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DYNAMICS OF THE PARABOLIC EQUATION DUE TO VAN DER POL NONLINEAR DISTRIBUTED ENERGY FLOWS

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Abstract. Consider a parabolic PDE with the van der Pol cubic nonlinearity. The existence, structure and dimension of the attractor by classical theory of infinite-dimensional dynamical systems are summarized. The existence and attraction of the positive equilibrium or the negative equilibrium when parameters enter some regimes are verified. The stability of the trivial equilibrium by energy inequality and classical theory of reaction-diffusion equations are proved. Furthermore, we also give an answer to the question whether an eigenfunction of the Laplacian is attracted or repelled by the trivial equilibrium.

Keywords. Parabolic PDE; Van der Pol nonlinearity; Dynamical systems; Eigenfunction.

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1. Introduction

During the past decade, progress has been made in dynamical system theory concerning the bifurcations and chaos in the 1D wave equations and the Klein-Gordon equations with a van der Pol cubic nonlinearity in one of the boundary conditions, see, for example, ([1]-[6]) and the references therein. The basic method is characteristic reflections, by which discrete dynamics are extracted. Then bifurcations and chaos were observed and proven in the discrete dynamics.

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The motivation or principle is the van der Pol oscillation, whose cubic nonlinearity has a self-adjust effect on the state and energy. The van der Pol oscillator hints that a self-adjust effect may cause chaos. So G. Chen, S.B. Hsu, and J.X. Zhou ([1]-[4]), added a cubic nonlinearity to one of the boundary conditions of the 1D wave equation as follows:

$$w_{tt}(x,t)-c^2w_{xx}(x,t)=0, \quad 0<x<1, \quad t>0, \quad c>0, \tag{1.1}$$

$$w_t(0,t)=-\eta w_x(0,t), \quad t>0, \quad \eta>0, \quad \eta\neq c,$$

$$w_x(1,t)=\alpha w_t(1,t)-\beta w_t^3(1,t), \quad t>0, \quad 0<\alpha<c, \quad \beta>0.$$

Then the energy functional

$$E(t)=\frac{1}{2}\int_0^1[w_x^2(x,t)+\frac{1}{c^2}w_t^2(x,t)]dx$$

rises if $|w_t(1,t)|$ is small, and *falls* if $|w_t(1,t)|$ is large. Thus the van der Pol boundary condition has a self-regulating effect. This causes chaos to occur in w_x and w_t if the parameters α, β, c and η enter certain regime. Then G. Chen, T.W. Huang and B. Sun [5] added a distributed antidumping term in (1.1) as follows:

$$w_{tt}+2kw_t-w_{xx}+k^2w=0, \quad \text{for} \quad (x,t)\in(0,1)\times(0,\infty),$$

$$w_t(0,t)+kw(0,t)=-\lambda w_x(0,t), \quad t>0, \quad \text{at} \quad x=0, \quad \text{for} \quad \text{given} \quad \lambda\in\mathbb{R},$$

$$w_x(1,t)=\alpha[w_t(1,t)+kw(1,t)]-\beta[w_t(1,t)+kw(1,t)]^3, \quad t>0.$$

Bifurcations and chaos are observed and proven by them too. Motivated by the bifurcations and chaos in the wave equations with van der Pol boundary conditions, we add nonlinearities of the van der Pol type to parabolic PDEs, and explore their dynamics. Consider

$$\frac{\partial u}{\partial t}-\triangle u=(\alpha-\beta u^2)u, \quad \text{in} \quad \Omega\times(0,\infty), \quad \alpha>0, \quad \beta>0, \tag{1.2}$$

$$u|_{\partial\Omega\times(0,\infty)}=0 \tag{1.3}$$

with the initial conditions $u(0)=u_0\in L^2(\Omega)$, where Ω is a bounded, sufficiently smooth domain. By multiplying (1.2) by u and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}|u|^2+\|u\|^2=\int_\Omega(\alpha-\beta u^2)u^2dx, \tag{1.4}$$

where $|\cdot|$ denotes the norm on $L^2(\Omega)$, and $\|\cdot\|$ denotes the "reduced" norm on $H_0^1(\Omega)$, i.e.,

$$|u|^2 = \int_\Omega u^2 dx, \quad \|u\|^2 = |\nabla u|^2.$$

It is easy to see that $|u|$ *falls* if it is large, and may *rises* if it is small. Thus the van der Pol nonlinearity may have a self-regulating effect. This may cause complex dynamics when the parameters α and β enter a certain regime. Actually, by a classical embedding theorem for L^p-spaces, we have

$$|u| \leq [m(\Omega)]^{1/2-1/4}|u|_4,$$

where $|\cdot|_4$ denotes the $L^4(\Omega)$-norm, $m(\Omega)$ denotes the measure or volume of Ω. It follows immediately that

$$\left(\int_\Omega u^2 dx\right)^2 \leq m(\Omega)\int_\Omega u^4 dx. \tag{1.5}$$

Combining (1.4) and (1.5) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|^2 + \|u\|^2 &\leq \int_\Omega u^2 dx \left(\alpha - \frac{\beta}{m(\Omega)}\int_\Omega u^2 dx\right)\\ &= |u|^2\left[\alpha - \frac{\beta}{m(\Omega)}|u|^2\right]. \end{aligned} \tag{1.6}$$

Now we use the Poincare inequality

$$\lambda_1 |u|^2 \leq \|u\|^2,$$

(where λ_1 is the first eigenvalue of the Laplacian on Ω) to obtain

$$\frac{1}{2}\frac{d}{dt}|u|^2 \leq |u|^2\left[\alpha - \lambda_1 - \frac{\beta}{m(\Omega)}|u|^2\right]. \tag{1.7}$$

It is easy to see that (1.2)-(1.3) is globally asymptotically stable when $\alpha \leq \lambda_1$. For $\alpha > \lambda_1$, $|u|$ decreases when $\frac{\beta}{m(\Omega)}|u|^2$ is greater than $\alpha - \lambda_1$. Therefore, the system has a global attractor. In this paper we will verify the existence of the global attractor, study the structure of the attractor. This paper is organized as follows. In Section 2, we recall some preliminary knowledge of the attractor for reaction-diffusion equations. In Section 3, we discuss the attractor and the equilibriums of the parabolic PDE with van der Pol distributed nonlinearity and Direchlet boundary conditions. In Section 4. we deal with one-dimensional parabolic PDEs with van der Pol distributed nonlinearity and Dirichlet boundary condition.

