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WEAK COPROXIMINALITY FOR BANACH SPACES

T. S. S. R. K. RAO

Theoretical Statistics and Mathematics Unit,

Indian Statistical Institute, R. V. College P.O. Bangalore 560059, India

Abstract. In this paper, we introduce a weaker form of the classical notion of coproximinality for Banach spaces. This is intended as a new tool to make the metric projection map linear. We show that if a weakly coproximinal subspace $Y \subset X$ is a semi-M-ideal in X, then the associated metric projection map is linear and Y is a M-ideal in X. This is also linked to the classical problem of identifying Banach spaces as a quotient space X/Y, where Y has certain non-linear geometric properties in X. We give a counterexample to the 3-space problem for weak coproximinality. We also study its stability properties for spaces of vector-valued continuous functions.

Keywords. Banach space; M-ideal; Semi-linear map; Weak coapproximation.

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1. Introduction

Let X be a real Banach space. We recall from [1] that a closed subspace $Y \subset X$ is said to be coproximinal, if for any $x \in X$, there is a $y_0 \in Y$ such that for all $y \in Y$, $\|y - y_0\| \leq \|y - x\|$ for all $y \in Y$. It is well known that this geometric notion works as a non-linear analogue of Y being constrained in X, i.e., there is a linear projection $P : X \to Y$ of norm one, onto Y.

Since most often it is difficult to produce a projection of norm one, one studies geometric properties that mimic the effect of being the range of a projection of norm one. Among such

E-mail address: tss@isibang.ac.in.

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interesting properties studied in the literature are those of the notion of a sun, co-sun and the notion of best coapproximation. See [1], [2], [3], and [4] for some recent applications of these notions. In this paper, we introduce another such non-linear notion. We use several ideas from geometry of Banach spaces to settle various problems arising in this non-linear context. Applications from the theory of vector-valued functions are indicated.

We always consider a non-reflexive Banach space X as canonically embedded in its bidual X^{**} and write $X \subset X^{**}$. We recall that for a closed subspace $Y \subset X$, its bi-annihilator $Y^{\perp\perp} \subset X^{**}$ can be isometrically identified with Y^{**}.

Definition 1.1. We say that a closed subspace $Y \subset X$ is weakly coproximinal, if for any $x \in X$ there is a $\tau \in Y^{\perp\perp}$ such that $\|\Lambda - \tau\| \leq \|\Lambda - x\|$ for all $\Lambda \in Y^{\perp\perp}$.

If there is a linear onto projection $P : X \to Y$ with $\|P\| = 1$, as $P^{**} : X^{**} \to Y^{\perp\perp}$ is again a linear projection of norm one, with $P = P^{**}$ on X, by taking $\tau = P(x)$, it is easy to see that Y is weakly coproximinal in X. It is also easy to see that any reflexive weakly coproximinal subspace is coproximinal.

We recall that canonical projection $Q : X^{***} \to X^{***}$ defined by $Q(\Lambda) = \Lambda|X$ is a linear projection of norm one, onto X^* and $ker(Q) = X^\perp$. Also Q^* is a projection of norm one with range $X^{\perp\perp}$.

Example 1.2. *Let X be a non-reflexive Banach space. Let $X^{(IV)}$ denote the fourth dual of X. Now consider $X^{\perp\perp} \subset X^{(IV)}$. It is well known that X is also weak*-dense in $X^{\perp\perp}$ and the canonical isometry of X^{**} with $X^{\perp\perp}$ is identity on X. Thus via this isometry, for $\Lambda \in X^{**}$, by our identifications, we can take $\tau = Q^*(\Lambda) \in X^{\perp\perp}$ and we have that, $\|\Gamma - \tau\| = \|Q^*(\Gamma - \Lambda)\| \leq \|\Gamma - \Lambda\|$ for all $\Gamma \in X^{\perp\perp}$. Thus X is weakly coproximinal in X^{**}. Let c_0 denote the space of sequences converging to 0 and let ℓ^∞ be the space of bounded sequences. It is easy to see that the constant sequence 1 has no best coapproximation in c_0. Since ℓ^∞ is the bidual of c_0, we have that c_0 is a weakly coproximinal subspace of ℓ^∞.*

For closed subspaces $Z \subset Y \subset X$, since $Z^{\perp\perp} \subset Y^{\perp\perp} \subset X^{**}$, it is easy to see that if Z is weakly coproximinal in X, then it is also weakly coproximinal in Y.

