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COMMON FIXED POINTS OF $\mathscr{L}$-FUZZY MAPPINGS ON NON-ARCHIMEDEAN ORDERED FUZZY METRIC SPACES

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Abstract. In this paper, we prove the existence of common fixed points of $\mathscr{L}$-fuzzy mappings on non-Archimedean ordered fuzzy metric spaces by using the integral type and contractive conditions. Examples are also given to illustrate the significance of these results.

Keywords. Fixed point; Ordered fuzzy metric space; Partial order; $\mathscr{L}$-fuzzy map.

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1. Introduction

Zadeh [20] introduced the notion of the fuzzy set which generalizes the classical set. Heilpern [11] proved fixed points for fuzzy mappings in metric linear spaces using the concept of α-levels set of a fuzzy set. Goguen [9] gave the notion of $\mathscr{L}$-fuzzy sets as a further generalization of fuzzy sets. Recently, Rashid *et al.* [13] obtained fixed point theorems for $\mathscr{L}$-fuzzy mappings on metric spaces and Beg *et al.* [5] extended these results to the framework of fuzzy metric spaces.

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We obtained sufficient conditions for the existence of fixed points of $\mathscr{L}$-fuzzy and crisp mappings in non-Archimedean ordered fuzzy metric spaces. We also used implicit relations and integral contractive conditions. Our results generalize the existing results in [1]- [5], [16].

2. Preliminaries

Definition 2.1. [17] An operation $* : [0,1] \times [0,1] \to [0,1]$ is called continuous t-norm if:

(I) $*(a,b) = *(b,a)$,

(II) $*(a, *(b,c)) = *(*(a,b),c)$,

(III) $*$ is continuous,

(IV) $a * 1 = a$ for all $a \in [0,1]$,

(V) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$,

for all $a,b,c,d \in [0,1]$.

Definition 2.2. [10] Let $\sup\limits_{0<a<1} *(a,a) = 1$. A t-norm $*$ is said to be of H-type if the sequence $\{*^m(s)\}_{m=1}^{\infty}$ is equi-continuous at $s = 1$, where $*^1(s) = s, *^{m+1}(s) = *(*^m(s), m = 1,2,3,...,\ s \in [0,1]$; i.e., for all $\varepsilon \in (0,1)$, there exists $\eta \in (0,1)$ such that if $s \in (1-\eta, 1]$, then $*^m(s) > 1-\varepsilon$ for all $m \in \mathbb{N}$.

Definition 2.3. [8] A fuzzy metric space is a triplet $(X,M,*)$ where X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions (for all $x,y,z \in X$ and $t,s > 0$):

(GV_1) $M(x,y,t) > 0$,

(GV_2) $M(x,y,t) = 1\ \forall t > 0$ if and only if $x = y$,

(GV_3) $M(x,y,t) = M(y,x,t)$,

(GV_4) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$,

(GV_5) $M(x,y,-) : (0,\infty) \to [0,1]$ is continuous.

If we replace $(GV4)$ with $M(x,z,\max\{t,s\}) \geq M(x,y,t) * M(y,z,s)$, for all $t,s > 0$, then the triplet $(X,M,*)$ is said to be a non-Archimedean fuzzy metric space. A non-Archimedean fuzzy metric space is also a fuzzy metric space.

Proposition 2.4. [15] *If $(X,M,*)$ is a fuzzy metric space then M is continuous on $X\times X\times(0,\infty)$.*

Definition 2.5. [8] Let $(X,M,*)$ be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in X is said to be convergent to some $x\in X$ if for all $t>0$, $\lim_{n\to\infty} M(x_n,x,t)=1$.

(b) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for all $t>0$, $n,p\in\mathbb{N}$ $\lim_{n\to\infty} M(x_{n+p},x_n,t)=1$.

(c) X is said to be complete if every Cauchy sequence in X converges to some point in X.

Definition 2.6. [15] Let $(X,M,*)$ be a fuzzy metric space and $CP(X)$ be collection of nonempty compact subsets of X. We define a function H_M on $CP(X)\times CP(X)\times(0,\infty)$ by

$$H_M(A,B,t)=M(A,B,t)=\min\{\inf_{a\in A}M(a,B,t),\inf_{b\in B}M(A,b,t)\},$$

for all $A,B\in CP(X)$ and $t>0$, also $(H_M,*)$ is a fuzzy metric on $CP(X)$.

