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POROSITY OF THE FREE BOUNDARY IN THE DEGENERATE PARABOLIC VARIATIONAL PROBLEM

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Abstract. The aim of this paper is to study the porosity of the free boundary for the degenerate quasilinear parabolic variational problem with identically zero constraint.

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1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^N (N \geq 2)$, $\Omega_T = \Omega \times (0,T)$ and $\partial_p \Omega_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,T])$. Denote the parabolic space by $V^{1,p}(\Omega_T)$, see [1],

$$V^{1,p}(\Omega_T) = L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)) \quad (1 < p < \infty).$$

The Steklov average v_h of a function is defined by

$$v_h(x,t) = \frac{1}{h}\int_t^{t+h} v(x,\tau)\mathrm{d}\tau \quad \text{for } t \in (0,T-h],$$

and $v_h = 0$ for $t > T-h$.

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We consider a variational inequality for the quasilinear parabolic operator

$$\operatorname{div} a(x,t,u,\nabla u) - \partial_t u,$$

giving rise to a free boundary. More precisely, given measurable function $a(x,t,\mu,\eta):\Omega_T\times\mathbb{R}^{N+1}\to\mathbb{R}^N$ satisfying standard structural conditions, and bounded functions f and θ, and the obstacle function $\psi\in V^{1,p}(\Omega_T)$, the variational problem is to find a function

$$u\in\mathscr{K}_\theta := \mathscr{K}_\theta(p) = \{w : w\in V^{1,p}(\Omega_T),\ \forall t\ w=\theta \text{ on } \partial_p\Omega_T,\ w\geq\psi \text{ a.e. in } \Omega_T\},$$

such that (for $h>0$ and $0<t<t+h<T$)

$$\int_\Omega \partial_t u_h(w-u)\mathrm{d}x + \int_\Omega (a(x,t,u,\nabla u))_h\cdot\nabla(w-u)\mathrm{d}x + \int_\Omega f_h(w-u)\mathrm{d}x \geq 0, \tag{1.1}$$

a.e. in $t\in(0,T)$, and for all $w\in\mathscr{K}_\theta$.

Under certain conditions on f and θ, we are expected to show that the free boundary of solutions to the variational problems (1.1) is porous for each $t-$level cut. Thus the $t-$cuts of the free boundary is of Lebesgue measure zero.

The obstacle problem has lots of applications in Mathematical Physics (see [2] for instance) and has been studied by a large number of mathematicians, Alt, Caffarelli, Frehse, Kinderlehrer, Phillips, Rodrigues, Shahgholion, Stampacchia, Weiss, and others. Porosity of the free boundary in the $p-$Laplacian type obstacle problem was considered in [3, 4, 5], and has been extended to the one governed by large class of heterogeneous quasilinear elliptic operators, see [6, 7]. It is based on the growth rate of solutions near the free boundary, which can be done by strong minimum principle or Harnack inequality, see [3, 4, 5, 6, 7]. As for $p-$parabolic variational problems $(p\neq 2)$, one cannot inherit each technique from the elliptic obstacle problem due to the lack of strong minimum principle or Harnack inequality. One needs further arguments to establish the growth rate of solutions near the free boundary. In 2003, Shahgholian overcame this difficulty by using Hölder's estimates for solutions in the $p-$parabolic variational problem $(p\geq 2)$. As a by-product, the author obtained the porosity of the free boundary for each t-level cut, see [8]. Recently, [9, 10] obtained porosity of the free boundary for the $p-$parabolic obstacle problem $(1<p<2)$. Motivated by the ideas of [8], in this paper, we are going to extend the results of [8] to a large class of variational problem governed by degenerate quasi-linear parabolic operators. Due to the non-invariant property of solutions under the operator div $a(x,t,u,\nabla u) - \partial_t u$, techniques of compactness will be applied to establish the growth when we use intrinsic scaling, which was also applied in [4, 5, 6, 7, 10, 11, 12]. It should be noticed that the finite $N-1$-Hausdorff measure of the free boundary in the elliptic obstacle problem was considered in [11, 12, 13, 14, 15] based on the property of porosity of the free boundary.