We show that a non-reflexive Banach space X is coproximinal in its bidual if and only if whenever X is isometrically embedded in a Banach space Z (still denoted by X), as a weakly coproximinal subspace, X is coproximinal in Z.

We give an example to show that for closed subspaces $Z \subset Y \subset X$, that Z is weakly coproximinal in X and the quotient space Y/Z is weakly coproximinal in X/Z need not imply that Y is weakly coproximinal in X, thus settling the 3-space problem in this context.

We recall from [5] that a closed subspace $Y \subset X$ is a M-ideal if there is a linear onto projection $P : X^* \to X^*$ such that $ker(P) = Y^\perp$ and $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. See Chapter I of [5] for several examples from classical analysis, of subspaces that are M-ideals and their geometric properties. Such a P is called a L-projection. In this case $P^* : X^{**} \to X^{**}$ is a projection such that $\|\Lambda\| = max\{\|P^*(\Lambda)\|, \|\Lambda - P(\Lambda)\|\}$ for $\Lambda \in X^{**}$, so that $\|P^*\| = 1$ and range of P^* is $M^{\perp\perp}$. In particular Y is weakly coproximinal in X.

We next recall the notion of a semi-M-ideal from [6]; see also [5].

Definition 1.3. A closed subspace $Y \subset X$ is said to be a semi-M-ideal, if there exists an onto projection (not necessarily linear) $P : X^* \to Y^\perp$, such that $P(\lambda x^* + P(y^*)) = \lambda P(x^*) + P(y^*)$, $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$, for $x^*, y^* \in X^*$ and $\lambda \in \mathbb{R}$.

We recall from [7] that the map P in the above definition is the metric projection. Using the 2-ball property characterization of semi-M-ideals (see, [8], [5], Chapter I and also page 43 of Notes and remarks), it is easy to see that if Y is a semi-M-ideal in X, then for any $x \notin Y$, Y is a semi-M-ideal in $span\{x, Y\}$.

It follows from the proof of Theorem 12 in [7] that any real Banach space is isometric to a quotient space X/Y where Y is a semi-M-ideal in X but not a M-ideal.

We are interested in studying new geometric conditions that Y has in X, under which P becomes a linear map. We show that if in addition Y is also weakly coproximinal then Y is a M-ideal in X and hence (see the proof of Proposition I.1.2 of [5]) the above P becomes a linear L-projection. We also show that weakly coproximinality is preserved by quotients and by c_0-direct sums. We show the stability of this property for spaces of vector-valued continuous functions in some special situations.

Our application is to the space $WC(K,X)$, the space of X-valued functions on K, that are continuous when X has the weak topology, equipped with the supremum norm. We recall that a closed subspace $Y \subset X$ is said to be factor reflexive if the quotient space X/Y is a reflexive space. Let K be a compact extremally disconnected space and let $Z \subset Y \subset X$ be closed subspaces such that Z is a factor reflexive subspace of Y and Y is a weakly coproximinal subspace of X. We show that $WC(K,Y/Z)$ is a coproximinal subspace of $WC(K,X/Z)$.

2. Main results

The first result illustrates the relationship between coproximinality and weak coproximinality.

Proposition 2.1. *Let X be a Banach space. X is coproximinal in its bidual X^{**} if and only if when ever X is isometrically embedded in a Banach space Y as a weakly coproximinal subspace, X is a coproximinal subspace of Y.*

Proof. Suppose X is a coproximinal subspace of X^{**}. Let Y be a Banach space and ignoring the embedding of X, we assume that $X \subset Y$ is a weakly coproximinal subspace. Using the canonical embedding, the hypothesis implies that X is a coproximinal subspace of $X^{\perp\perp} \subset Y^{**}$. Now for $y \in Y$, choose $\tau \in X^{\perp\perp}$ such that $\|\Lambda - \tau\| \leq \|\Lambda - y\|$ for all $\Lambda \in X^{\perp\perp}$. Now since X is coproximinal in $X^{\perp\perp}$, choose $x_0 \in X$ corresponding to $\tau \in X^{\perp\perp}$. We thus have $\|x - x_0\| \leq \|x - \tau\| \leq \|x - y\|$ for all $x \in X$. Thus X is coproximinal in Y. Since X is always a weakly coproximinal subspace of X^{**}, the converse implication is easy to see.