Definition 2.7. A partially ordered set consists of a set X and a binary relation $\preceq$ on X which satisfies the following conditions for all $x,y,z\in X$:

(1) $x\preceq x$ (Reflexivity),

(2) if $x\preceq y$ and $y\preceq x$ then $x=y$ (Antisymmetry),

(3) if $x\preceq y$ and $y\preceq z$ then $x\preceq z$ (Transitivity).

A set with a partial order $\preceq$ is called a partially ordered set. Let $(X,\preceq)$ be a partially ordered set and $x,y\in X$. Elements x and y are said to be comparable elements of X if either $x\preceq y$ or $y\preceq x$.

Definition 2.8. [9] A partially ordered set $(L,\leq_L)$ is called:

(I) A lattice, if $a\vee b\in L$ and $a\wedge b\in L$ for any $a,b\in L$.

(II) A complete lattice, if $\vee A\in L$ and $\wedge A\in L$ for any $A\subseteq L$.

(III) Distributive if $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$ for any $a,b,c\in L$.

Definition 2.9. [9] Let A and B be two nonempty subsets of $(X,\preceq)$, the relation $\preceq_1$ between A and B defined as $A\preceq_1 B$: if for every $a\in A$ there exists $b\in B$ such that $a\preceq b$.

Definition 2.10. [9] An $\mathscr{L}$-fuzzy set A on a nonempty set X is a function $A: X \to L$, where L is complete distributive lattice with $1_{\mathscr{L}}$ and $0_{\mathscr{L}}$. In $\mathscr{L}$-fuzzy sets if $L = [0,1]$, then we obtained fuzzy sets.

Definition 2.11. [13] The $\alpha_{\mathscr{L}}$-level set of $\mathscr{L}$-fuzzy set A is denoted by $A_{\alpha_{\mathscr{L}}}$ and is defined as follows:

$$A_{\alpha_{\mathscr{L}}} = \{x : \alpha_{\mathscr{L}} \preceq_L A(x)\} \quad \text{if} \quad \alpha_{\mathscr{L}} \in L \backslash \{0_{\mathscr{L}}\}, \quad A_{0_{\mathscr{L}}} = \overline{\{x : 0_{\mathscr{L}} \preceq_L A(x)\}}.$$

Here $\overline{B}$ denotes the closure of the set B. The characteristic function $\chi_{\mathscr{L}_A}$ of an $\mathscr{L}$-fuzzy set A is as follows:

$$\chi_{\mathscr{L}_A}(x) = \begin{cases} 0_{\mathscr{L}}, & \text{if} \quad x \notin A, \\ 1_{\mathscr{L}}, & \text{if} \quad x \in A. \end{cases}$$

Definition 2.12. [13] Let X and Y be two arbitrary nonempty sets, $\Im_{\mathscr{L}}(Y)$ the collection of all $\mathscr{L}$-fuzzy sets in Y. A mapping F is called $\mathscr{L}$-fuzzy mapping if F is a mapping from X into $\Im_{\mathscr{L}}(Y)$. An $\mathscr{L}$-fuzzy mapping F is an $\mathscr{L}$-fuzzy subset on $X \times Y$ with membership function $F(x)(y)$. The function $F(x)(y)$ is the grade of membership of y in $F(x)$.

Definition 2.13. [13] Let F, G are $\mathscr{L}$-fuzzy mappings from an arbitrary nonempty set X into $\Im_{\mathscr{L}}(X)$. A point $z \in X$ is called an $\mathscr{L}$-fuzzy fixed point of F if $z \in \{Fz\}_{\alpha_{\mathscr{L}}}$, where $\alpha_{\mathscr{L}} \in L \backslash \{0_{\mathscr{L}}\}$. The point $z \in X$ is called a common $\mathscr{L}$-fuzzy fixed point of F and G if $z \in \{Fz\}_{\alpha_{\mathscr{L}}} \cap \{Gz\}_{\alpha_{\mathscr{L}}}$.

Definition 2.14. [12] Let $(X, M, *)$ be a fuzzy metric space, $Y \subseteq X$ and $CL(X)$ is the collection of nonempty closed subsets of X. A map $f: Y \to X$ is called coincidentally idempotent with respect to a mapping $F: Y \to CL(X)$ if f is idempotent at the coincidence points of (f, F), i.e., $ffx = fx$ for all $x \in Y$ with $fx \in Fx$ provided that $fx \in Y$.