2. Global attractor for semilinear reaction-diffusion equations

In this section, we recall some preliminary knowledge on the attractor for the reaction-diffusion equations

$$\frac{\partial u}{\partial t} - \triangle u = f(u), \quad u|_{\partial\Omega} = 0, \tag{2.1}$$

where the nonlinearity satisfies

$$-k - \alpha_1 |s|^p \le f(s)s \le k - \alpha_2 |s|^p, \quad p > 2, \tag{2.2}$$

and

$$f'(s) \le l, \tag{2.3}$$

for all $s \in \mathbb{R}$. The following results are quoted from Robinsons work ([7]).

Proposition 2.1. *Equation (2.1) with f a C^1 function satisfying (2.2) and (2.3) has a unique weak solution: for any $T > 0$ given $u_0 \in L^2(\Omega)$, there exists a solution u with*

$$u \in L^2\left(0, T; H_0^1(\Omega)\right) \cap L^p(\Omega_T), \quad u \in C^0\left([0,T]; L^2(\Omega)\right),$$

and $u_0 \mapsto u(t)$ is continuous on $L^2(\Omega)$.

Proposition 2.2. *Equation (2.1)-(2.2)-(2.3) has an absorbing set in $L^2(\Omega)$: there is a constant ρ_H and a time $t_0(|u_0|)$ such that, for the solution $u(t) = S(t)u_0$,*

$$|u(t)| \le \rho_H \quad for \quad all \quad t \ge t_0(|u_0|),$$

where $S(t) : L^2(\Omega) \to L^2(\Omega)$ denotes the corresponding C^0-semigroup.

Proposition 2.3. *Equation (2.1)-(2.2)-(2.3) has an absorbing set in $H_0^1(\Omega)$: there is a constant ρ_V and a time $t_1(|u_0|)$ such that*

$$\|u(t)\| \le \rho_V \quad for \quad all \quad t \ge t_1(|u_0|).$$

Proposition 2.4. *Equation (2.1)-(2.2)-(2.3) has a connected global attractor $\mathscr{A}$, which is uniformly bounded in $L^\infty(\Omega)$, with*

$$\|u\|_\infty \le \left(\frac{k}{\alpha_2}\right)^{1/p} \quad for \quad all \quad u \in \mathscr{A}.$$

Proposition 2.5. *The global attractor of (2.1)-(2.2)-(2.3) is bounded in $H^2(\Omega)$. Furthermore, if Ω is a bounded C^∞ domain and f is a C^∞ function, then the attractor is a bounded subset of $H^k(\Omega)$ for every $k = 1, 2, \cdots$. In particular, if $u \in \mathscr{A}$ then $u \in C^\infty(\overline{\Omega})$.*

Proposition 2.6. *Equation (2.1)-(2.2)-(2.3) has the injectivity properties on the attractor: if $u(t)$ and $v(t)$ are two trajectories on $\mathscr{A}$ with $u(T) = v(T)$ for some $T > 0$, then $u(t) = v(t)$ for all $0 \le t \le T$.*

Corollary 2.7. *The restriction of the semigroup $\{S(t)\}_{t\ge 0}$ to $\mathscr{A}$ gives rise to a dynamical system*

$$(\mathscr{A}, \{S(t)\}_{t\in\mathscr{R}}),$$

where the norm on $\mathscr{A}$ inherited from $L^2(\Omega)$ is used.

Proposition 2.8. *$\mathscr{A}$ is the unstable manifold of the set of all the fixed points:*

$$\mathscr{A} = W^u(\varepsilon),$$

where ε is the set of fixed points, which are the solutions of the equation $-\triangle u = f(u(x))$. If ε is discrete then

$$\mathscr{A} = \bigcup_{z\in\varepsilon} W^u(z),$$

and also

$$\mathscr{A} = \bigcup_{z\in\varepsilon} W^s(z).$$

3. Dynamics of parabolic PDEs with a van der Pol distributed nonlinearity

Let $f(u) = (\alpha - \beta u^2)u$, $\alpha > 0$, $\beta > 0$, then $f'(u) = \alpha - 3\beta u^2$, and f satisfies

$$-k - \alpha_1 s^4 \le f(s)s \le k - \alpha_2 s^4,$$

$$f'(s) \le \alpha,$$

for some $\alpha_1 > 0$, $\alpha_2 > 0$, $k > 0$ and all $s \in \mathbb{R}$.

We may take $\alpha_1 = \beta$, $\alpha_2 = \beta - \varepsilon$, and $k = \alpha^2/(4\varepsilon)$ for any $\varepsilon \in (0, \beta)$. It is easy to see that f satisfies (2.2)-(2.3), so equation (1.2)-(1.3) is solvable uniquely, it has absorbing sets and an attractor as Section 2 describes. Let us summarize them as follows.