Throughout this paper, we always assume $2 < p < \infty$ unless specified otherwise. We make the standard structural conditions on the function $a(x,t,\mu,\eta)$ for some positive constants γ_0, γ_1, namely,

$$(a_1)\ a_i(x,t,\mu,0) = 0; \tag{1.2}$$

$$(a_2)\ \sum_{i,j=1}^N \frac{\partial a_i}{\partial \eta_j}(x,t,\mu,\eta)\xi_i\xi_j \geq \gamma_0|\eta|^{p-2}|\xi|^2; \tag{1.3}$$

$$(a_3)\ \sum_{i,j=1}^N \left|\frac{\partial a_i}{\partial \eta_j}(x,t,\mu,\eta)\right| \leq \gamma_1|\eta|^{p-2}; \tag{1.4}$$

$$(a_4)\ \sum_{i,j=1}^N \left|\frac{\partial a_i}{\partial x_j}(x,t,\mu,\eta)\right| + \sum_{i,j=1}^N \left|\frac{\partial a_i}{\partial \mu}(x,t,\mu,\eta)\right| \leq \gamma_1|\eta|^{p-1}. \tag{1.5}$$

for a.e. $(x,t) \in \Omega_T$, all $\mu \in \mathbb{R}, \eta \in \mathbb{R}^N \backslash \{0\}$, and all $\xi \in \mathbb{R}^N$.

Remark 1.1. Assumptions $(a_1)-(a_4)$ imply that (see [16, 17] for instance)

(i) $a(x,t,\mu,\eta)\eta \geq \frac{\gamma_0}{p-1}|\eta|^p$,

(ii) $|a(x,t,\mu,\eta)| \leq \frac{\gamma_1}{p-1}|\eta|^{p-1}$,

(iii) $\langle a(x,t,\mu,\eta_1) - a(x,t,\mu,\eta_2), \eta_1 - \eta_2\rangle \geq 0$,

for a.e. $(x,t) \in \Omega_T$ and all $\mu \in \mathbb{R}, \eta, \eta_1, \eta_2 \in \mathbb{R}^N$. Thus the structural conditions for quasilinear operators in [1] are satisfied, which are needed in this paper.

Suppose that f and θ are bounded continuous functions on the closure of Ω_T. To establish the results obtained in this paper, further conditions on f, θ and ψ are imposed as follows:

(f) $0 < \lambda_0 \leq f \leq \Lambda$ in Ω_T, $f(x,t)$ is monotone non-increasing in t.

(θ) $\theta(x,0) = 0$, $\theta(x,t)$ is monotone non-decreasing in t.

(ψ) $\psi \equiv 0$ in Ω_T.

Let us gather some properties for the solution u to the variational inequality (1.1). The following theorem can be proven by classical techniques, we refer readers to [3, 5, 8] for sketch of proofs.

Classical Theorem. *There exists a unique solution u to the variational problem (1.1) in $\mathscr{K}_\theta$ with*

$$0 \le u \le \|\theta\|_{\infty,\Omega_T} \text{ in } \Omega_T, \text{ and } \partial_t u \ge 0 \text{ in } \{u>0\}.$$

Moreover u satisfies

$$div\, a(x,t,u,\nabla u) - \partial_t u = g \quad \text{in } \{u>0\}.$$

weakly in Ω_T with $g \in L^\infty(\Omega_T)$ satisfying

$$f\chi_{\{u>0\}} \le g \le f\chi_{\overline{\{u>0\}}} \text{ a.e. in } \Omega_T.$$

We recall the concept of porosity, see [3, 8].

Porosity. A set E in $\mathbb{R}^N$ is called porous with porosity constant δ if there is a constant $r_0 > 0$ such that for each $x \in E$ and $0 < r < r_0$ there is a point y such that $B_{\delta r}(y) \subset B_r(x) \setminus E$.

According to [18], a porous set has Hausdorff dimension not exceeding $N - C\delta^N$, thus it is of Lebesgue measure zero.

Now we state the main theorem in this paper.

Theorem 1.2. *Let u be the solution to problem (1.1) in $\mathscr{K}_\theta$. Then for every compact set $K \subset \Omega_T$ there holds*

$$c_0 r^{\frac{p}{p-1}} \le \sup_{B_r(x_0)} u(\cdot,t_0) \le C_0 r^{\frac{p}{p-1}}, \quad \forall\, (x_0,t_0) \in \partial\{u>0\} \cap K.$$

Consequently, the intersection $\partial\{u>0\} \cap K \cap \{t = t_0\}$ is porous (in $\mathbb{R}^N$) with the porosity constant

$$\delta = \delta(\|\theta\|_{\infty,\Omega_T}, \lambda_0, \Lambda_0, dist(K, \partial_p\Omega_T), \gamma_0, \gamma_1, p).$$

Here c_0 depends on p, λ_0, γ_1 and N, C_0 depends on $p, \lambda_0, \Lambda_0, \gamma_0, \gamma_1, \|\theta\|_{\infty,\Omega_T}$ and N.