Definition 2.15. [18] Let f, g be two mappings from a metric space X into itself and F, G be fuzzy mappings from X into $W(X)$ (The set of all fuzzy sets of X which its α-level sets are nonempty compact subsets of X). If for some $x_0 \in X$, there exist a sequence $\{y_n\}$ in X such that

$$\{y_{2n+1}\} = \{gx_{2n+1}\} \subset Fx_{2n}, \quad \{y_{2n+2}\} = \{fx_{2n+2}\} \subset Gx_{2n+1},$$

then $O(F,G,f,g,x_0)$ is called the orbit for the mappings (F,G,f,g). Metric space X is called x_0 joint orbitally complete, if every Cauchy sequence of each orbit at x_0 is convergent in X.

Definition 2.16. [1] Let $(X,M,*)$ be a fuzzy metric space, $Y \subseteq X$. A map $f : Y \to X$ is called F-weakly commuting at $x \in Y$ if $ffx \in Ffx$ provided that $fx \in Y$ for all $x \in Y$.

Let Φ be the family of all continuous mappings $\phi : [0,1]^6 \to [0,1]$, which are non-decreasing in the 1^{st} and non-increasing in the 3^{rd}, 4^{th}, 5^{th}, 6^{th} coordinate variable and satisfying the following properties:

(ϕ_1) $\phi(u,v,v,u,u*v,1) \geq 0$ or $\phi(u,v,u,v,1,u*v) \geq 0$,
(ϕ_2) $\int_0^{\phi(u,v,v,u,u*v,1)} \varphi(s)ds \geq 0$ or $\int_0^{\phi(u,v,u,v,1,u*v)} \varphi(s)ds \geq 0$,
(ϕ_3) $\int_0^{\phi(\int_0^u \psi(s)ds, \int_0^v \psi(s)ds, \int_0^v \psi(s)ds, \int_0^u \psi(s)ds, \int_0^{u*v} \psi(s)ds, 1)} \varphi(s)ds \geq 0$ or
$\int_0^{\phi(\int_0^u \psi(s)ds, \int_0^v \psi(s)ds, \int_0^u \psi(s)ds, \int_0^v \psi(s)ds, 1, \int_0^{u*v} \psi(s)ds)} \varphi(s)ds \geq 0$,

$\forall u,v \in (0,1]$ implies $u = 1$, where $\varphi, \psi : [0,\infty) \to [0,\infty)$ is a summable non negative Lebesgue integrable functions such that for each $\varepsilon \in (0,1]$, $\int_0^{\varepsilon} \varphi(s)ds > 0$ and $\int_0^{\varepsilon} \psi(s)ds > 0$.

Note that if $\psi(s) = 1$, then $(\phi_3) \Rightarrow (\phi_2)$, if $\varphi(s) = 1$, then $(\phi_2) \Rightarrow (\phi_1)$ and if $\varphi(s) = \psi(s) = 1$, then $(\phi_3) \Rightarrow (\phi_1)$.

3. Main results

We rewrite the basic notion of joint orbitally complete for crisp and $\mathscr{L}$-fuzzy mappings in non-Archimedean fuzzy metric spaces.

Definition 3.1. Let $(X,M,*)$ be a non-Archimedean fuzzy metric space, $f,g : X \to X$ and F,G be $\mathscr{L}$-fuzzy mappings from X into $\Im_{\mathscr{L}}(X)$. Let $x_0 \in X$. If there exist a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = gx_{2n+1} \in \{Fx_{2n}\}_{\alpha_{\mathscr{L}}}, \quad y_{2n+2} = fx_{2n+2} \in \{Gx_{2n+1}\}_{\alpha_{\mathscr{L}}},$$

then $O(F,G,f,g,x_0)$ is called the orbit for the mappings (F,G,f,g). Non-Archimedean fuzzy metric space X is called x_0 joint orbitally complete, if every Cauchy sequence of each orbit at x_0 is convergent in X.