Theorem 3.1. *Equation (1.2)-(1.3) has an absorbing set in $L^2(\Omega)$ and $H_0^1(\Omega)$, and a connected global attractor $\mathscr{A}$, which is uniformly bounded in $L^\infty(\Omega)$, with $\|u\|_\infty \leq \sqrt{\alpha/\beta}$ for all $u \in \mathscr{A}$. Moreover, the attractor is also bounded in $H^2(\Omega)$. Furthermore, if Ω is a bounded C^∞ domain, then the attractor is a bounded subset of $H^k(\Omega)$ for every $k = 1, 2, \cdots$. In particular, if $u \in \mathscr{A}$ then $u \in C^\infty(\overline{\Omega})$.*

Proof. It suffices to verify the L^∞-bound of the attractor. It follows Proposition 2.4 that any $u \in \mathscr{A}$ satisfies

$$\|u\|_\infty \leq \left(\frac{k}{\alpha_2}\right)^{1/4} = \left[\frac{\alpha^2}{4\varepsilon(\beta-\varepsilon)}\right]^{1/4}. \tag{3.1}$$

Substituting $\varepsilon = \beta/2$ to the right hand side of (3.1) leads to the minimal L^∞-bound $\sqrt{\alpha/\beta}$. This completes the proof.

Theorem 3.2. *The attractor of (1.2)-(1.3) is the unstable manifold of the set of all the fixed points, and its stable manifold if the set of all the fixed points is discrete. Furthermore, the dimension of the global attractor is bounded.*

Theorems 3.1 and 3.2 are direct corollaries of the propositions in Section 2. However, special system has special properties.

Theorem 3.3. *If $\alpha \leq \lambda_1$, then 0 is globally asymptotically stable.*

Proof. Case 1. $\alpha < \lambda_1$. It follows (1.7) that

$$\frac{d}{dt}|u|^2 + 2(\lambda_1 - \alpha)|u|^2 \leq 0,$$

and thus

$$\exp(2(\lambda_1-\alpha)t)\left[\frac{d}{dt}|u|^2 + 2(\lambda_1-\alpha)|u|^2\right] \leq 0,$$

$$\frac{d}{dt}\left[\exp(2(\lambda_1-\alpha)t)|u|^2\right] \leq 0,$$

$$\exp(2(\lambda_1-\alpha)t)|u(t)|^2 \leq |u(0)|^2,$$

$$|u(t)|^2 \leq \exp(-2(\lambda_1-\alpha)t)|u(0)|^2.$$

Case 2. $\alpha = \lambda_1$. It follows from (1.7) that

$$\frac{1}{2}\frac{d}{dt}|u|^2 \leq -\frac{\beta}{m(\Omega)}|u|^4.$$

For convenience of statement, let $y(t) = |u(t)|^2$. Then

$$\frac{dy}{dt} \leq -\frac{2\beta}{m(\Omega)} y^2,$$

$$\frac{dy}{y^2} \leq -\frac{2\beta}{m(\Omega)} dt.$$

So, we have

$$\int_{y(0)}^{y(t)} \frac{dy}{y^2} \leq -\int_0^t \frac{2\beta}{m(\Omega)} dt,$$

$$-\frac{1}{y(t)} + \frac{1}{y(0)} \leq -\frac{2\beta}{m(\Omega)} t,$$

$$y(t) \leq \frac{m(\Omega) y(0)}{m(\Omega) + 2\beta t y(0)}.$$

Therefore

$$|u(t)|^2 \leq \frac{m(\Omega)|u(0)|^2}{m(\Omega) + 2\beta t |u(0)|^2}.$$

This completes the proof.

Theorem 3.4. *If $\alpha > \lambda_1$, then for any $\varepsilon > 0$, $N\left(0, \sqrt{m(\Omega)(\alpha - \lambda_1)/\beta} + \varepsilon\right)$ is an absorbing set in $L^2(\Omega)$ for problem (1.2)-(1.3), i.e., for any initial data $u_0 \in L^2(\Omega)$ there exists a $t_0 = t_0(|u_0|)$ such that*

$$|u(t)| < \sqrt{m(\Omega)(\alpha - \lambda_1)/\beta} + \varepsilon \quad \textit{for} \quad t \geq t_0.$$

Proof. Suppose initially that $|u(t_0)| < \sqrt{m(\Omega)(\alpha - \lambda_1)/\beta} + \varepsilon$ for some $t_0 > 0$. Then we have

$$|u(t)| < \sqrt{m(\Omega)(\alpha - \lambda_1)/\beta} + \varepsilon \quad \text{for} \quad t \geq t_0.$$

Otherwise, there exists $t_1 > t_0$ such that

$$|u(t_1)| \geq \sqrt{m(\Omega)(\alpha - \lambda_1)/\beta} + \varepsilon.$$

Let t_1 be the first time after t_0 for $|u(t)| = \sqrt{m(\Omega)(\alpha - \lambda_1)/\beta} + \varepsilon$, i.e.,

$$|u(t)| < \sqrt{m(\Omega)(\alpha - \lambda_1)/\beta} + \varepsilon \quad \text{for} \quad t_0 < t < t_1.$$

Then by the continuity of $|u(t)|$ with regard to t, there exists $t_2 < t_1$ such that

$$|u(t)| > \sqrt{m(\Omega)(\alpha - \lambda_1)/\beta} + \varepsilon/2 \quad \text{for} \quad t_2 < t < t_1.$$

It follows from (1.7) that

$$|u(t_1)|^2 \leq |u(t_2)|^2 + 2\int_{t_2}^{t_1} |u(\tau)|^2 \left[\alpha - \lambda_1 - \frac{\beta}{m(\Omega)}|u(\tau)|^2\right] d\tau$$
$$< |u(t_2)|^2,$$

which leads to a contraction.