We next give a new criteria for a closed subspace $Y \subset X$ to be a M-ideal in X.

Proposition 2.2. *Y is a M-ideal in X if and only if for any $x \in X$, $Y^{\perp\perp}$ is a M-ideal in $span\{x, Y^{\perp\perp}\}$.*

Proof. Suppose Y is a M-ideal in X. Then since $X^* = Y^{\perp} \bigoplus_1 N$ for some closed subspace N of X^*, we have that $X^{**} = Y^{\perp\perp} \bigoplus_\infty N^{\perp}$. Now if $x = \Lambda + \tau$ for $\Lambda \in Y^{\perp\perp}$ and $\tau \in N^{\perp}$, since $X \cap Y^{\perp\perp} = Y$, we may assume that $x \notin Y$. We have $span\{x, Y^{\perp\perp}\} = span\{\tau, Y^{\perp\perp}\} = Y^{\perp\perp} \bigoplus_\infty span\{\tau\}$. Thus in particular $Y^{\perp\perp}$ is a M-ideal in $span\{x, Y^{\perp\perp}\}$.

Conversely suppose for all $x \in X$, $Y^{\perp\perp}$ is a M-ideal in $span\{x, Y^{\perp\perp}\}$. As $span\{x, Y^{\perp\perp}\}$ is weak*-closed, it is in particular a dual space and also $(span\{x,Y\})^{\perp\perp} = span\{x, Y^{\perp\perp}\}$. Thus by Corollary II.3.6 in [5] we get that $Y^{\perp\perp}$ is a M-summand in $span\{x, Y^{\perp\perp}\}$ and thus Y is a M-ideal in $span\{x, Y\}$ for all $x \notin Y$. It now follows from the remarks made after Proposition II.3.2 in [5] that Y is a M-ideal in X.

Theorem 2.3. *Let $Y \subset X$ be a weakly coproximinal subspace. If Y is a semi-M-ideal in X, then it is a M-ideal in X.*

Proof. Let $x \in X$. We show that $Y^{\perp\perp}$ is a M-ideal in $span\{x, Y^{\perp\perp}\}$. It would then follow from the above proposition that Y is a M-ideal in X.

Let $x \in X$. By hypothesis there exists a $\tau \in Y^{\perp\perp}$ such that $\|\Lambda - \tau\| \leq \|\Lambda - x\|$ for all $\Lambda \in Y^{\perp\perp}$. Define $P : span\{x, Y^{\perp\perp}\} \to Y^{\perp\perp}$ by $P(\alpha x + \Lambda) = \alpha\tau + \Lambda$, for $\Lambda \in Y^{\perp\perp}$ and scalar α. It is easy to see that P is a linear projection onto $Y^{\perp\perp}$. Also for $\alpha \neq 0$, $\|P(\alpha x + \Lambda\| = |\alpha|\|\tau + \frac{\Lambda}{\alpha}\| \leq |\alpha|\|x + \frac{\Lambda}{\alpha}\| = \|\alpha x + \Lambda\|$. Thus $\|P\| = 1$.

It follows from Theorem 6.14 in [6] that $Y^{\perp\perp}$ is a semi-M-ideal in X^{**}. By our remarks made after the definition of semi-M-ideal, we have that $Y^{\perp\perp}$ is a semi-M-ideal in $span\{x, Y^{\perp\perp}\}$. Since $P : span\{x, Y^{\perp\perp}\} \to Y^{\perp\perp}$ is a projection of norm one, it again follows from Proposition I.1.2 of [5] and the remarks on page 43, that $Y^{\perp\perp}$ is a M-ideal in $span\{x, Y^{\perp\perp}\}$. Thus by the above proposition we get that Y is a M-ideal in X.

Remark 2.4. It is clear from the above proof that if $Y \subset X$ is a weakly coproximinal subspace and is such that Y is a semi-M-ideal in $span\{x, Y\}$ for all $x \notin Y$, then it is a M-ideal in X. We do not know if this statement is true without the assumption of weak coproximinality?

Remark 2.5. As remarked earlier, by using Theorem 12 in [7], we have that any real Banach space is isometric to a quotient space X/Y where Y is a semi-M-ideal in X but not a weakly coproximinal subspace. Since any Banach space X, under the canonical embedding is a weakly coproximinal subspace, this can be compared with the well studied problem of identifying a Banach space as X^{**}/X (see [10]). In particular it would be interesting to know if every real Banach space can be realized as X/Y where Y is now a weakly coproximinal subspace of X.