2. A class of functions on the unit cylinder

We first denote $u \in W^{1,p}(\Omega_T)$ if u and its gradient are both in $L^p(\Omega_T) = L^p(0,T;L^p(\Omega))$. Let $q = \frac{p}{p-1}$ and $Q_r(z,s) = B_r(z) \times (-r^q + s, r^q + s)$ be the cylinder in $\mathbb{R}^{N+1}$. Write $Q_1 = Q_1(0,0)$, the unit cylinder. Due to the local character of the results obtained in this paper (Theorem 1.1), we may consider the following local formulation. We say that a function $u \in W^{1,p}(Q_1)$ belongs to the class $\mathscr{G}_a = \mathscr{G}_a(p, \gamma_0, \gamma_1)$ if

(2a) $\|\text{div } a(x,t,u,\nabla u) - \partial_t u\|_{\infty,Q_1} \le 1$;

(2b) $0 \le u \le 1$, a.e. in Q_1;

(2c) $u(0,0) = 0$;

(2d) $\partial_t u \ge 0$ a.e. in Q_1.

Condition (2a) should be understood in the weak sense, i.e., div $a(x,t,u,\nabla u) - \partial_t u = h$ weakly for $h \in L^\infty(Q_1)$ with $\|h\|_{\infty,Q_1} \le 1$. Condition (2c) makes sense since (2a) and (2b) provide that $u \in C^{0,\alpha}(Q_{\frac{1}{2}})$ for some $\alpha \in (0,1)$ (see e.g. [1]).

In this section, we discuss the behavior of solution to (1.1) and functions in $\mathscr{G}_a$ near the free boundary. Firstly we establish the following non-degeneracy of the solution, showing that it cannot grow too slowly near the free boundary.

Lemma 2.1. *Let $u \in W^{1,p}(Q_1)$ be a non-negative continuous function in Q_1, satisfying*

$$\text{div } a(x,t,u,\nabla u) - \partial_t u = f$$

weakly in $U^+ = \{u > 0\}$. Then for every $(z,s) \in \overline{U^+}$ and $r > 0$ with $Q_r(z,s) \subset Q_1$,

$$\sup_{(x,t)\in\partial_p Q_r^-(z,s)} u(x,t) \ge c_0 r^{\frac{p}{p-1}} + u(z,s),$$

where $Q_r^-(z,s) = B_r(z) \times (s - r^q, s)$, c_0 is a positive constant depending only on p, λ_0, γ_1.

Proof. First suppose that $(z,s) \in U^+$, and for small $\varepsilon > 0$ set $u_\varepsilon(x,t) = u(x,t) - (1-\varepsilon)u(z,s)$, and $v(x,t) = C_1|x-z|^{\frac{p}{p-1}} - C_2(t-s)$, where C_1, C_2 are positive constants, depending only on p, λ_0, γ_1, such that

$$\gamma_1 \left(\frac{pC_1}{p-1}\right)^{p-1} \left(1 + \frac{pC_1}{p-1} + \frac{2p}{p-1}\right) + C_2 \le \lambda_0.$$

We claim that for C_1, C_2 there holds

$$\text{div } a(x,\nabla v) - \partial_t v \le \lambda_0, \quad \forall\ (x,t) \in U^+ \cap Q_r^-(z,s). \tag{2.1}$$

To prove (2.1), we need to calculate ∇v and divergence of $a(x,\nabla v)$. Indeed,

$$\nabla v(x,t) = \frac{pC_1}{p-1}|x-z|^{\frac{2-p}{p-1}}(x-z), \quad |D_{ij}v(x,t)| \le \frac{2p^2C_1}{(p-1)^2}|x-z|^{\frac{2-p}{p-1}}.$$

