{ in } \left(H_0^1(\Omega)\right)'.$$

Since $0<m<\lambda_1(b)$, the operator $(-\Delta-mb)^{-1}:L^\infty(\Omega)\to H_0^1(\Omega)\subset L^\infty(\Omega)$ is well defined, bounded and positive. Let $v:=(-\Delta-mb)^{-1}\left(mau^{m-1-\alpha}\right)$. Then

$$\|u\|_\infty^m\leq\|v\|_\infty\leq\left\|(-\Delta-mb)^{-1}\right\|_{\infty,\infty}\left\|mau^{m-1-\alpha}\right\|_\infty=c\|u\|_\infty^{m-1-\alpha}$$

for some positive constant c independent of M and u; therefore the lemma holds with $\mathscr{M}=c^{\frac{1}{1+\alpha}}$.

Assume now that *H4")* holds. Let m be an integer such that $m\geq\max\{2,1+\alpha\}$, let $\lambda_1:=\lambda_1(\mathbf{1})$, let $\varepsilon\in\left(0,\frac{\lambda_1}{m}\right)$, and let $s_0>0$ be such that $\sup_{s\geq s_0}\frac{f(.,s)}{s}\leq\varepsilon$. From Lemma 2.6 we have, for $0\leq\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$, that

$$\begin{aligned}(2.16)\qquad &\int_\Omega\langle\nabla(u^m),\nabla(\varphi)\rangle\\ &\leq m\int_\Omega\chi_{\{u>0\}}au^{m-1-\alpha}\varphi+m\int_{\{u<s_0\}}u^{m-1}f(.,u)\varphi+m\int_{\{u\geq s_0\}}u^{m-1}f(.,u)\varphi\\ &\leq m\int_\Omega\chi_{\{u>0\}}au^{m-1-\alpha}\varphi+ms_0^{m-1}\int_{\{u<s_0\}}\sup_{s<s_0}|f(.,s)|\varphi+m\int_{\{u\geq s_0\}}u^m\frac{f(.,u)}{u}\varphi\\ &\leq m\int_\Omega\left(au^{m-1-\alpha}+A+m\varepsilon u^m\right)\varphi,\end{aligned}$$

where

$$A:=ms_0^{m-1}\sup_{(x,s)\in\overline{\Omega}\times[0,s_0]}|f(x,s)|$$

is a constant independent of M and u. Thus, by Remark 2.7,

$$-\Delta(u^m)\leq ma(u^m)^{\frac{m-1-\alpha}{m}}+m\varepsilon u^m+A \text{ in } \left(H_0^1(\Omega)\right)'. \tag{2.17}$$

Since $m\varepsilon<\lambda_1(\mathbf{1})$ we have that $(-\Delta-m\varepsilon\mathbf{1})^{-1}$ is a bounded and positive operator on $L^\infty(\Omega)$; and so, from (2.17), $u^m\leq(-\Delta-m\varepsilon\mathbf{1})^{-1}\left(ma(u^m)^{\frac{m-1-\alpha}{m}}+A\right)$; which gives

$$\|u^m\|_\infty\leq c'\|u^m\|_\infty^{\frac{m-1-\alpha}{m}}+c' \tag{2.18}$$

for some c' independent of M and u. Since $0 < \frac{m-1-\alpha}{m} < 1$, the lemma follows.

Remark 2.9. Let $w \in L^1(\Omega)$ such that $|\{w > 0\}| > 0$, and let $\beta \in [0,1)$. Then there exists a nonnegative $\Phi \in C_c^\infty(\Omega)$ such that $\int w\Phi^{1-\beta} > 0$. Indeed, consider a nonnegative radial function $h \in C_c^\infty(\mathbb{R}^n)$ with support in the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ and such that $\int_B h = 1$. For $\varepsilon > 0$ let $h_\varepsilon(x) := \frac{1}{\varepsilon^n} h\left(\frac{x}{\varepsilon}\right)$ and for $\delta > 0$ let $\Omega_\delta := \{x \in \Omega : dist(x, \partial\Omega) > \delta\}$. Then $|\{w > 0\} \cap \Omega_\delta| > 0$ for δ positive and small enough. Fix such a δ and define $E = \{x \in \Omega : w(x) > 0\} \cap \Omega_\delta$. For $\varepsilon > 0$ define $\Phi_\varepsilon := h_\varepsilon * \chi_E$. Then $\Phi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $supp(\Phi_\varepsilon) \subset \Omega$ for $\varepsilon < \delta$. Also, $\lim_{\varepsilon \to 0^+} \Phi_\varepsilon = \chi_E$ with convergence in $L^1(\Omega)$ (see e.g., [2], Theorem 4.22), and so $\lim_{j\to\infty} \Phi_{\varepsilon_j} = \chi_E$ *a.e.* in Ω for some sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} \varepsilon_j = 0$. Then, by Fatou's Lemma, $0 < \int_\Omega w\chi_E \leq \underline{\lim}_{j\to\infty} \int_\Omega w\Phi_{\varepsilon_j}^{1-\beta}$. Thus $\int_\Omega w\Phi_{\varepsilon_j}^{1-\beta} > 0$ for j large enough.