If, instead, $|u(t)| \geq \sqrt{m(\Omega)(\alpha-\lambda_1)/\beta} + \varepsilon$ for all $t \geq 0$, then (1.7) implies that

$$|u(t)|^2 \leq |u(0)|^2 + 2\int_0^t |u(\tau)|^2 \left[\alpha - \lambda_1 - \frac{\beta}{m(\Omega)}|u(\tau)|^2\right] d\tau$$
$$< |u(0)|^2 - 2\varepsilon^2\sqrt{\beta(\alpha-\lambda_1)/m(\Omega)}t.$$

Letting $t \to \infty$ leads to a contraction.

Combining the two aspects above, we have $t_0 > 0$ such that

$$|u(t)| < \sqrt{m(\Omega)(\alpha-\lambda_1)/\beta} + \varepsilon$$

for all $t \geq t_0$. This completes the proof.

As a direct corollary of Theorem 3.4, we have the following.

Theorem 3.5. *The attractor of (1.2)-(1.3) is in the L^2-ball centered at the origin with radius*

$$\sqrt{m(\Omega)(\alpha-\lambda_1)/\beta}.$$

Since the attractor consists of the unstable manifolds or the stable manifolds of all the fixed points, the dynamics is determined by the properties of these fixed points. An equilibrium of (1.2)-(1.3) is a solution of the following elliptic equation

$$\begin{cases} -\triangle u(x) = u(x)g(u(x)), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{3.2}$$

where $g(u) = \alpha - \beta u^2$. We have the following.

Theorem 3.6. *Assume that $\partial\Omega$ is $C^{2+\gamma}$-smooth for some $\gamma \in (0,1)$.*
(i) If $\alpha \leq \lambda_1$, then (3.2) has no positive solution or negative solution;
(ii) If $\alpha > \lambda_1$, then (3.2) has a unique positive and a unique negative solution in $C^2(\overline{\Omega})$,

they both satisfy

$$\|u\|_\infty \leq \sqrt{\alpha/\beta}$$

and

$$|u| \leq \sqrt{m(\Omega)(\alpha-\lambda_1)/\beta}.$$

Proof. If $\alpha \leq \lambda_1$, then 0 is globally asymptotically stable by Theorem 3.3, so (1.2)-(1.3) has no equilibrium except 0. Therefore, (3.2) has no positive or negative solution. (ii) is a direct corollary of Theorem 3.4.1 in [8] and Theorem 3.5.

Remark 3.7. Since any equilibrium is in the attractor, so the positive solution and the negative solution of (3.2) are in the attractor of (1.2)-(1.3). The L^∞-bound and the L^2-bound of the attractor are also L^∞-bound and L^2-bound of the equilibriums (solutions of (3.2)). Theorem 3.1 tells that $\sqrt{\alpha/\beta}$ is an L^∞-bound of the attractor, which agrees with the bound of the positive equilibrium and the negative equilibrium described by (ii) of Theorem 3.6.

Theorem 3.8. *Assume that $\alpha > \lambda_1$, then if $u(x,0) \geq 0$ (≤ 0), $u(x,0) \neq 0$, then (1.2)-(1.3) has a unique positive (negative) solution , and*

$$\lim_{t\to\infty} u(x,t) = u^+(x)(u^-(x)),$$

where u^+ and u^- denote the positive equilibrium and the negative equilibrium respectively.

Proof. See Theorem 4.2.6 and its examples in [9].

By Theorem 3.8 we know that (1.2)-(1.3) has a positive (negative) equilibrium when $\alpha > \lambda_1$, and it attracts all positive (negative) data. But little is known about the other equilibriums. In next section, we will consider parabolic PDE with van der Pol distributed nonlinearity over an interval, and try to get more information.

4. One-dimensional system with Direchlet boundary values and van der Pol distributed nonlinearity

In this section, we consider the 1D parabolic PDE with van der Pol distributed nonlinearity over interval $[0,1]$:

$$\begin{cases} u_t - u_{xx} = \left(\alpha - \beta u^2\right) u, \quad 0 < x < 1, \\ u(0,t) = u(1,t) = 0. \end{cases} \tag{4.1}$$

Of course, (4.1) possesses all the properties stated in Sections 2 and 3. The corresponding equilibrium problem can be described as follows

$$\begin{cases} -u''(x) = \left(\alpha - \beta u^2\right) u, \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{4.2}$$

Letting $\xi = \pi x$, denote $u(x) \equiv u(\xi/\pi)$ by $w(\xi)$, then (4.2) is equivalent to

$$\begin{cases} -w''(\xi) = \frac{1}{\pi^2}\left(\alpha - \beta w(\xi)^2\right) w(\xi), \quad 0 < \xi < \pi, \\ w(0) = w(\pi) = 0, \end{cases} \tag{4.3}$$

which is the equilibrium problem of the Chafee-Infante equation([9, 10]).

Theorem 4.1. *(4.1) has no equilibrium except $u = 0$ when $0 < \alpha \leq \pi^2$ and has equilibriums $u_n^{\pm}$ when $\alpha > n^2\pi^2$, each has $n+1$ zeros in $[0,1]$.*

Proof. Let $\lambda = \alpha/\pi^2$, $f(w) = \left(1 - \frac{\beta}{\alpha} w^2\right) w$. Then Theorem 4.1 is a direct corollary of Theorem 10.7.1 in [9] and Theorem 5.5 in [10].

Theorem 4.2. *For each integer $n \geq 1$, let $u_n^{\pm}$, $n^2\pi^2 < \alpha < +\infty$ be as in Theorem 4.1. Then for any $\alpha \in (\pi^2, +\infty)$, the equilibrium point $u_1^{\pm}$ is stable, $u_n^{\pm}$ $(n = 2, 3, \cdots)$ is unstable.*

Proof. Straightforward verification by Theorem 10.7.4 in [9].