Theorem 3.2. *Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with H-type t-norm $*$ and $\lim_{t\to\infty} M(y_0,y_1,t)=1$. Let $\preceq$ be a partial order defined on X, $f,g: X\to X$ such that $f(X)$ and $g(X)$ are closed. Suppose that F,G are two $\mathscr{L}$-fuzzy mappings from X into $\Im_{\mathscr{L}}(X)$ such that for each $x\in X$ and $\alpha_{\mathscr{L}}\in L\backslash\{0_{\mathscr{L}}\}$, $\{Fx\}_{\alpha_{\mathscr{L}}}$ and $\{Gx\}_{\alpha_{\mathscr{L}}}$ are nonempty closed subsets of X satisfying the following conditions:*

1. $\{Fx\}_{\alpha_{\mathscr{L}}}\preceq_1 g(X)$ *and* $\{Gx\}_{\alpha_{\mathscr{L}}}\preceq_1 f(X)$,
2. $gy\in\{Fx\}_{\alpha_{\mathscr{L}}}$ *or* $fy\in\{Gx\}_{\alpha_{\mathscr{L}}}$ *implies* $x\preceq y$,
3. *if* $y_n\to y$, *then* $y_n\preceq y$ *for all* n,
4. (f,F) *and* (g,G) *are weakly commuting and coincidentally idempotent,*
5. *one of* $f(X)$ *or* $g(X)$ *is* x_0 *joint orbitally complete for some* $x_0\in X$.

If for all comparable elements $x,y\in X$ there exist $\phi\in\Phi$ such that

$$\phi\begin{pmatrix} M(\{Fx\}_{\alpha_{\mathscr{L}}},\{Gy\}_{\alpha_{\mathscr{L}}},t), M(fx,gy,t), M(fx,\{Fx\}_{\alpha_{\mathscr{L}}},t),\\ M(gy,\{Gy\}_{\alpha_{\mathscr{L}}},t), M(fx,\{Gy\}_{\alpha_{\mathscr{L}}},t), M(gy,\{Fx\}_{\alpha_{\mathscr{L}}},t)\end{pmatrix}\geq 0,$$

then f,g,F and G have a common fixed point.

Proof. Let $x_0\in X$ and $y_0=fx_0$. By (1), there exist $x_1,x_2\in X$ such that $y_1=gx_1\in\{Fx_0\}_{\alpha_{\mathscr{L}}}$ and $y_2=fx_2\in\{Gx_1\}_{\alpha_{\mathscr{L}}}$. From (2), $x_0\preceq x_1\preceq x_2$. Now, $M(\{Fx_0\}_{\alpha_{\mathscr{L}}},\{Gx_1\}_{\alpha_{\mathscr{L}}},t)\leq M(gx_1,fx_2,t)=M(y_1,y_2,t)$. Since

$$\begin{aligned}
&\phi\begin{pmatrix} M(y_1,y_2,t), M(y_0,y_1,t), M(y_0,y_1,t),\\ M(y_1,y_2,t), M(y_0,y_1,t)*M(y_1,y_2,t), 1\end{pmatrix}\\
=\;&\phi\begin{pmatrix} M(y_1,y_2,t), M(y_0,y_1,t), M(y_0,y_1,t),\\ M(y_1,y_2,t), M(y_0,y_1,t)*M(y_1,y_2,t), M(y_1,y_1,t)\end{pmatrix}\\
\geq\;&\phi\begin{pmatrix} M(y_1,y_2,t), M(y_0,y_1,t), M(y_0,y_1,t),\\ M(y_1,y_2,t), M(y_0,y_2,t), M(y_1,y_1,t)\end{pmatrix}\\
\geq\;&\phi\begin{pmatrix} M(\{Fx_0\}_{\alpha_{\mathscr{L}}},\{Gx_1\}_{\alpha_{\mathscr{L}}},t), M(fx_0,gx_1,t), M(fx_0,\{Fx_0\}_{\alpha_{\mathscr{L}}},t),\\ M(gx_1,\{Gx_1\}_{\alpha_{\mathscr{L}}},t), M(fx_0,\{Gx_1\}_{\alpha_{\mathscr{L}}},t), M(gx_1,\{Fx_0\}_{\alpha_{\mathscr{L}}},t)\end{pmatrix}\\
\geq\;&0,
\end{aligned}$$