Next set of results exhibit more classes of Banach spaces that are weakly coproximinal. We first show that weakly coproximinality is preserved by quotients.

Proposition 2.6. *Let X be a Banach space and let $Z \subset Y \subset X$. If Y is weakly coproximinal in X, then Y/Z is weakly coproximinal in X/Z.*

Proof. Let $\pi : X \to X/Z$ be the quotient map. We recall that $(X/Z)^{**} = X^{**}/Z^{\perp\perp}$ and $(Y/Z)^{\perp\perp} = Y^{\perp\perp}/Z^{\perp\perp}$. Let $x \in X$ and $x \notin Y$. By hypothesis there exists a $\tau \in Y^{\perp\perp}$ such that $\|\Lambda - \tau\| \leq \|\Lambda - x\|$ for all $\Lambda \in Y^{\perp\perp}$. Now $\|\pi(\Lambda)\pi(\tau)\| = \|\pi(\Lambda - \tau)\| = inf\{\|\Lambda - \tau - \Gamma\| : \Gamma \in Z^{\perp\perp}\} \leq inf\{\|\Lambda - \Gamma - x\| : \Gamma \in Z^{\perp\perp}\} = \|\pi(\Lambda) - \pi(x)\|$ for any $\Lambda \in Y^{\perp\perp}$. Thus Y/Z is weakly coproximinal in X/Z.

Corollary 2.7. *Let $Z \subset X$ be a closed subspace. $W \subset X/Z$ be a closed subspace such that $\pi^{-1}(W)$ is a weakly coproximinal subspace of X. Then W is a weakly coproximinal subspace of X/Z.*

Proof. Let $Y = \pi^{-1}(W)$. Clearly $Z \subset Y \subset X$ and Y is a weakly coproximinal subspace of X. Thus by the above proposition, $Y/Z = W$ is a weakly coproximinal subspace of X/Z.

The 3-space problem can now be stated as, for closed subspaces $Z \subset Y \subset X$, if Z is weakly coproximinal in X and the quotient space Y/Z is weakly coproximinal in X/Z, is Y weakly coproximinal in X?

We next give an example using the space of bounded sequences, ℓ^∞, for settling the 3-space problem in the negative in the context of weak coproximinality. It is an easy consequence of Theorem 3 in Section 11 of [9], that if ℓ^∞ is isometrically embedded in a Banach space L then it is the range of a projection of norm one in L and hence it is weakly coproximinal in L. This also answers a question raised in [11].

Example 2.8. *Consider the ℓ^1-direct sum $\ell^\infty \oplus_1 \ell^\infty$. Let K be any compact Hausdorff space such that $\ell^\infty \oplus_1 \ell^\infty \subset C(K)$. In view of our remarks above, for the inclusion $\ell^\infty \subset \ell^\infty \oplus_1 \ell^\infty \subset C(K)$, ℓ^∞ is a weakly coproximinal subspace of $C(K)$. Also since $(\ell^\infty \oplus_1 \ell^\infty)/\ell^\infty$ is isometric to ℓ^∞, this quotient space is weakly coproximinal in $C(K)/\ell^\infty$. We next note that $\ell^\infty \oplus_1 \ell^\infty$ is not weakly coproximinal in $C(K)$. It is easy to see that $\ell^\infty \oplus_1 \ell^\infty$ is a coproximinal subspace of its bidual. Suppose $\ell^\infty \oplus_1 \ell^\infty$ is a weakly coproximinal subspace of $C(K)$. Then by Proposition 2.1, it is* *a coproximinal subspace of $C(K)$. Using the finite binary intersection property of balls as in Theorem 6 in Section 21 of* [9]*, as $C(K)$ has the finite binary intersection property, we can show that $\ell^\infty \oplus_1 \ell^\infty$ has the finite binary intersection property. Hence by Theorem 6 in Section 21 of* [9] *again, the dual of $\ell^\infty \oplus_1 \ell^\infty$, is isometric to $L^1(\mu)$, for a positive measure μ. Thus $L^1(\mu) = (\ell^\infty)^* \oplus_\infty (\ell^\infty)^*$. Since $L^1(\mu)$ always has non-trivial ℓ^1-decompositions, by Theorem I.1.8 of* [5]*, it can not have a non-trivial ℓ^∞-decomposition. Hence we get a contradiction. Thus $\ell^\infty \oplus_1 \ell^\infty$ is not a weakly coproximinal subspace of $C(K)$.*