One may verify that

$$\begin{aligned}
\operatorname{div} a(x,t,v,\nabla v) - \partial_t v &= \sum_{i=1}^{N} \left[\frac{\partial a_i}{\partial x_i}(x,t,v,w) + \frac{\partial a_i}{\partial \mu}(x,t,v,w) \frac{\partial v}{\partial x_i}(x,t) \right] \\
&\quad + \sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(x,t,v,w) \frac{\partial w_j}{\partial x_i}(x) + C_2 \\
&\leq \gamma_1 |w|^{p-1} + \gamma_1 |w|^{p-1}|w| + \gamma_1 |w|^{p-2} \frac{2p^2 C_1}{(p-1)^2} |x-z|^{\frac{2-p}{p-1}} + C_2 \\
&\leq \gamma_1 \left(\frac{pC_1}{p-1} \right)^{p-1} \left(|x-z| + \frac{pC_1}{p-1} |x-z|^{\frac{p}{p-1}} + \frac{2p}{p-1} \right) + C_2 \\
&\leq \gamma_1 \left(\frac{pC_1}{p-1} \right)^{p-1} \left(1 + \frac{pC_1}{p-1} + \frac{2p}{p-1} \right) + C_2 \\
&\leq \lambda_0,
\end{aligned}$$

where $w(x,t) = \nabla v(x,t) = \frac{pC_1}{p-1}|x-z|^{\frac{2-p}{p-1}}(x-z)$.

Notice that $\operatorname{div}\ a(x,t,u,\nabla u) - \partial_t u = \operatorname{div}\ a(x,,t,u_\varepsilon,\nabla u_\varepsilon) - \partial_t u_\varepsilon$ in $U^+ \cap Q_r^-(z,s)$. Recall condition (f), it follows

$$\operatorname{div} a(x,t,v,\nabla v) - \partial_t v \leq \operatorname{div} a(x,t,u,\nabla u_\varepsilon) - \partial_t u_\varepsilon \quad \text{in } \Omega_+ \cap Q_r^-(z,s).$$

It is easy to see $u_\varepsilon(x,t) = -(1-\varepsilon)u(z,s) \leq 0$ on ∂U^+ and $v(x,t) \geq 0$ for any $t \leq s$, thus $u_\varepsilon \leq v$ on $\partial U^+ \cap Q_r^-(z,s)$. If also $u_\varepsilon \leq v$ on $\partial Q_r^-(z,s) \cap U^+$, then we get by comparison principle

$$u_\varepsilon \leq v \ \text{ in } Q_r^-(z,s) \cap U^+.$$

But $u_\varepsilon(z,s) = \varepsilon u(z,s) > 0 = v(z,s)$, which is a contradiction. Therefore there exists some point $(y,\tau) \in \partial Q_r^-(z,s)$ such that

$$u_\varepsilon(y,\tau) \geq v(y,\tau) = c_0 r^{\frac{p}{p-1}},$$

where $c_0 = \min\{C_1, C_2\}$. Letting $\varepsilon \to 0$ we obtain the desired result for all $(z,s) \in U^+$, and by continuity for all $(z,s) \in \overline{U^+}$. This completes the proof.

Secondly, we will show that every function u in $\mathscr{G}_a$ cannot grow too fast near the free boundary, but has a growth rate of order $\frac{p}{p-1}$ (Theorem 2.3).

Define the supremum norm of u over the cylinder $Q_r^-(z,s)$ as in [8] by setting

$$S(r,u,z,s) = \sup_{x \in Q_r^-(z,s)} u(x,t), \ \text{ and } \ S(r,u) = \sup_{x \in Q_r^-(0,0)} u(x,t)\ .$$

For each $u \in \mathscr{G}_a$, define the set $\mathbb{M}_a(u,z,s)$ by setting

$$\mathbb{M}_a(u,z,s) = \{j \in \mathbb{N};\ AS(2^{-j-1},u,z,s) \geq S(2^{-j},u,z,s)\},$$

where $A = 2^q \max\{1, \frac{1}{c_0}\}$ with $q = \frac{p}{p-1}$, and c_0 as in Lemma 2.1. For simplicity, we write $\mathbb{M}_a(u) = \mathbb{M}_a(u,0,0)$.

It should be noticed that $\mathbb{M}_a(u) \neq \emptyset$ for all $u \in \mathscr{G}_a$ since $0 \in \mathbb{M}_a(u)$. Indeed, it follows by Lemma 2.1 that $S(1,u) \leq 1 = (\frac{1}{c_0 2^{-q}})c_0 2^{-q} \leq (\frac{1}{c_0 2^{-q}})S(2^{-1},u) \leq AS(2^{-1},u)$.

We have the following property for the elements is the class $\mathscr{G}_a$.

Lemma 2.2. *There is a positive constant $M_1 = M_1(p,\gamma_0,\gamma_1)$ such that*

$$S(2^{-j-1},u) \leq M_1(2^{-j})^q,$$

for all $u \in \mathscr{G}_a$ and $j \in \mathbb{M}_a(u)$.