Lemma 2.10. *Let $\alpha \in (0,3)$. Assume H1)-H4). Let $\mathscr{M}$ be as in Lemma 2.8, and let $M > \mathscr{M}$. If u is a minimizer for J on D_M^α, then $\chi_{\{u>0\}} a u^{-\alpha} \not\equiv 0$. In particular, $u \not\equiv 0$.*

Proof. If $1 \leq \alpha < 3$, $u \in D_M^\alpha$ implies $u > 0$ *a.e.* in $\{a > 0\}$, and so $\chi_{\{u>0\}} a u^{-\alpha} \not\equiv 0$. To prove the lemma when $0 < \alpha < 1$ we proceed by contradiction. Suppose that u is a minimizer for J on D_M^α and that $\chi_{\{u>0\}} a u^{-\alpha} = 0$. Let $\Phi \in C_c^\infty(\Omega)$ such that $\Phi \geq 0$ and $\int a\Phi^{1-\alpha} > 0$. By Lemma 2.8, $u \leq \mathscr{M} < M$; thus $u + t\Phi \in D_M^\alpha$ for t positive and small enough, and so $J(u) \leq J(u + t\Phi)$. Also, $\chi_{\{u>0\}} a u^{-\alpha} = 0$ implies that $u = 0$ *a.e.* in $\{a > 0\}$. Then $\int_\Omega a u^{1-\alpha} = 0$, and $\int_\Omega a(u + t\Phi)^{1-\alpha} = \int_\Omega a(t\Phi)^{1-\alpha}$. Thus the inequality $J(u) \leq J(u + t\Phi)$ can be written as

$$0 \leq t \int_\Omega \langle \nabla u, \nabla \Phi \rangle + \frac{t^2}{2} \int_\Omega |\nabla \Phi|^2 - \frac{t^{1-\alpha}}{1-\alpha} \int_{\{a>0\}} a\Phi^{1-\alpha} - \int_\Omega (F(.,u+t\Phi) - F(.,u)).$$

From this inequality, dividing by $t^{1-\alpha}$, taking the limit as $t \to 0^+$, using that, by Lemma 2.3 ii), $\lim_{t\to 0^+} \frac{1}{t} \int_\Omega (F(.,u+t\Phi) - F(.,u)) = \int_\Omega \Phi f(.,u)$, and recalling that $\int_{\{a>0\}} a\Phi^{1-\alpha} > 0$, we obtain a contradiction.

In order to emphasize the dependence on f, we will sometimes write J_f for the functional J.

Lemma 2.11. *Let $\alpha \in (0,3)$. Assume H1)-H3), H5), and that either H6) or H7) holds. When H7) holds assume also that there exists $s_0 > 0$ such that $\underset{(x,s)\in\Omega\times(s_0,\infty)}{ess\sup} \frac{f(x,s)}{s} < \infty$ Then there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, there exists $\mathscr{M}_\lambda > 0$ such that $\|u_\lambda\|_\infty \leq \mathscr{M}_\lambda$ for any $M > 0$, and any minimizer u_λ for $J_{\lambda f}$ on D_M^α. If, in addition, $f \leq 0$ in $\Omega \times (0,\infty)$, then $\lambda^* = \infty$.*

Proof. Consider the case when *H7)* holds. Let $M>0$ and let u be a minimizer for $J_{\lambda f}$ on D_M^α. Let m be an integer such that $m \geq \max\{2, 1+\alpha\}$ and let $k > \max\left\{0, \underset{x\in\Omega\times(s_0,\infty)}{ess\sup} \frac{f(x,s)}{s}\right\}$. For $\lambda>0$ we can repeat the computations performed in (2.16), with λf and λmk in place of f and $m\varepsilon$ respectively, to obtain, for $0 \leq \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, that

$$\begin{aligned}\int_\Omega \langle \nabla(u^m), \nabla(\varphi)\rangle &\leq m\int_\Omega a u^{m-1-\alpha}\varphi + A\int_\Omega \varphi + \int_{\Omega\cap\{u>s_0\}} m\lambda u^m \frac{f(x,u)}{u}\varphi \\ &\leq m\int_\Omega a u^{m-1-\alpha}\varphi + A\int_\Omega \varphi + \delta\int_\Omega m\lambda k u^m\varphi,\end{aligned} \tag{2.20}$$