For any sequence of Banach spaces $\{X_n\}_{n\geq 1}$, by $\bigoplus_{c_0} X_n$, we denote the space of all sequences $\{\{x_n\}_{n\geq 1} : x_n \in X_n,\ \lim \|x_n\| = 0\}$, equipped with the supremum norm. We recall that $(\bigoplus_{c_0} X_n)^{**}$ is the space of bounded sequences, $\bigoplus_\infty X_n^{**}$.

Proposition 2.9. *Let $Y_n \subset X_n$ be a closed weakly coproximinal subspace for all n. Then $\bigoplus_{c_0} Y_n$ is weakly coproximinal in $\bigoplus_\infty X_n$.*

Proof. We will show that $\bigoplus_{c_0} Y_n$ is weakly coproximinal in $\bigoplus_{c_0} X_n$. Now for the inclusion, $\bigoplus_{c_0} X_n \subset \bigoplus_\infty X_n \subset \bigoplus_\infty X_n^{**}$, as the last space is the bidual of the first, we get that $\bigoplus_{c_0} X_n$ is weakly coproximinal in $\bigoplus_\infty X_n$ and as this property is transitive, we get the desired conclusion.

Let $\{x_n\}_{n\geq 1} \in \bigoplus_{c_0} X_n$. For every n, for $x_n \in X_n$, by hypothesis, there is a $\tau_n \in Y_n^\perp$ such that $\|\Lambda - \tau_n\| \leq \|\Lambda - x_n\|$ for all $\Lambda \in Y_n^{\perp\perp}$. In particular $\|\tau_n\| \leq \|x_n\|$ for all n. Thus $\{\tau_n\}_{n\geq 1} \in \bigoplus_{c_0} Y_n^{\perp\perp}$. It is easy to see that $(\bigoplus_{c_0} Y_n)^{\perp\perp} = \bigoplus_\infty Y_n^{\perp\perp}$. Now for any $\{\Lambda_n\}_{n\geq 1} \in \bigoplus_\infty Y_n^{\perp\perp}$, as $\Lambda_n \in Y_n^{\perp\perp}$ for all n, we have, $\|\Lambda_n - \tau_n\| \leq \|\Lambda_n - x_n\|$, for all n. Thus $\|\{\Lambda_n\}_{n\geq 1} - \{\tau_n\}_{n\geq 1}\| \leq \|\{\Lambda_n\}_{n\geq 1} - \{x_n\}_{n\geq 1}\|$. Therefore $\bigoplus_{c_0} Y_n$ is weakly coproximinal in $\bigoplus_{c_0} X_n$.

We recall that a closed subspace $Y \subset X$ is a constrained subspace if there is a linear projection $P : X \to Y$ such that $\|P\| = 1$. Our next result addresses weak coproximinality for subspaces of constrained subspaces.

Proposition 2.10. *Let $Y \subset X$ be a closed subspace and let $P : X \to Y$ be a surjective linear projection of norm one. Let $Z_1 \subset X$ be a closed subspace such that $P(Z_1) \subset Z_1$ and let $Z_2 = Y \cap Z_1$. If Z_1 is weakly coproximinal in X, then Z_2 is weakly coproximinal in Y.*

Proof. Let $y \in Y$. Since Z_1 is coproximinal in X, there is a $\tau \in Z_1^{\perp\perp}$ such that $\|\Lambda - \tau\| \leq \|\Lambda - y\|$ for all $\Lambda \in Z_1^{\perp\perp}$. It is easy to see that $P^{**} : X^{**} \to Y^{\perp\perp}$ is a surjective projection of norm one and has the property, $P^{**}(Z_1^{\perp\perp}) \subset Z_1^{\perp\perp}$ and also $Z_2^{\perp\perp} = Z_1^{\perp\perp} \cap Y^{\perp\perp}$. Thus $P^{**}(\tau) \in Z_2^{\perp\perp}$ and for any $\Lambda \in Z_2^{\perp\perp}$,

$$\|\Lambda - P^{**}(\tau)\| = \|P^{**}(\Lambda - \tau)\| \leq \|\Lambda - \tau\| \leq \|\Lambda - y\|.$$

Thus Z_2 is weakly coproximinal in Y.