from (ϕ_1), we have $M(y_1,y_2,t) \geq M(y_0,y_1,t)$. Similarly, we can find $x_3 \in X$ and $x_2 \preceq x_3$ such that $y_3 = gx_3 \in \{Fx_2\}_{\alpha_{\mathscr{L}}}$. Also, $M(\{Fx_2\}_{\alpha_{\mathscr{L}}},\{Gx_1\}_{\alpha_{\mathscr{L}}},t) \leq M(fx_2,gx_3,t)$ and $M(y_2,y_3,t) \geq M(y_1,y_2,t)$. continuing in this way, we have a sequence $\{y_n\}$ such that $\{y_{2n+1}\} = \{gx_{2n+1}\} \preceq_1 \{Fx_{2n}\}_{\alpha_{\mathscr{L}}}$ and $\{y_{2n+2}\} = \{fx_{2n+2}\} \preceq_1 \{Gx_{2n+1}\}_{\alpha_{\mathscr{L}}}$. By induction, we obtain $M(y_{n+1},y_{n+2},t) \geq M(y_n,y_{n+1},t)$. Thus $\{M(y_n,y_{n+1},t)\}$ is a non decreasing sequence in $(0,1]$. Since $M(y_n,y_m,t) \geq M(y_n,y_{n+1},t) * M(y_{n+1},y_{n+2},t) * \ldots * M(y_{m-1},y_m,t)$, we have

$$\begin{aligned} M(y_n,y_m,t) &\geq M(y_n,y_{n+1},t) * M(y_{n+1},y_{n+2},t) * \ldots * M(y_{m-1},y_m,t) \\ &\geq M(y_n,y_{n+1},t) * M(y_n,y_{n+1},t) * \ldots * M(y_n,y_{n+1},t) \\ &= *^{m-n} M(y_n,y_{n+1},t). \end{aligned}$$

Since $*$ is a t-norm of H-type, for any $\varepsilon \in (0,1)$, we see that there exists $\eta \in (0,1)$ such that if $s \in (\eta,1]$. Then $*^{m-n}s > 1-\varepsilon$ for all $n,m \in N$. Since $M(y_0,y_1,t) = 1$ as $t \to \infty$, we find that there exist $n_0 \in N$ such that $M(y_0,y_1,t) > \eta$. Now, in view of $*^{m-n}M(y_n,y_{n+1},t) \geq *^{m-n}M(y_0,y_1,t) > 1-\varepsilon$, we have $\lim_{n\to\infty} M(y_n,y_m,t) = 1$. So $\{y_n\}$ is a Cauchy sequence. Similarly $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ are also Cauchy sequences. Now, if one of $f(X)$ or $g(X)$ is x_0 joint orbitally complete, then $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ converge to $z \in X$. From (3), $y_{2n+1} \preceq z$ and $y_{2n+2} \preceq z$. As $y_{2n+2} \to z$, $y_{2n+2} = fx_{2n+2}$ and $f(X)$ is closed, we find that there exists $v \in X$ such that $z = fv$. Next we show that $fv \in \{Fv\}_{\alpha_{\mathscr{L}}}$. Since

$$\begin{aligned} &\phi\left(\begin{array}{c} M(\{Fv\}_{\alpha_{\mathscr{L}}},y_{2n+2},t), M(z,y_{2n+1},t), M(z,\{Fv\}_{\alpha_{\mathscr{L}}},t), \\ M(y_{2n+1},y_{2n+2},t), M(z,y_{2n+1},t) * M(y_{2n+1},y_{2n+2},t), M(y_{2n+1},\{Fv\}_{\alpha_{\mathscr{L}}},t) \end{array}\right) \\ \geq\ &\phi\left(\begin{array}{c} M(\{Fv\}_{\alpha_{\mathscr{L}}},y_{2n+2},t), M(z,y_{2n+1},t), M(z,\{Fv\}_{\alpha_{\mathscr{L}}},t), \\ M(y_{2n+1},y_{2n+2},t), M(z,y_{2n+2},t), M(y_{2n+1},\{Fv\}_{\alpha_{\mathscr{L}}},t) \end{array}\right) \\ \geq\ &\phi\left(\begin{array}{c} M(\{Fv\}_{\alpha_{\mathscr{L}}},\{Gx_{2n+1}\}_{\alpha_{\mathscr{L}}},t), M(fv,gx_{2n+1},t), M(fv,\{Fv\}_{\alpha_{\mathscr{L}}},t), \\ M(gx_{2n+1},\{Gx_{2n+1}\}_{\alpha_{\mathscr{L}}},t), M(fv,\{Gx_{2n+1}\}_{\alpha_{\mathscr{L}}},t), M(gx_{2n+1},\{Fv\}_{\alpha_{\mathscr{L}}},t) \end{array}\right) \\ \geq\ &0, \end{aligned}$$

when $n \to \infty$, we have that

$$\phi(M(\{Fv\}_{\alpha_{\mathscr{L}}}, fv, t), 1, M(fv, \{Fv\}_{\alpha_{\mathscr{L}}}, t), 1, 1, M(fv, \{Fv\}_{\alpha_{\mathscr{L}}}, t)) \geq 0.$$