where $\delta := 0$ if $f \leq 0$ in $\Omega\times[0,\infty)$, and $\delta := 1$ otherwise; and where

$$A := m\lambda s_0^{m-1} \left\| f_{|\Omega\times(0,s_0)} \right\|_{L^\infty(\Omega\times(0,s_0))}$$

is a constant independent of M and u. Then, as in Lemma 2.8, we arrive to

$$-\Delta(u^m) \leq ma(u^m)^{\frac{m-1-\alpha}{m}} + \delta m\lambda k u^m + A \text{ in } \left(H_0^1(\Omega)\right)'. \tag{2.21}$$

If $\delta = 1$ and $0<\lambda<\frac{\lambda_1(\mathbf{1})}{mk}$, then $\lambda_1(\lambda\delta mk\mathbf{1}) = \frac{\lambda_1(\mathbf{1})}{\lambda mk} > 1$; and so, from (2.21),

$$u^m \leq (-\Delta - \lambda mk)^{-1}\left(ma(u^m)^{\frac{m-1-\alpha}{m}} + A\right),$$

which implies (2.18) for some positive constant c' independent of M and u; therefore the lemma holds with $\lambda^* = \frac{\lambda_1(\mathbf{1})}{m}$. If $\delta = 0$ (i.e., if $f \leq 0$), (2.21) gives $u^m \leq (-\Delta)^{-1}\left(ma(u^m)^{\frac{m-1-\alpha}{m}} + A\right)$, which implies that (2.18) holds for all $\lambda \geq 0$; therefore, in this case, the lemma holds with $\lambda^* = \infty$.

When *H6)* holds the proof is similar: let $k > \max\left\{0, \underset{x\in\Omega\times(0,\infty)}{ess\sup} \frac{f(x,s)}{s}\right\}$. Instead of (2.20) we now have

$$\begin{aligned}\int_\Omega \langle \nabla(u^m), \nabla(\varphi)\rangle &\leq m\int_\Omega a u^{m-1-\alpha}\varphi + \delta\int_{\Omega\cap\{u>0\}} m\lambda u^m \frac{f(x,u)}{u}\varphi \\ &\leq m\int_\Omega a u^{m-1-\alpha}\varphi + \delta\int_\Omega m\lambda k u^m\varphi,\end{aligned}$$

with δ as before. Thus (2.21) holds with $A=0$, and the proof ends as in the previous case.

3. Proofs of the main results

Proof of Theorem 1.2. Let $\mathscr{M}$ be as given in Lemma 2.8. Let $M = \mathscr{M}+1$, and let u be a minimizer for J on D_M^α. Thus, by Lemma 2.10, $\chi_{\{u>0\}} a u^{-\alpha} \not\equiv 0$ (and so $u \not\equiv 0$). Let ψ be a nonnegative function in $H_0^1(\Omega)\cap L^\infty(\Omega)$, and let $\varepsilon>0$. Thus $\frac{\psi}{u+\varepsilon}\in H^1(\Omega)\cap L^\infty(\Omega)$, and $\nabla\left(u\frac{\psi}{u+\varepsilon}\right)=\varepsilon\frac{\nabla u}{(u+\varepsilon)^2}\psi+\frac{u}{u+\varepsilon}\nabla\psi$. Then Lemma 2.5 gives

$$\begin{aligned}(3.1)\qquad &\varepsilon\int_\Omega \psi\frac{|\nabla u|^2}{(u+\varepsilon)^2}+\int_\Omega\frac{u}{u+\varepsilon}\langle\nabla u,\nabla\psi\rangle\\ &\le\int_\Omega\chi_{\{a>0\}}au^{1-\alpha}\frac{\psi}{u+\varepsilon}+\int_\Omega f(.,u)\,u\frac{\psi}{u+\varepsilon}.\end{aligned}$$

Since $\nabla u=0$ *a.e.* in $\{u=0\}$, (3.1) can be written as

$$\begin{aligned}(3.2)\qquad &\varepsilon\int_{\{u>0\}}\psi\frac{|\nabla u|^2}{(u+\varepsilon)^2}+\int_{\{u>0\}}\frac{u}{u+\varepsilon}\langle\nabla u,\nabla\psi\rangle\\ &-\int_{\{u>0\}}f(.,u)\frac{u}{u+\varepsilon}\psi\le\int_{\{u>0\}}au^{-\alpha}\frac{u}{u+\varepsilon}\psi.\end{aligned}$$