Proof. Arguing by contradiction, assume that for every $k \in \mathbb{N}$, there exists $u_k \in \mathscr{G}_a$ and $j_k \in \mathbb{M}_a(u_k)$ such that

$$S(2^{-j_k-1},u_k) \geq k(2^{-j_k})^q. \tag{2.2}$$

Observe that by the uniform boundedness of u_k and (2.2) it follows $j_k \to \infty$ as $k \to \infty$.

Consider the function

$$\widetilde{u}_k(x,t) = \frac{u_k(2^{-j_k}x, \alpha_k t)}{S(2^{-j_k-1},u_k)}$$

defined in the unit cylinder, where $\alpha_k = (2^{-j_k})^p (S(2^{-j_k-1},u_k))^{2-p}$. Note that by (2.2) we have

$$\begin{aligned}\alpha_k &\leq \frac{1}{k^{p-1}}(S(2^{-j_k-1},u_k))^{p-1} \cdot (S(2^{-j_k-1},u_k))^{2-p}\\ &= \frac{1}{k^{p-1}} S(2^{-j_k-1},u_k)\\ &\leq \frac{1}{k^{p-1}} \to 0 \ \text{ as } k \to \infty.\end{aligned}$$

By the definition of $\mathbb{M}_a(u_k)$ and $\mathscr{G}_a$ it follows

$$
\begin{aligned}
&0 \le \widetilde{u}_k \le A \quad \text{in } Q_1^-, \\
&\sup_{Q^-_{\frac{1}{2}}} \widetilde{u}_k \ge 1 \quad \text{(by (2d) and } (2^{-1})^q \alpha_k \ge (2^{-j_k-1})^q), \\
&\widetilde{u}_k(0,0) = 0, \\
&\partial_t \widetilde{u}_k \ge 0 \quad \text{in } Q_1^-.
\end{aligned}
$$

Now, define for $(x,t,\mu,\eta) \in B_1 \times (-1,1) \times \mathbb{R}^{N+1}$

$$
a^k(x,t,\mu,\eta) = \left(\frac{2^{-j_k}}{S(2^{-j_k-1},u_k)} \right)^{p-1} \cdot a\left(2^{-j_k}x, \alpha_k t, S(2^{-j_k-1},u_k)\mu, \frac{S(2^{-j_k-1},u_k)}{2^{-j_k}}\eta \right).
$$

We claim that $a^k(x,t,\mu,\eta)$ satisfies the same structural conditions as $a(x,t,\mu,\eta)$ for large k. Indeed, letting $s_k = \frac{2^{-j_k}}{S(2^{-j_k-1},u_k)}$, one may verify directly that

$$
\begin{aligned}
\sum_{i,j=1}^N \frac{\partial a_i^k}{\partial \eta_j}(x,t,\mu,\eta)\xi_i\xi_j &= \sum_{i,j=1}^N s_k^{p-2} \frac{\partial a_i}{\partial \eta_j}(2^{-j_k}x, \alpha_k t, S(2^{-j_k-1},u_k)\mu, s_k^{-1}\eta)\xi_i\xi_j \\
&\ge \gamma_0 s_k^{p-2} |s_k^{-1}\eta|^{p-2}|\xi|^2 \\
&= \gamma_0 |\eta|^{p-2}|\xi|^2,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i,j=1}^N |\frac{\partial a_i^k}{\partial \eta_j}(x,t,\mu,\eta)| &= \sum_{i,j=1}^N s_k^{p-2} \left| \frac{\partial a_i}{\partial \eta_j}(2^{-j_k}x, \alpha_k t, S(2^{-j_k-1},u_k)\mu, s_k^{-1}\eta) \right| \\
&\le \gamma_1 s_k^{p-2} |s_k^{-1}\eta|^{p-2} \\
&= \gamma_1 |\eta|^{p-2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i,j=1}^N [|\frac{\partial a_i^k}{\partial x_j}(x,t,\mu,\eta)| + |\frac{\partial a_i^k}{\partial \mu}(x,t,\mu,\eta)|] \\
&= \sum_{i,j=1}^N s_k^{p-1} 2^{-j_k} |\frac{\partial a_i}{\partial x_j}\left(2^{-j_k}x, \alpha_k t, S(2^{-j_k-1},u_k)\mu, s_k^{-1}\eta\right)| \\
&\quad + \sum_{i,j=1}^N s_k^{p-1} S(2^{-j_k-1},u_k) |\frac{\partial a_i^k}{\partial \mu}(2^{-j_k}x, \alpha_k t, S(2^{-j_k-1},u_k)\mu, s_k^{-1}\eta)| \qquad (2.3) \\
&\le \gamma_1 |\eta|^{p-1} \max\{2^{-j_k}, S(2^{-j_k-1},u_k)\} \\
&\le \gamma_1 |\eta|^{p-1}.
\end{aligned}
$$