By (ϕ_1), this gives $M(z, \{Fv\}_{\alpha_{\mathscr{L}}}, t) \geq 1$. Thus $fv \in \{Fv\}_{\alpha_{\mathscr{L}}}$. Similarly, one find $z = gw \in \{Gw\}_{\alpha_{\mathscr{L}}}$ for $w \in X$. Also, $ffv = fv$ and $ffv \in \{Ffv\}_{\alpha_{\mathscr{L}}}$ so that $z = fz \in \{Fz\}_{\alpha_{\mathscr{L}}}$. Also, $ggw = gw$ and $ggw \in \{Ggw\}_{\alpha_{\mathscr{L}}}$ implies $z = gz \in \{Gz\}_{\alpha_{\mathscr{L}}}$. Thus f, g, F and G have a common fixed point.

Example 3.3. *Let $X = [0,1]$, $a * b = ab$ for all $a, b \in (0,1]$ and $M(x,y,t) = e^{-\frac{d(x,y)}{t}}$ for each $x, y \in X$ and $t > 0$. Define the partial order $x \preceq y$ as $x \leq y$ for each $x, y \in X$ and $X \preceq_1 Y$ as: for each $x \in X$ there exist $y \in Y$ such that $x \leq y$. Define the maps f, g, F, G on X as $gx = \frac{x}{2}$, $fx = \frac{x}{3}$,*

$$(Fx)(y) = \begin{cases} 0, & \textit{if} \quad 0 \leq y \leq \frac{1}{5}, \\ \frac{1}{3}, & \textit{if} \quad \frac{1}{5} < y < \frac{x}{4}, \\ \frac{2}{3}, & \textit{if} \quad \frac{x}{4} \leq y \leq 1. \end{cases}$$

and

$$(Gx)(y) = \begin{cases} 0, & \textit{if} \quad 0 \leq y \leq \frac{1}{5}, \\ \frac{1}{6}, & \textit{if} \quad \frac{1}{5} < y < \frac{x}{6}, \\ \frac{1}{4}, & \textit{if} \quad \frac{x}{6} \leq y \leq 1. \end{cases}$$

Define the sequences x_{2n}, x_{2n+1} and x_{2n+2} in X such that $x_{2n} = \{\frac{1}{2n+1}\}$, $x_{2n+1} = \{\frac{1}{2n+2}\}$ and $x_{2n+2} = \{\frac{1}{2n+3}\}$, $n \in \mathbb{N}$, then $y_{2n+1} = gx_{2n+1} = \frac{1}{2(2n+2)}$ and $y_{2n+2} = fx_{2n+2} = \frac{1}{3(2n+3)}$. Now, $\{Fx_{2n}\}_{\frac{2}{3}} = [\frac{1}{4(2n+1)}, 1]$, $\{Gx_{2n+1}\}_{\frac{1}{4}} = [\frac{1}{6(2n+2)}, 1]$, $gx_{2n+1} \in \{Fx_{2n}\}_{\frac{2}{3}}$ and $fx_{2n+2} \in \{Gx_{2n+1}\}_{\frac{1}{4}}$. Thus, $\lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = 0$ and $\lim_{n\to\infty} \{Fx_{2n}\}_{\frac{2}{3}} = \lim_{n\to\infty} \{Gx_{2n+1}\}_{\frac{1}{4}} = [0,1]$. Let $\phi(t_1, ..., t_6) = t_6$, then $\phi(t_1, ..., t_6) = M(gx_{2n+1}, \{Fx_{2n}\}_{\alpha_{\mathscr{L}}}, t) = 1$. Finally, (f, F) and (g, G) are weakly commuting and coincidentally idempotent. Now, f, g, F and G satisfy all conditions of Theorem 3.2 and $f0 = g0 = 0 \in [0,1] = \{F0\}_{\frac{2}{3}} = \{G0\}_{\frac{1}{4}}$ is a common fixed point.