Also $\lim_{\varepsilon\to0^+}\frac{u}{u+\varepsilon}\langle\nabla u,\nabla\psi\rangle=\chi_{\{u>0\}}\langle\nabla u,\nabla\psi\rangle=\langle\nabla u,\nabla\psi\rangle$ *a.e.* in Ω, and $\left|\frac{u}{u+\varepsilon}\langle\nabla u,\nabla\psi\rangle\right|\le|\langle\nabla u,\nabla\psi\rangle|\in L^1(\Omega)$, and so Lebesgue's dominated convergence theorem gives

$$(3.3)\qquad \lim_{\varepsilon\to0^+}\int_{\{u>0\}}\frac{u}{u+\varepsilon}\langle\nabla u,\nabla\psi\rangle=\int_\Omega\langle\nabla u,\nabla\psi\rangle.$$

Since $\lim_{\varepsilon\to0^+}au^{-\alpha}\frac{u}{u+\varepsilon}\psi=au^{-\alpha}\psi$ *a.e.* in $\{u>0\}$, and $au^{-\alpha}\frac{u}{u+\varepsilon}\psi$ is nonincreasing in ε, the monotone convergence theorem gives

$$(3.4)\qquad \lim_{\varepsilon\to0^+}\int_{\{u>0\}}au^{-\alpha}\frac{u}{u+\varepsilon}\psi=\int_{\{u>0\}}au^{-\alpha}\psi=\int_\Omega\chi_{\{u>0\}}au^{-\alpha}\psi$$

Also, $\left|\frac{u}{u+\varepsilon}f(.,u)\,\psi\right|\le\sup_{0\le s\le M}|f(.,s)|\,\psi\in L^1(\Omega)$ and then, by Lebesgue's dominated convergence theorem,

$$(3.5)\qquad \lim_{\varepsilon\to0^+}\int_{\{u>0\}}f(.,u)\frac{u}{u+\varepsilon}\psi=\int_\Omega\chi_{\{u>0\}}f(.,u)\,\psi\le\int_\Omega f(.,u)\,\psi,$$

the last equality because, by *H5)*, $f(.,0)\ge0$. Then, from (3.2), (3.3), (3.4) and (3.5), we have

$$\begin{aligned}(3.6)\qquad &\int_\Omega\langle\nabla u,\nabla\psi\rangle-\int_\Omega f(.,u)\,\psi\\ &\le\int_\Omega\langle\nabla u,\nabla\psi\rangle-\int_\Omega\chi_{\{u>0\}}f(.,u)\,\psi\\ &\le\lim_{\varepsilon\to0^+}\left(\int_{\{u>0\}}\frac{u}{u+\varepsilon}\langle\nabla u,\nabla\psi\rangle-\int_{\{u>0\}}f(.,u)\frac{u}{u+\varepsilon}\psi\right)\\ &\le\overline{\lim}_{\varepsilon\to0^+}\left(\int_{\{u>0\}}\frac{\varepsilon\psi|\nabla u|^2}{(u+\varepsilon)^2}+\int_{\{u>0\}}\frac{u}{u+\varepsilon}\langle\nabla u,\nabla\psi\rangle-\int_{\{u>0\}}f(.,u)\frac{u}{u+\varepsilon}\psi\right)\\ &\le\overline{\lim}_{\varepsilon\to0^+}\int_{\{u>0\}}au^{-\alpha}\frac{u}{u+\varepsilon}\psi=\int_\Omega\chi_{\{u>0\}}au^{-\alpha}\psi.\end{aligned}$$

Thus $\int_\Omega \langle \nabla u, \nabla \psi \rangle - \int_\Omega f(.,u)\psi \le \int_\Omega \chi_{\{u>0\}} a u^{-\alpha} \psi$. To prove the existence assertion of the theorem it remains to see that$\chi_{\{u>0\}} a u^{-\alpha} \psi \in L^1(\Omega)$, and that

$$\int_\Omega \langle \nabla u, \nabla \psi \rangle - \int_\Omega f(.,u)\psi \ge \int_\Omega \chi_{\{u>0\}} a u^{-\alpha} \psi \tag{3.7}$$

for any nonnegative $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Assume temporarily that ψ satisfies the additional condition $\|\psi\|_\infty \le \frac{1}{2}$, and let $t \in (0,1)$. Note that $u + t\psi \in D_M^\alpha$. Indeed, by Lemma 2.8 we have $u \le \mathscr{M}$, and so $0 \le u + t\psi \le \mathscr{M} + 1 \le M$. Also $u + t\psi \in H_0^1(\Omega)$. Thus $u + t\psi \in D_M^\alpha$ when $0 < \alpha < 1$.