We simulate the dynamics of (4.1) by the pdepe function in Matlab, with spatial step length $\delta x = 0.025$. Fix $\beta = 1$, and take $\alpha = 8, 9, 11, 12$, $u(x,0) = \pm 10 sin(\pi x)$ respectively. Our experiments show that the origin is globally asymptotically stable when $0 < \alpha \leq \pi^2$, and unstable when $\alpha > \pi^2$. Furthermore, any positive (negative) solution converges to the positive (negative) equilibrium when $\alpha > \pi^2$. The spatiotemporal profiles of $u(x,t)$ are plotted in Figs. 4.1-4.8.

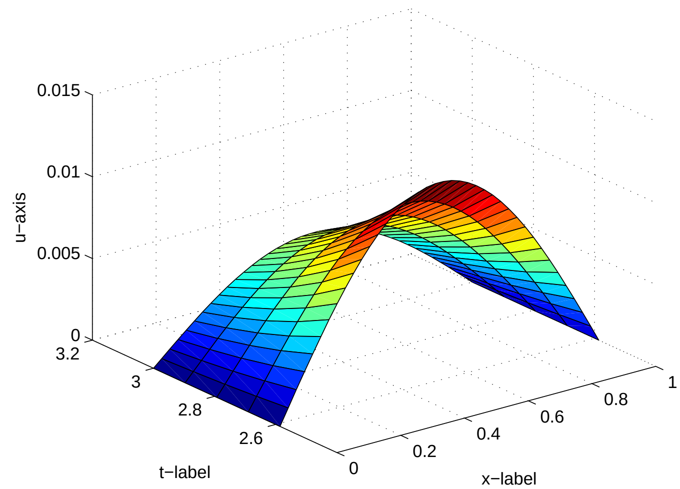

Fig. 4.1. The spatiotemporal profile of $u(x,t)$ with $\alpha = 8$, $\beta = 1$, $u(x,0) = 10\sin(\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to 0 as $t \to +\infty$.

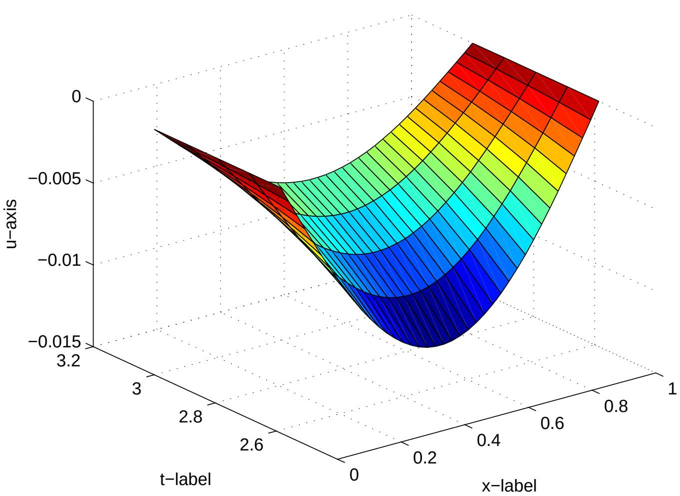

Fig. 4.2. The spatiotemporal profile of $u(x,t)$ with $\alpha = 8$, $\beta = 1$, $u(x,0) = -10\sin(\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to 0 as $t \to +\infty$.

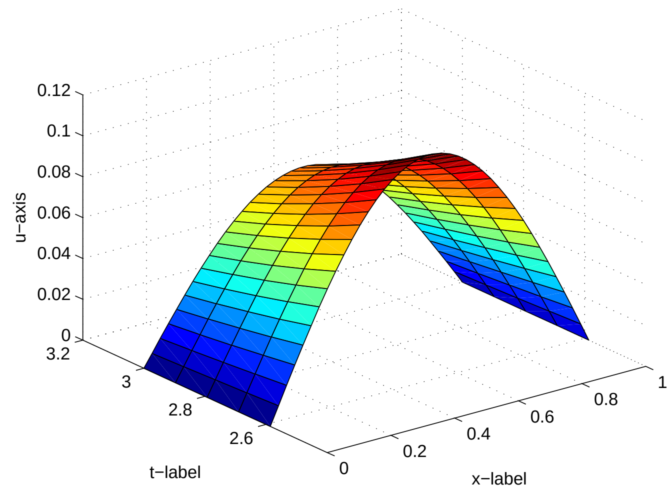

Fig. 4.3. The spatiotemporal profile of $u(x,t)$ with $\alpha = 9$, $\beta = 1$, $u(x,0) = 10\sin(\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to 0 as $t \to +\infty$.

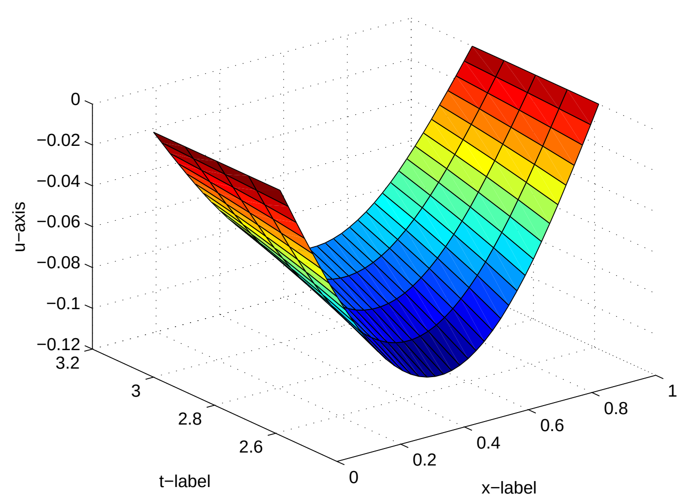

Fig. 4.4. The spatiotemporal profile of $u(x,t)$ with $\alpha = 9$, $\beta = 1$, $u(x,0) = -10\sin(\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to 0 as $t \to +\infty$.