Now by (2a) and (2.2), we obtain

$$\begin{aligned}\|\text{div } a^k(x,t,\widetilde{u}_k(x,t),\nabla\widetilde{u}_k(x,t))-\partial_t\widetilde{u}_k(x,t)\|_\infty &= 2^{-j_k}s_k^{p-1}\|(Au_k-\partial_t u_k)(2^{-j_k}x,\alpha_k t)\|_\infty\\ &\le 2^{-j_k}\left(\frac{2^{-j_k}}{S(2^{-j_k-1},u_k)}\right)^{p-1}\\ &\le \frac{1}{k^{p-1}}\to 0 \quad \text{as } k\to\infty,\end{aligned}$$

where $(Au)(x,t)$ is defined by $(Au)(x,t)=\text{div } a(x,t,u,\nabla u(x,t))$. Observe that by (2.3), for any $M>0$, we have

$$|\frac{\partial a_i^k}{\partial x_j}(x,t,\mu,\eta)|+|\frac{\partial a_i^k}{\partial \mu}(x,t,\mu,\eta)|\to 0 \quad \text{as } k\to\infty,$$

uniformly in $(x,t,\mu,\eta)\in B_1\times(-1,1)\times(-1,1)\times B_M$. Therefore the pointwise limit of $a^k(x,t,\mu,\eta)$ does not depend on x and μ:

$$a^k(x,t,\mu,\eta)\to\widetilde{a}(t,\eta),$$

with $\widetilde{a}$ satisfying the same structural conditions as $(a_1)-(a_4)$. Then invoking compactness arguments (see Lemma 14.1 on page 75 of [1]), we deduce that, up to subsequence, $\widetilde{u}_k$ converges locally uniformly in Q_1^- to a function u. Moreover, the limit function u satisfies

$$\text{div } \widetilde{a}(t,\nabla u)-\partial_t u=0,\ \ u\ge 0,\ \ u(0,0)=0,\ \ \sup_{Q^-_{\frac{1}{2}}} u\ge 1,\ \ \partial_t u\ge 0 \ \text{ in } Q_1^-.$$

Due to the lack of strong minimum principle, we need further discussion to get a contradiction. At this junction, we show that u is time independent, i.e.,

$$\partial_t u=0 \ \text{ in } Q_1^-.$$

Then u is nonzero, nonnegative $A-$harmonic function in the unit ball and it vanishes at the origin. Indeed, for any fixed $t=\tau\in(-1,1)$, this is a contradiction to the strong minimum principle applied for div $\widetilde{a}(\tau,\nabla u)=0$, see [19] for instance. To this end, let us recall the following Hölder's estimate for solutions of our problem, see [1]. Let $G_T=G\times(0,T]$, where G is a bounded domain in $\mathbb{R}^N$. For every pair $(x,t)\in K$ (a compact set in G_T) there exist constants $\gamma>1$ and $\alpha\in(0,1)$ such that

$$|u(x,t)-u(u,\tau)|\le\gamma\|u_k\|_{\infty,G_T}\left(\frac{|x-y|+\|u_k\|_{\infty,G_T}^{\frac{p-2}{p}}|t-\tau|^{\frac{1}{p}}}{\text{dist}_p(K,\partial_p G_T;p)}\right)^\alpha,$$

where

$$\operatorname{dist}_p(K,\partial_p G_T;p)=\inf_{(x,t)\in K,(y,\tau)\in\partial_p G_T}\left(|x-y|+\|u\|_{\infty,G_T}^{\frac{2-p}{p}}|t-\tau|^{\frac{1}{p}}\right).$$

Observe that γ does not depend on $\|u_k\|_{\infty,G_T}$ in our case, see [1].