If $1 < \alpha < 3$, then $0 \le a(u+t\psi)^{1-\alpha} \le a u^{1-\alpha} \in L^1(\{a > 0\})$, and so $u + t\psi \in D_M^\alpha$.

If $\alpha = 1$, we have $a|\ln(u+t\psi)| \le a(u+t\psi)$ in $\{a>0\} \cap \{u + t\psi \ge 1\}$, and $a|\ln(u+t\psi)| \le a|\ln(u)|$ in $\{a>0\} \cap \{u+t\psi < 1\}$. Thus $a|\ln(u+t\psi)| \in L^1(\{a>0\})$, which implies that $u + t\psi \in D_M^\alpha$.

To prove (3.7) we consider first the case $\alpha \ne 1$: Using $J(u) \le J(u+t\psi)$ we obtain

$$\begin{aligned} 0 &\le \frac{1}{t}\left(J(u+t\psi) - J(u)\right) \\ &= \int_\Omega \langle \nabla u, \nabla \psi \rangle + \frac{t}{2}\int_\Omega |\nabla \psi|^2 - \int_{\{a>0\}} \frac{1}{(1-\alpha)t} a\left((u+t\psi)^{1-\alpha} - u^{1-\alpha}\right) \\ &\quad - \frac{1}{t}\int_\Omega \left(F(.,u+t\psi) - F(.,u)\right). \end{aligned} \tag{3.8}$$

If $1 < \alpha < 3$ we have $u > 0$ *a.e.* in $\{a > 0\}$, and so

$$\begin{aligned} &\int_{\{a>0\}} \frac{1}{(1-\alpha)t} a\left((u+t\psi)^{1-\alpha} - u^{1-\alpha}\right) \\ &= \int_{\{a>0\}\cap\{u>0\}} \frac{1}{(1-\alpha)t} a\left((u+t\psi)^{1-\alpha} - u^{1-\alpha}\right) \\ &= \int_{\{a>0\}\cap\{u>0\}\cap\{\psi>0\}} \frac{1}{(1-\alpha)t} a\left((u+t\psi)^{1-\alpha} - u^{1-\alpha}\right). \end{aligned} \tag{3.9}$$

By the mean value theorem $(u+t\psi)^{1-\alpha} - u^{1-\alpha} = (1-\alpha)(u+\sigma_t)^{-\alpha}\psi$ holds *a.e.* on $\{u>0\} \cap \{\psi > 0\}$, where σ_t is a measurable function (that depends on t, u and ψ) such that $0 < \sigma_t < t\psi$. Thus

$$\begin{aligned} &\frac{1}{(1-\alpha)t}\int_{\{a>0\}\cap\{u>0\}\cap\{\psi>0\}} a\left((u+t\psi)^{1-\alpha} - u^{1-\alpha}\right) \\ &= \int_{\{a>0\}\cap\{u>0\}\cap\{\psi>0\}} a(u+\sigma_t)^{-\alpha}\psi. \end{aligned} \tag{3.10}$$

Note that $a(u+\sigma_t)^{-\alpha}\psi \geq 0$ and $\lim_{t\to 0^+} a(u+\sigma_t)^{-\alpha}\psi = au^{-\alpha}\psi$ hold $a.e.$ on the set where $a>0$, $u>0$, and $\psi>0$; therefore, from (3.9), (3.10) and Fatou's Lemma, we get

$$\begin{aligned}(3.11)\qquad & \underline{\lim}_{t\to 0^+}\int_{\{a>0\}}\frac{1}{(1-\alpha)t}a\left((u+t\psi)^{1-\alpha}-u^{1-\alpha}\right)\\ &\geq \int_{\{a>0\}\cap\{u>0\}\cap\{\psi>0\}}\underline{\lim}_{t\to 0^+} a(u+\sigma_t)^{-\alpha}\psi\\ &= \int_{\{a>0\}\cap\{u>0\}\cap\{\psi>0\}} au^{-\alpha}\psi = \int_\Omega \chi_{\{u>0\}}au^{-\alpha}\psi.\end{aligned}$$