Figs. 4.1-4.4 show that the orign is globally asymptotically stable when $0 < \alpha \leq \pi^2$, which coincides with Theorem 4.1. Moreover, the solutions decay slowly as α increases.

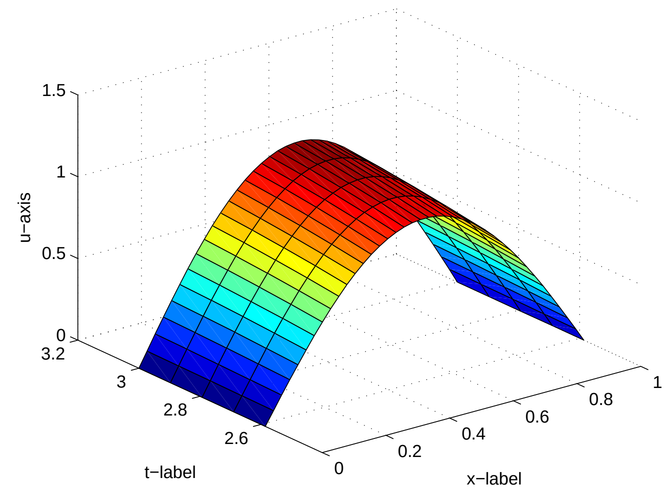

Fig. 4.5. The spatiotemporal profile of $u(x,t)$ with $\alpha = 11$, $\beta = 1$, $u(x,0) = 0.1\sin(\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to a positive equilibrium as $t \to +\infty$.

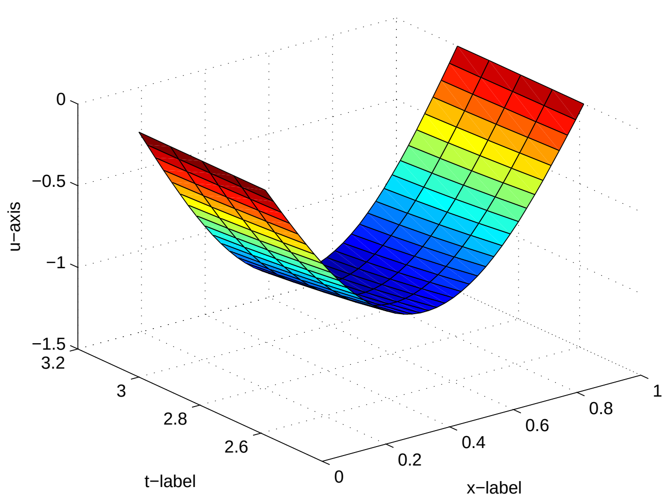

Fig. 4.6. The spatiotemporal profile of $u(x,t)$ with $\alpha = 11$, $\beta = 1$, $u(x,0) = -0.1\sin(\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to a negative equilibrium as $t \to +\infty$.

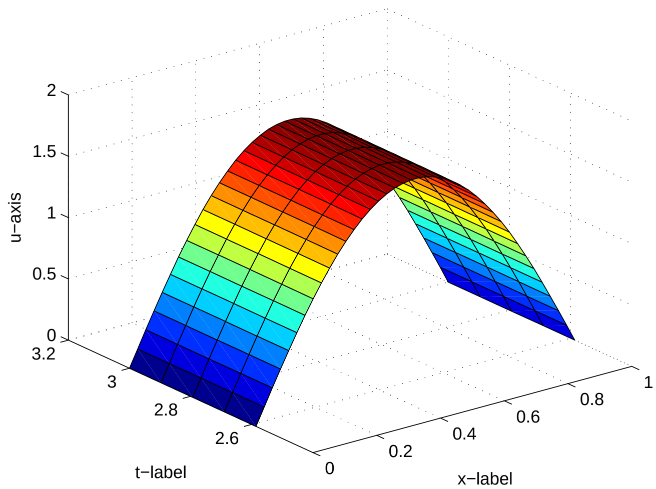

Fig. 4.7. The spatiotemporal profile of $u(x,t)$ with $\alpha = 12$, $\beta = 1$, $u(x,0) = 0.1\sin(\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to a positive equilibrium as $t \to +\infty$.

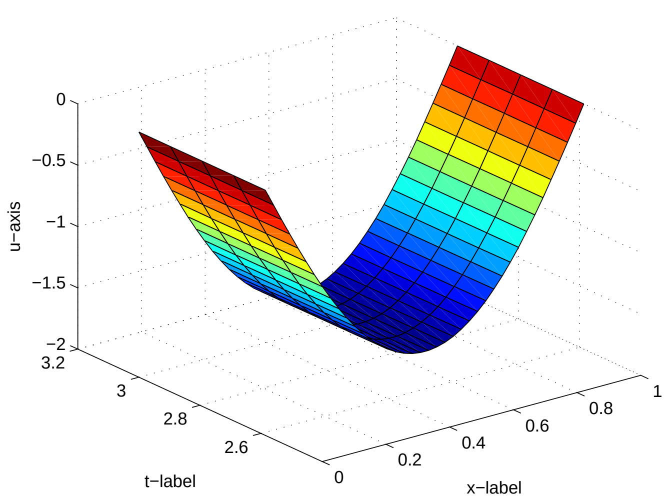

Fig. 4.8. The spatiotemporal profile of $u(x,t)$ with $\alpha = 12$, $\beta = 1$, $u(x,0) = -0.1\sin(\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to a negative equilibrium as $t \to +\infty$.

Figs. 4.5-4.8 show that the equilibrium 0 is unstable when $\alpha > \pi^2$, and the positive (negative) solution tends to the positive (negative) equilibriums as $t \to +\infty$. The positive (or negative) equilibrium becomes *larger* as α increases, which coincides with the bound of the attractor described in Sections 2 and 3.