For a compact Hausdorff space K, let $C(K,X)$ denote the space of X-valued continuous functions on K, equipped with the supremum norm. Embedding X as the space of constant functions in $C(K,X)$, for any fixed $k_0 \in K$, $P : C(K,X) \to X$ defined by $P(f) = f(k_0)$ is a projection of norm one. Now it is easy to see that for a closed subspace $Y \subset X$, if $C(K,Y)$ is weakly coproximinal in $C(K,X)$, then Y is weakly coproximinal in X.

We next give an application of Proposition 2.10, for retracts in compact Hausdorff space. Let $E \subset K$ and $\phi : K \to E$ be a continuous map such that ϕ is identity on E. Then $\Phi : C(E,X) \to C(K,X)$ defined by $\Phi(f) = f \circ \phi$ for $f \in C(E,X)$ is an into isometry and $P : C(K,X) \to \Phi(C(E,X))$ defined by $P(f) = \Phi(f|E)$ is a surjective linear projection of norm one.

Proposition 2.11. *Let $E \subset K$ be a retract and let $Y \subset X$ be a weakly coproximinal subspace of X. If $C(K,Y)$ is weakly coproximinal in $C(K,X)$, then $C(E,Y)$ is weakly coproximinal in $C(E,X)$.*

Proof. We will show that $\Phi(C(E,Y))$ is weakly coproximinal in $\Phi(C(E,X))$. Using the projection P described above, we see that $P(C(K,Y)) \subset C(K,Y)$. Thus in order to apply Proposition 2.10, we need to verify that $\Phi((C(E,Y)) = \Phi(C(E,X)) \cap C(K,Y)$. Let $f \in \Phi(C(E,X)) \cap C(K,Y)$, then $f = \Phi(h) = h \circ \phi$ for some $h \in C(E,X)$. As f takes values in Y we get that $h \in C(E,Y)$ so that $f \in \Phi(C(E,Y)$. On the other hand for $g \in C(E,Y)$, $\Phi(g) = g \circ \phi \in \Phi(C(E,X)) \cap C(K,Y)$. Hence the conclusion follows.

Question 2.12. For a finite dimensional coproximinal subspace $F \subset X$ and for an infinite compact set K, we do not know if, $C(K,F)$ is a weakly coproximinal subspace of $C(K,X)$?

We recall from Section 11 of [9] that a compact Hausdorff space K is extremally disconnected if and only if it is homeomorpic to a retract of the Stone-Čech compactification $\beta(S)$ of a discrete set S. In what follows we treat K as a retract of $\beta(S)$.

Proposition 2.13. *Let $Z \subset Y \subset X$ be closed subspaces such that $Y \subset X$ is a weakly coproximinal subspace of X and Z is a factor reflexive subspace of Y. Let K be a compact extremally disconnected space. $WC(K, Y/Z)$ is a coproximinal subspace of $WC(K, X/Z)$.*

Proof. Let K be a retract of $\beta(S)$ for some discrete set S. Arguments similar to those given during the proof of Proposition 2.11 can be used to see that it is enough to verify that $WC(\beta(S), Y/Z)$ is a coproximinal subspace of $WC(\beta(S), X/Z)$.

Since Z is a factor reflexive subspace of Y, by Proposition 2.6 we have that Y/Z is a coproximinal subspace of X/Z. For any $f \in WC(\beta(S), X/Z)$, for each $s \in S$, there exists a $g_0(s) \in Y/Z$ such that $\|\Lambda - g_0(s)\| \leq \|\Lambda - f(s)\|$ for all $\Lambda \in Y/Z$. It is easy to see that $g_0 : S \to Y/Z$ is a bounded function and as Y/Z is reflexive, bounded sets in Y/Z are relatively weakly compact. Hence g_0 has an extension, still denoted by g_0 in $WC(\beta(S), Y/Z)$. Now it is easy to see that for any $h \in WC(\beta(S), Y/X)$, $\|h - g_0\| \leq \|h - f\|$. Thus $WC(\beta(S), Y/Z)$ is a coproximinal subspace of $WC(\beta(S), X/Z)$ and the same conclusion holds for $WC(K, Y/Z) \subset WC(K, X/Z)$.

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