Consider now the case $0<\alpha<1$: we again apply the mean value theorem to get a measurable function $\sigma_t : \{\psi>0\}\to\mathbb{R}$ (which depends on t,u and ψ) that satisfies $0<\sigma_t<t\psi$, and

$$\begin{aligned}& \underline{\lim}_{t\to 0^+}\int_{\{a>0\}}\frac{1}{(1-\alpha)t}a\left((u+t\psi)^{1-\alpha}-u^{1-\alpha}\right)\\ &= \underline{\lim}_{t\to 0^+}\int_{\{a>0\}\cap\{\psi>0\}}\frac{1}{(1-\alpha)t}a\left((u+t\psi)^{1-\alpha}-u^{1-\alpha}\right)\\ &\geq \int_{\{a>0\}\cap\{\psi>0\}}\underline{\lim}_{t\to 0^+}\left(a(u+\sigma_t)^{-\alpha}\psi\right)\\ &= \int_{\{a>0\}\cap\{\psi>0\}} au^{-\alpha}\psi,\end{aligned}$$

where $(au^{-\alpha}\psi)(x) := \infty$ if $a(x)>0$, $\psi(x)>0$, and $u(x)=0$. Thus, for $\alpha\in(0,1)\cup(1,3)$, we have

$$(3.12)\qquad \underline{\lim}_{t\to 0^+}\frac{1}{(1-\alpha)t}\int_{\{a>0\}} a\left((u+t\psi)^{1-\alpha}-u^{1-\alpha}\right) \geq \int_\Omega \chi_{\{u>0\}}au^{-\alpha}\psi.$$

Also, by Lemma 2.3 ii), we have

$$(3.13)\qquad \lim_{t\to 0^+}\frac{1}{t}\int_\Omega (F(.,u+t\psi)-F(.,u)) = \int_\Omega f(.,u)\psi.$$

Now, from (3.8),

$$\begin{aligned}(3.14)\qquad & \frac{1}{(1-\alpha)t}\int_{\{a>0\}\cap\{u>0\}\cap\{\psi>0\}} a\left((u+t\psi)^{1-\alpha}-u^{1-\alpha}\right)\\ &\leq \int_\Omega \langle\nabla u,\nabla\psi\rangle - \frac{1}{t}\int_\Omega (F(.,u+t\psi)-F(.,u)) + \frac{t}{2}\int_\Omega |\nabla\psi|^2\end{aligned}$$

and so, for $1<\alpha<3$, taking $\underline{\lim}_{t\to 0^+}$ in (3.14), and using (3.11), (3.12), and (3.13), we get

$$(3.15)\qquad \int_\Omega \langle\nabla u,\nabla\psi\rangle - \int_\Omega f(.,u)\psi \geq \int_\Omega \chi_{\{u>0\}}au^{-\alpha}\psi \text{ if } 1<\alpha<3,$$

which, in particular, gives $\chi_{\{u>0\}}au^{-\alpha}\psi \in L^1(\Omega)$. Since both sides in (3.15) are linear on ψ, the additional assumption $\|\psi\|_\infty \leq \frac{1}{2}$ can be removed. Then u is a solution to (1.1) when $1<\alpha<3$; and since $u \in D_M^\alpha$, it satisfies $u>0$ *a.e.* in $\{a>0\}$.

Similarly, if $0<\alpha<1$, taking $\underline{\lim}_{t\to 0^+}$ in (3.14), and using (3.11), (3.12) and (3.13), we get

$$\int_\Omega \langle \nabla u, \nabla \psi \rangle - \int_\Omega f(.,u)\,\psi \geq \int_{\{a>0\}\cap\{\psi>0\}} au^{-\alpha}\psi \text{ if } 0<\alpha<1, \tag{3.16}$$

for any nonnegative $\psi \in H_0^1(\Omega)\cap L^\infty(\Omega)$. In particular (3.16) gives that $u>0$ *a.e.* in $\{a>0\}\cap\{\psi>0\}$. From this fact, we conclude (using Remark 2.9 applied with $w=a\chi_{\{u=0\}}$) that $u>0$ *a.e.* in $\{a>0\}$. Then $\int_{\{a>0\}\cap\{\psi>0\}} au^{-\alpha}\psi = \int_\Omega \chi_{\{u>0\}}au^{-\alpha}\psi$; and so, if $0<\alpha<1$, (3.16) becomes

$$\int_\Omega \langle \nabla u, \nabla \psi \rangle - \int_\Omega f(.,u)\,\psi \geq \int_\Omega \chi_{\{u>0\}}au^{-\alpha}\psi, \tag{3.17}$$

which, in particular, gives $\chi_{\{u>0\}}au^{-\alpha}\psi \in L^1(\Omega)$. Summing up, when $\alpha \neq 1$, $u>0$ *a.e.* in $\{a>0\}$; and, for any nonnegative $\psi \in H_0^1(\Omega)\cap L^\infty(\Omega)$, $\chi_{\{u>0\}}au^{-\alpha}\psi \in L^1(\Omega)$, and (3.7) holds.