We take $\alpha = 12$ and $\beta = 1$, but a different initial data $u(x,0) = 10\sin(2\pi x)$, which has a more zero in $[0,1]$, the spatiotemporal profile of $u(x,t)$ is plotted in Fig. 4.9.

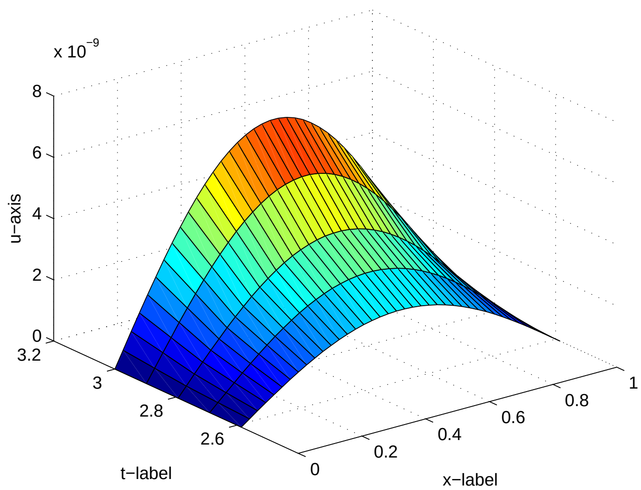

Fig. 4.9. The spatiotemporal profile of $u(x,t)$ with $\alpha = 12$, $\beta = 1$, $u(x,0) = 10\sin(2\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to 0 as $t \to +\infty$.

Fig. 4.9 shows that 0 attracts some solutions though it is unstable, it seems that $c\sin(k\pi x)$ is in its stable manifold for $k = 2,3,\cdots$ and $c \in \mathbb{R}$. This experiment result motivates us to explore the stable manifold and unstable manifold of the trivial equilibrium.

Let $\varphi(x) = c\sin(2\pi x)$, note that $\varphi(0) = \varphi(1/2) = \varphi(1) = 0$, $\varphi''(0) = \varphi''(1/2) = \varphi''(1) = 0$, and that

$$f(\varphi(0)) = f(\varphi(1/2)) = f(\varphi(1)) = 0.$$

So the initial-boundary value problem

$$\begin{cases} u_t = u_{xx} + \alpha u - \beta u^3, \quad 0 < x < 1, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = c\sin(2\pi x) \end{cases} \tag{4.4}$$

can be decomposed into two subsystems:

$$\begin{cases} u_t = u_{xx} + \alpha u - \beta u^3, \quad 0 < x < 1/2, \\ u(0,t) = u(1/2,t) = 0, \\ u(x,0) = c\sin(2\pi x) \end{cases} \tag{4.5}$$

and

$$\begin{cases} u_t = u_{xx} + \alpha u - \beta u^3, \quad 1/2 < x < 1, \\ u(1/2,t) = u(1,t) = 0, \\ u(x,0) = c\sin(2\pi x). \end{cases} \tag{4.6}$$

The initial data $u(x,0) = c\sin(2\pi x)$ is compatible with the Direchlet boundary condition and the equation over interval $[0,1/2]$ and $[1/2,1]$. It follows a classical theory on the reaction-diffusion equation ([9]) that each of (4.5) or (4.6) has a unique solution in $C^{2,1}(\overline{Q_T})$ for any $T > 0$, where $C^{2,1}(\overline{Q_T}) = \{u(x,t)|D_t^r D_x^s u \in C(\overline{Q_T}), 2r+s \leq 2\}$. Moreover, the solution of (4.5) is positive, the solution of (4.6) is negative for $c > 0$, and reversely for $c < 0$. Denote the solution of (4.5) by $u_1(x,t)$, and that of (4.6) by $u_2(x,t)$. Then their union

$$u(x,t) = \begin{cases} u_1(x,t), \quad x \in (0,1/2], \\ u_2(x,t), \quad x \in (1/2,1), \end{cases} \qquad t \in (0,\infty) \tag{4.7}$$

is the solution of (4.4). Therefore the asymptotic behavior of (4.4) is reduced to those of (4.5) and (4.6). In fact, suppose $u_1(x,t)$ is the solution of (4.5), then it is easy to verify that

$$u_2(x,t) = -u_1(1-x,t)$$

satisfies (4.6). Moreover,

$$\frac{\partial}{\partial t}u_2(x,t) = -\frac{\partial}{\partial t}u_1(1-x,t),$$
$$\frac{\partial^2}{\partial x^2}u_2(x,t) = -\frac{\partial^2}{\partial x^2}u_1(1-x,t).$$

So, we have

$$\begin{aligned} \lim_{x \to \frac{1}{2}^+} \frac{\partial}{\partial t}u_2(x,t) &= -\lim_{x \to \frac{1}{2}^-} \frac{\partial}{\partial t}u_1(x,t) \\ &= -\frac{\partial}{\partial t}u_1(1/2,t) = 0, \end{aligned}$$

$$\begin{aligned}
\lim_{x\to\frac{1}{2}^+}\frac{\partial^2}{\partial x^2}u_2(x,t) &= -\lim_{x\to\frac{1}{2}^+}\frac{\partial^2}{\partial x^2}u_1(1-x,t)\\
&= \lim_{x\to\frac{1}{2}^+}\left[\alpha u_1(1-x,t)-\beta u_1(1-x,t)^3-\frac{\partial}{\partial t}u_1(1-x,t)\right]\\
&= \alpha u_1(1/2,t)-\beta u_1(1/2,t)^3-\frac{\partial}{\partial t}u_1(1/2,t)\\
&= 0.
\end{aligned}$$

Therefore, the $u(x,t)$ defined by (4.7) is $C^{2,1}$-smooth in $[0,1]\times[0,\infty)$, and thus is a classical solution of (4.4).