Now choosing $(x,t),(x',t')\in Q^-_{\frac{1}{2}}$ and using the definition of $\mathbb{M}_a(u_k)$ and Hölder's estimates for solution with $G_T=Q^-_{2^{-j_k}}$ and $K=Q^-_{2^{-j_k-1}}$ (see Theorem 1.1 on page 41 of [1]), we arrive at

$$\begin{aligned}|\widetilde{u}_k(x,t)-\widetilde{u}_k(x,t')| &= \frac{|u_k(2^{-j_k}x,\alpha_k t)-u_k(2^{-j_k}x,\alpha_k t')|}{S(2^{-j_k-1},u_k)}\\ &\le A\frac{|u_k(2^{-j_k}x,\alpha_k t)-u_k(2^{-j_k}x,\alpha_k t')|}{S(2^{-j_k},u_k)}\\ &\le A\gamma\frac{\|u_k\|_{\infty,G_T}}{S(2^{-j_k},u_k)}\left(\frac{\|u_k\|_{\infty,G_T}^{\frac{p-2}{p}}\alpha_k^{\frac{1}{p}}|t-t'|^{\frac{1}{p}}}{\operatorname{dist}_p(K,\partial_p G_T;p)}\right)^{\alpha}.\end{aligned}$$

Notice that $\operatorname{dist}_p(K,\partial_p G_T;p)\ge(2^{-j_k-1})^{\frac{q}{p}}\|u\|_{\infty,G_T}^{\frac{p-2}{p}}$. It follows

$$\begin{aligned}|\widetilde{u}_k(x,t)-\widetilde{u}_k(x,t')| &\le A\gamma|t-t'|^{\frac{\alpha}{p}}(\alpha_k\cdot 2^{(j_k+1)q})^{\frac{\alpha}{p}}\\ &= A\gamma|t-t'|^{\frac{\alpha}{p}}\left[(2^{-j_k})^p\cdot(S(2^{-j_k-1},u_k))^{2-p}\cdot 2^{(j_k+1)q}\right]^{\frac{\alpha}{p}}\\ &\le A\gamma|t-t'|^{\frac{\alpha}{p}}\left[(2^{-j_k})^{p-1}\cdot k^{2-p}\cdot(2^{-j_k})^{q(2-p)}\cdot(2^{j_k})^q\cdot 2^q\right]^{\frac{\alpha}{p}}\\ &= 2^{\frac{\alpha}{p-1}}A\gamma|t-t'|^{\frac{\alpha}{p}}k^{2-p}\to 0 \text{ as } k\to\infty.\end{aligned}$$

Hence u is $t-$independent, and the proof is completed.

Now we establish the growth for the elements of the class $\mathscr{G}_a$.

Theorem 2.3. *There is a positive constant $M_0=M_0(p,\gamma_0,\gamma_1)$ such that for every $u\in\mathscr{G}_a$, there holds*

$$|u(x,t)|\le M_0(d(x,t))^q\quad\forall\,(x,t)\in Q_{\frac{1}{2}}$$

where $d(x,t)=\sup\{r;Q_r(x,t)\subset U^+\}$ for $(x,t)\in U^+$, and $d(x,t)=0$ otherwise.

Proof. The proof of this theorem is standard (see [8]). For convenience, we recover the process. Let us take the first j for which

$$S(2^{-j},u)>2^q M_1 2^{-qj}.$$

It follows that

$$S(2^{-j+1},u)\leq 2^qM_12^{-q(j-1)}<2^qS(2^{-j},u)\leq AS(2^{-j},u), \tag{2.4}$$

i.e. $j-1\in\mathbb{M}_a(u)$, so Lemma 2.2 holds for $j-1$. Now we arrive at the following obvious contradiction to (2.4)

$$S(2^{-j},u)\leq S(2^{-j+1},u)\leq M_12^{-q(j-1)}=2^qM_12^{-qj}.$$

Therefore $S(2^{-j},u)\leq 2^qM_12^{-qj}$, $\forall\ j$, which implies

$$\sup_{Q_r^-(0,0)} u\leq 2^qM_1r^q,\quad \forall\ r\leq 1.$$

To obtain a similar estimate for u over the whole cylinder (and not only over the lower half part) we use an upper barrier. Define $w(x,t)=C_3|x|^q+C_4t$, where $C_4=1+\gamma_1(qC_3)^{p-1}\left(1+\frac{pC_1}{p-1}+\frac{2p}{p-1}\right)$ and $C_3>0$. Let now $Q_1^+=B_1(0)\times(0,1)$. Then proceeding as Lemma 2.1, we deduce

$$\begin{aligned}\operatorname{div} a(x,t,w,\nabla w)-\partial_t w &\leq \gamma_1(qC_3)^{p-1}\left(1+\frac{pC_1}{p-1}+\frac{2p}{p-1}\right)-C_4\\ &=-1\leq \operatorname{div} a(x,t,u,\nabla u)-\partial_t u\quad \text{in } Q_1^+.\end{aligned}$$

Since by choosing C_3 large, we will have $w\geq u$ on $\partial_pQ_1^+$, where for the estimate on $\{t=0\}$ we have used the previous discussion, i.e., $S(r,u)\leq Cr^q$. Hence by the comparison principle we have $w\geq u$ in Q_1^+. Therefore

$$\sup_{Q_r(0,0)} u\leq M_2r^q.$$

The proof is completed.