When $\alpha=1$, the same facts can be proved proceeding, line by line, as in the case $1<\alpha<3$, but with $\frac{1}{1-\alpha}a\left((u+t\psi)^{1-\alpha}-u^{1-\alpha}\right)$ replaced by $a(\ln(u+t\psi)-\ln u)$; and using that, on the set $\{u>0\}\cap\{\psi>0\}$, we have

$$\ln(u+t\psi)-\ln u = \frac{1}{2}(u+\sigma)^{-1}t\psi$$

for some measurable function σ, that depends on t,u and ψ, and satisfies $0<\sigma<t\psi$.

Finally, if $f(.,0)>0$ *a.e.* in Ω, by (3.6) we have, for any nonnegative $\psi \in H_0^1(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega \langle \nabla u, \nabla \psi \rangle - \int_\Omega \chi_{\{u>0\}} f(.,u)\,\psi \leq \int_\Omega \chi_{\{u>0\}}au^{-\alpha}\psi.$$

which, jointly with (3.15), implies $\int_\Omega f(.,u)\,\psi \leq \int_\Omega \chi_{\{u>0\}} f(.,u)\,\psi$; and then $\int_{\{u=0\}} f(.,u)\,\psi \leq 0$. Since $f(.,0)>0$ *a.e.* in Ω, it follows that $\chi_{\{u=0\}} f(.,0)\,\psi = 0$ *a.e.* in Ω for any nonnegative $\psi \in C_c^\infty(\Omega)$. Thus, by Remark 2.9, $\chi_{\{u=0\}} f(.,0)=0$ *a.e.* in Ω, and then $|\{u=0\}|=0$.

Remark 3.1. If $f \leq 0$, condition *H4)* is automatically fulfilled; indeed, in this case *H4')* holds.

Proof of Theorem 1.3. Since for $\lambda \geq 0$, λf satisfies the same assumptions fulfilled by f, the first assertion of the theorem follows from Theorem 1.2. If, in addition, $\lim_{s\to\infty}\frac{f(x,s)}{s}=0$ uniformly on Ω, then $-f$ satisfies *H1)-H3), H5),* and *H4");* and so, for $\lambda<0$, writing $\lambda f =$

$-\lambda(-f)$, the second assertion of the theorem follows from the first one. Finally, if $\lambda \geq 0$ and $f(.,0) > 0$ *a.e.* in Ω, the statement $u_\lambda > 0$ *a.e.* in Ω follows from Theorem 1.2, again.

Proof of Theorem 1.4. Assume that *H6)* holds. Let m be an integer such that $m \geq \max\{2, 1+\alpha\}$, and let $k \in \mathbb{R}$ satisfy $k > \max\left\{0, ess\sup_{\Omega\times(0,\infty)} \frac{f(x,s)}{s}\right\}$. Thus $\frac{\lambda f(.,s)}{s} \leq \lambda k$ and, since $\lambda_1(\lambda k\mathbf{1}) = \frac{\lambda_1(\mathbf{1})}{\lambda k} > m$ for $0 < \lambda < \frac{\lambda_1(\mathbf{1})}{mk}$, Theorem 1.2 gives, for such λ, the sought weak solution of (1.2).Note also that, if $\lambda = 0$, (1.2) reduces to $-\Delta u = \chi_{\{u>0\}} a u^{-\alpha}$ in Ω, $u \geq 0$ in Ω, $u = 0$ on $\partial\Omega$; and this problem has a positive weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (see [10]). Then the lemma holds with $\lambda^* := \frac{\lambda_1(\mathbf{1})}{mk}$.

Assume now that *H7)* holds. Let $V := \{k \in (0,\infty) : f(.,k) \in L^\infty(\Omega)\}$. Since $f \in L^\infty(\Omega \times (0,\sigma))$ for any $\sigma > 0$, we have that $\mathbb{R}\setminus V$ has zero Lebesgue's measure. For $k \in V$ to be chosen latter, let $f_k : \Omega \times [0,\infty)$ be defined by $f_k(.,s) := f(.,s)$ if $0 \leq s \leq k$, and by $f_k(.,s) := f(.,k)$ otherwise. Let $\lambda > 0$; clearly λf_k satisfies the conditions *H2), H3)* and *H5)*. Since $f(.,k) \in L^\infty(\Omega)$, we have $\overline{\lim}_{s\to\infty} \frac{\lambda f_k(.,s)}{s} = 0$ uniformly on Ω, and so λf_k satisfies also *H4")*. Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the solution to the problem:

$$\begin{cases} -\Delta u = \chi_{\{u>0\}} a u^{-\alpha} + \lambda f_k(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega,\ u \not\equiv 0 \text{ in } \Omega, \end{cases}$$

provided by Theorem 1.2. Thus u satisfies $\chi_{\{u>0\}} a u^{-\alpha} \not\equiv 0$. Let m be an integer such that $m \geq \max\{2, 1+\alpha\}$, let λ_1 be the first eigenvalue for $-\Delta$ on Ω with homogeneous Dirichlet condition, let $\eta \in (0,1)$, and let $\varepsilon := \eta \frac{\lambda_1}{\lambda m}$. Take $\Lambda \in (0,\infty)$, and define

$$s_0 := \max\left\{\frac{m\Lambda \|f(.,k)\|_\infty}{\eta\lambda_1}, k\right\}.$$