Corollary 3.4. *Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with H-type t-norm $*$ and $\lim_{t\to\infty} M(y_0,y_1,t)=1$. Let $\preceq$ be a partial order defined on X, $f:X\to X$ such that $f(X)$ is closed. Let F be an $\mathscr{L}$-fuzzy mappings from X into $\Im_{\mathscr{L}}$ such that for each $x\in X$ and $\alpha_{\mathscr{L}}\in L\backslash\{0_{\mathscr{L}}\}$, $\{Fx\}_{\alpha_{\mathscr{L}}}$ is nonempty closed subset of X satisfying the following conditions:*

1. $\{Fx\}_{\alpha_{\mathscr{L}}} \preceq_1 f(X)$,
2. $fy\in\{Fx\}_{\alpha_{\mathscr{L}}}$ *implies* $x\preceq y$,
3. *if* $y_n\to y$, *then* $y_n\preceq y$ *for all* n,
4. (f,F) *are weakly commuting and coincidentally idempotent.*
5. $f(X)$ *is* x_0 *joint orbitally complete for some* $x_0\in X$.

If for all comparable elements $x,y\in X$ there exist $\phi\in\Phi$ such that

$$\phi\left(\begin{array}{c} M(\{Fx\}_{\alpha_{\mathscr{L}}},\{Fy\}_{\alpha_{\mathscr{L}}},t), M(fx,fy,t), M(fx,\{Fx\}_{\alpha_{\mathscr{L}}},t),\\ M(fy,\{Fy\}_{\alpha_{\mathscr{L}}},t), M(fx,\{Fy\}_{\alpha_{\mathscr{L}}},t), M(fy,\{Fx\}_{\alpha_{\mathscr{L}}},t)\end{array}\right)\geq 0,$$

then f and F have a common fixed point.

Theorem 3.5. *Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with H-type t-norm $*$ and $\lim_{t\to\infty} M(y_0,y_1,t)=1$. Let $\preceq$ be a partial order defined on X, $f,g:X\to X$ and $\{F_n\}$ be a sequence of $\mathscr{L}$-fuzzy mappings from X into $\Im_{\mathscr{L}}(X)$ such that for each $x\in X$ and $\alpha_{\mathscr{L}}\in L\backslash\{0_{\mathscr{L}}\}$, $\{F_nx\}_{\alpha_{\mathscr{L}}}$ are nonempty closed subsets of X satisfying the following conditions:*

1. $\{F_kx\}_{\alpha_{\mathscr{L}}} \preceq_1 g(X)$ *and* $\{F_lx\}_{\alpha_{\mathscr{L}}} \preceq_1 f(X)$,
2. $gy\in\{F_kx\}_{\alpha_{\mathscr{L}}}$ *or* $fy\in\{F_lx\}_{\alpha_{\mathscr{L}}}$ *implies* $x\preceq y$,
3. *if* $y_n\to y$, *then* $y_n\preceq y$ *for all* n,
4. (f,F_k) *and* (g,F_l) *are weakly commuting and coincidentally idempotent,*
5. *one of* $f(X)$ *or* $g(X)$ *is* x_0 *joint orbitally complete for some* $x_0\in X$.

If for all comparable elements $x,y\in X$ there exist $\phi\in\Phi$ such that for all $x,y\in X$, $k=2n+1$ and $l=2n+2$, $n\in\mathbb{N}$

$$\phi\left(\begin{array}{c} M(\{F_kx\}_{\alpha_{\mathscr{L}}},\{F_ly\}_{\alpha_{\mathscr{L}}},t), M(fx,gy,t), M(fx,\{F_kx\}_{\alpha_{\mathscr{L}}},t),\\ M(gy,\{F_ly\}_{\alpha_{\mathscr{L}}},t), M(fx,\{F_ly\}_{\alpha_{\mathscr{L}}},t), M(gy,\{F_kx\}_{\alpha_{\mathscr{L}}},t)\end{array}\right)\geq 0,$$

then (f,F_k) and (g,F_l) have a common fixed point.

Corollary 3.6. *Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with H- type t-norm $*$ and $\lim_{t\to\infty} M(y_0,y_1,t)=1$, let $\preceq$ be a partial order defined on X, $f:X\to X$ and $\{F_n\}$ be a sequence of $\mathscr{L}$-fuzzy mappings from X into $\Im_{\mathscr{L}}(X)$ such that for each $x\in X$, $\alpha_{\mathscr{L}}\in L\backslash\{0_{\mathscr{L}}\}$, $\{F_nx\}_{\alpha_{\mathscr{L}}}$ are nonempty closed subsets of X satisfying the following conditions:*

(1) $\{F_nx\}_{\alpha_{\mathscr{L}}}\preceq_1 f(X)$,

(2) $fy\in\{F_nx\}_{\alpha_{\mathscr{L}}}$ *implies* $x\preceq y$,

(3) *if* $y_n\to y$, *then* $y_n\preceq y$ *for all n*,

(4) (f,F_n) *are weakly commuting and coincidentally idempotent*,

(5) $f(X)$ *is* x_0 *joint orbitally complete for some* $x_0\in X$.