On the other hand, the first eigenvalue of $-\triangle$ over interval $[0,1/2]$ or $[1/2,1]$ is $4\pi^2$ ($\pm\sin(2\pi x)$ is the first eigenfunction). So the solution of each subsystem converges to 0 as $t\to+\infty$ for $\alpha\le 4\pi^2$. Therefore 0 attracts $c\sin(2\pi x)$ for $\alpha\le 4\pi^2$ in (4.4). In reverse, the positive equilibrium of (4.5) or the negative equilibrium of (4.6) attracts $c\sin(2\pi x)$ for $c>0$ and $\alpha>4\pi^2$, the negative equilibrium of (4.5) or the positive equilibrium of (4.6) attracts $c\sin(2\pi x)$ for $c<0$ and $\alpha>4\pi^2$. Therefore, $c\sin(2\pi x)$ is repelled by 0 in (4.4) for $\alpha>4\pi^2$. By similar arguments we conclude that 0 attracts $c\sin(n\pi x)$ for $\alpha\le n^2\pi^2$, but repells it for $\alpha>n^2\pi^2$.

Let us summarize this conclusion as follows.

Theorem 4.3. *Let $u_n(x,t)$ be the solution of (4.4) with $u_n(x,0)=c\sin(n\pi x)$, $c\neq 0$, then $\lim\limits_{t\to\infty}|u_n(\cdot,t)|=0$ when $\alpha\le n^2\pi^2$, and $\lim\limits_{t\to\infty}|u_n(\cdot,t)-u_n(\cdot)|=0$, where $u_n(\cdot)|_{[\frac{k-1}{n},\frac{k}{n}]}$ is the positive solution of*

$$\begin{cases}-u''(x)=\alpha u-\beta u^3, & \frac{k-1}{n}<x<\frac{k}{n},\\ u(\frac{k-1}{n})=u(\frac{k}{n})=0,\end{cases}\tag{4.8}$$

for $c>0$ and odd $k\le n$, or $c<0$ and even $k\le n$; $u_n(\cdot)|_{[\frac{k-1}{n},\frac{k}{n}]}$ is the negative solution of (4.8) for $c>0$ and even $k\le n$, or $c<0$ and odd $k\le n$. Furthermore, $\sum\limits_{l=0}^{m}c_l\sin((n+l)\pi x)$ is also attracted by 0 for $\alpha\le n^2\pi^2$, $c_l\in\mathbb{R}$ and $m=1,2,\cdots$.

Remark 4.4. In fact, one may verify that $u_n(x)$ in Theorem 4.3 is either of the two equilibriums of (4.1) which have exactly $n+1$ zeros in $[0,1]$: 0, $1/n$, $2/n$, $\cdots$, 1. So an alternative statement of Theorem 4.3 may be as follows.

Theorem 4.5. *$\phi_n(x) = c\sin(n\pi x)$ is attracted by 0 when $\alpha \le n^2\pi^2$, but is attracted by u_n^+ or u_n^- when $\alpha > n^2\pi^2$.*

Theorems 4.3 and 4.5 show that 0 attracts more eigenfunctions of the Laplacian when α is small, but attracts fewer eigenfunctions when α is large. As α increases, more eigenfunctions of the Lappacian go to the unstable manifold of the origin and the stable manifold of the nontrivial equilibriums. It follows this fact that each equilibrium has nonempty stable manifold.

Fig. 4.10 shows that 0 does not attract $c\sin(2\pi x)$ when $\alpha > 4\pi^2$ and $c \neq 0$.

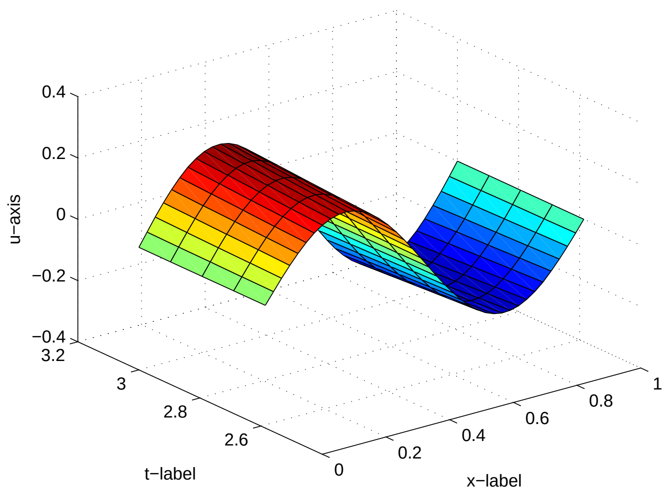

Fig. 4.10. The spatiotemporal profile of $u(x,t)$ with $\alpha = 40$, $\beta = 1$, $u(x,0) = 0.1\sin(2\pi x)$, $x \in [0,1]$, $t \in [2.6,3]$. One can see that the solution tends to a nontrivial equilibrium as $t \to +\infty$.

5. Conclusions

The van der Pol distributed nonlinearity leads to a global attractor in a parabolic PDE with Dirichlet boundary condition. The attractor consists of the unstable manifold or the stable manifold of fixed points. The PDE has only trivial equilibrium 0 when parameter α is small enough, which is globally attractive. The trivial equilibrium becomes unstable, and other equilibriums appear as α increases. The PDE has more equilibriums for larger α. There are a positive equilibrium and a negative equilibrium which are stable, the positive equilibrium attracts all positive data, and the negative equilibrium attracts all negative data. Though 0 is not stable for large α, it attracts infinite many eigenfunctions of the Laplacian. However, the van der Pol distributed nonlinearity does not cause chaos in a parabolic PDE with Direchlet boundary values.

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