Thus, for $s > s_0$ and $0 \leq \lambda < \Lambda$,

$$\frac{\lambda |f_k(.,s)|}{s} \leq \frac{\Lambda |f(.,k)|}{s_0} \leq \frac{\eta\lambda_1}{m} < \frac{\lambda_1}{m} \ a.e. \text{ in } \Omega.$$

From the proof of Theorem 1.2 we know that, for M positive and large enough, u is a minimizer for $J_{\lambda f_k}$ on D_M^α; and so, by Lemma 2.6, we have, in the weak sense stated there (i.e., for test functions in $H_0^1(\Omega)\cap L^\infty(\Omega)$),

$$-\Delta(u^m) \le ma(u^m)^{\frac{m-1-\alpha}{m}} + mu^{m-1}\lambda f_k(.,u) \tag{3.18}$$
$$\le ma(u^m)^{\frac{m-1-\alpha}{m}} + \lambda mAs_0^{m-1} + mu^m\chi_{u>s_0}\frac{\lambda|f_k(.,u)|}{u}$$
$$\le mau^{m-1-\alpha} + \lambda mAs_0^{m-1} + \eta\lambda_1 u^m$$

for $0\le\lambda<\Lambda$ and with $A := 1+\left\|f_{|\Omega\times(0,s_0)}\right\|_{L^\infty(\Omega\times(0,s_0))}<\infty$. As $ma(u^m)^{\frac{m-1-\alpha}{m}} + \lambda mAs_0^{m-1} + \eta\lambda_1 u^m \in L^2(\Omega)$, Remark 2.7 says

$$-\Delta(u^m) \le ma(u^m)^{\frac{m-1-\alpha}{m}} + \lambda mAs_0^{m-1} + \eta\lambda_1 u^m \tag{3.19}$$

in the usual $H_0^1(\Omega)$ weak sense (i.e., for arbitrary test functions in $H_0^1(\Omega)$). Now, $\eta\lambda_1<\lambda_1$, and so $(-\Delta-\eta\lambda_1)^{-1} : L^\infty(\Omega)\to L^\infty(\Omega)$ is a well defined, bounded, and positive operator; let $c := \left\|(-\Delta-\eta\lambda_1)^{-1}\right\|_{L^\infty(\Omega),L^\infty(\Omega)}$. Then, since u is nonnegative, from (3.19) we get

$$\|u\|_\infty^m \le c\left(m\|a\|_\infty\|u\|_\infty^{m-1-\alpha} + \lambda mAs_0^{m-1}\right).$$

Then, either $\|u\|_\infty^m \le 2cm\|a\|_\infty\|u\|_\infty^{m-1-\alpha}$, or $\|u\|_\infty^m \le 2c\lambda mAs_0^{m-1}$. Now we choose $k\in V$ such that $k>(2cm\|a\|_\infty)^{\frac{1}{1+\alpha}}$. If $\|u\|_\infty^m \le 2cm\|a\|_\infty\|u\|_\infty^{m-1-\alpha}$, then $\|u\|_\infty\le k$; therefore $f_k(,.u) = f(,.u)$, and so u is a solution to (1.2). If $\|u\|_\infty^m \le 2c\lambda mAs_0^{m-1}$, then $\|u\|_\infty \le \lambda^{\frac{1}{m}}\left(2cmAs_0^{m-1}\right)^{\frac{1}{m}}$; and so, if $\lambda\in[0,\lambda^*)$ with $\lambda^* := \min\left\{\Lambda, \frac{k^m}{2cmAs_0^{m-1}}\right\}$, we have $u\le k$, which implies that u solves (1.2). Finally, the conclusion $u_\lambda>0$ $a.e.$ in $\{a>0\}$ follows from Theorem 1.2 used with f_k instead of f.

Remark 3.2. Assume *H1)-H3)*, *H5)* and $f\le 0$. Then (1.2) has a weak solution (in the sense of Definition 1.1) for all $\lambda\ge 0$. Indeed, this follows from Theorem 1.2 applied with λf instead of f.

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