If for all comparable elements $x,y\in X$ there exist $\phi\in\Phi$ such that for all $x,y\in X$, $k=2n+1$ and $l=2n+2$, $n\in\mathbb{N}$

$$\phi\begin{pmatrix} M(\{F_kx\}_{\alpha_{\mathscr{L}}},\{F_ly\}_{\alpha_{\mathscr{L}}},t),M(fx,fy,t),M(fx,\{F_kx\}_{\alpha_{\mathscr{L}}},t),\\ M(fy,\{F_ly\}_{\alpha_{\mathscr{L}}},t),M(fx,\{F_ly\}_{\alpha_{\mathscr{L}}},t),M(fy,\{F_kx\}_{\alpha_{\mathscr{L}}},t)\end{pmatrix}\geq 0,$$

then (f,F_n) have a common fixed point.

4. Integral type

Integral contractive type mappings are a generalization of contractive mappings. Recently several results on fixed points of integral contractive types have appeared in the literature [6], [7], [14], [19]. In this section, we prove an integral type contractive condition with implicit relations for two pairs of $\mathscr{L}$-fuzzy and crisp mappings in non-Archimedean ordered fuzzy metric spaces.

Theorem 4.1. *Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with H-type t-norm $*$ and $\lim_{t\to\infty} M(y_0,y_1,t)=1$. Let $\preceq$ be a partial order defined on X, $f,g:X\to X$ and $\{F_n\}$ be a sequence of $\mathscr{L}$-fuzzy mappings from X into $\Im_{\mathscr{L}}(X)$ such that for each $x\in X$ and $\alpha_{\mathscr{L}}\in L\backslash\{0_{\mathscr{L}}\}$, $\{F_nx\}_{\alpha_{\mathscr{L}}}$ are nonempty closed subsets of X satisfying the following conditions:*

(1) $\{F_kx\}_{\alpha_{\mathscr{L}}}\preceq_1 g(X)$ *and* $\{F_lx\}_{\alpha_{\mathscr{L}}}\preceq_1 f(X)$,

(2) $gy\in\{F_kx\}_{\alpha_{\mathscr{L}}}$ *or* $fy\in\{F_lx\}_{\alpha_{\mathscr{L}}}$ *implies* $x\preceq y$,

(3) *if* $y_n\to y$, *then* $y_n\preceq y$ *for all n*,

(4) (f,F_k) *and* (g,F_l) *are weakly commuting and coincidentally idempotent,*

(5) *one of* $f(X)$ *or* $g(X)$ *is* x_0 *joint orbitally complete for some* $x_0 \in X$.

If for all comparable elements $x,y \in X$ *there exist* $\phi \in \Phi$ *such that for all* $x,y \in X$, $k = 2n+1$ *and* $l = 2n+2$, $n \in \mathbb{N}$

$$\int_0^{\phi\left(\begin{array}{c} \int_0^{M(\{F_kx\}_{\alpha_{\mathscr{L}}},\{F_ly\}_{\alpha_{\mathscr{L}}},t)}\psi(s)ds, \int_0^{M(fx,gy,t)}\psi(s)ds, \int_0^{M(fx,\{F_kx\}_{\alpha_{\mathscr{L}}},t)}\psi(s)ds, \\ \int_0^{M(gy,\{F_ly\}_{\alpha_{\mathscr{L}}},t)}\psi(s)ds, \int_0^{M(fx,\{F_ly\}_{\alpha_{\mathscr{L}}},t)}\psi(s)ds, \int_0^{M(gy,\{F_kx\}_{\alpha_{\mathscr{L}}},t)}\psi(s)ds \end{array}\right)} \varphi(s)ds \geq 0,$$

then (f,F_k) *and* (g,F_l) *have a common fixed point.*

Proof. Following the proof in Theorem 3.2 with (ϕ_3), we find the desired conclusion immediately.

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