3. Proof of the main theorem

Due to the non-degeneracy and optimal growth of the solution near the free boundary, one can prove the main result as in [8]. For completeness of the proof, we provide the details with minor changes of [8].

Proof of Theorem 1.2. Without loss of generality, we assume that the compact set K in the main theorem is the closed unit cylinder $\overline{Q}_1$, and moreover that $\overline{Q}_2\subset\Omega_T$.

For $(x,t) \in U^+ \cap \overline{Q}_1$, let $d(x,t)$ be defined as in Theorem 2.3 and take $(x^0,t^0) \in \partial U^+ \cap \overline{Q}_1$ which realizes this distance. Next define $\widetilde{u}(y,s) = u(x^0+y,t^0+s)$ in Q_1. Let $M = \max\{\|\theta\|_{\infty,\Omega_T}, \Lambda_0\}$, $\widetilde{a}(y,s,\mu,\eta) = \frac{a(x^0+y,t^0+s,M\mu,M\eta)}{M}$ and $\widetilde{A}v(y,s) = \text{div } \widetilde{a}(y,s,v(y,s),\nabla v(y,s))$. We claim that $\frac{\widetilde{u}}{M} \in \mathscr{G}_{\widetilde{a}}$. Indeed, one may verify directly that $\widetilde{a}$ satisfies all structural conditions (not necessarily with the same constants as a). Furthermore, we have

$$
\begin{aligned}
\widetilde{A}\left(\frac{\widetilde{u}}{M}\right) - \partial_s\left(\frac{\widetilde{u}}{M}\right) &= \frac{1}{M}\text{div } a(x^0+y,t^0+s,u(x^0+y,t^0+s)),\nabla u(x^0+y,t^0+s)) \\
&\quad - \frac{1}{M}\partial_s u(x^0+y,t^0+s) \\
&= \frac{1}{M}[(Au) - \partial_s u](x^0+y,t^0+s) \\
&\leq \frac{\Lambda_0}{M} \\
&\leq 1,
\end{aligned}
$$

and

$$0 \leq \frac{\widetilde{u}}{M} \leq \frac{\|\theta\|_{\infty,\Omega_T}}{M} \leq 1 \quad \text{and} \quad \frac{\widetilde{u}(0,0)}{M} = 0.$$

Therefore we infer by Theorem 2.3 that

$$u(x,t) = \widetilde{u}(x-x^0,t-t^0) \leq MM_0(d(x,t))^{\frac{p}{p-1}}. \tag{3.1}$$

Let $(z,\tau) \in \partial U^+ \cap \overline{Q}_1$. Then for $0 < r < 1$, by Lemma 2.1, there exists $x^1 \in \partial B_r(z)$ such that

$$u(x^1) \geq c_0 r^{\frac{p}{p-1}}.$$

It follows from (3.1) that

$$c_0 r^{\frac{p}{p-1}} \leq u(x^1,\tau) \leq MM_0(d(x,\tau))^{\frac{p}{p-1}}.$$

Let $\delta = (\frac{c_0}{MM_0})^{\frac{p-1}{p}}$. Then $d(x,\tau) \geq \delta r$, and $0 < \delta \leq 1$. Therefore

$$B_{\delta r}(x^1) \cap B_r(z) \subset U^+.$$

Now choose $y \in [z,x^1]$ such that $|y-x^1| = \frac{\delta r}{2}$. Then we have

$$B_{\frac{\delta r}{2}}(y) \subset B_{\delta r}(x^1) \cap B_r(z) \subset B_r(z) \setminus \partial U^+.$$

Indeed, for any $y_0 \in B_{\frac{\delta r}{2}}(y)$, we have

$$|y_0 - x^1| \leq |y_0 - y| + |y - x^1| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r.$$

Moreover, since $|y - z| = |z - x^1| - |y - x^1|$, we have

$$|y_0 - z| \leq |y_0 - y| + (|z - x^1| - |y - x^1|) \leq \frac{\delta r}{2} + (r - \frac{\delta r}{2}) = r.$$

This shows that $\partial U^+ \cap \{t = \tau\} \cap \overline{B}_1$ is porous with the porosity constant $\frac{\delta}{2}$.

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