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COMPUTING SINGULAR POINT QUANTITIES FOR QUASI ANALYTIC SYSTEMS

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Abstract. We describe a new algorithm to compute singular point quantities for quasi analytic systems. It is transparent conceptually and efficient computationally with computer algebra software *Mathematica 8.0*. We demonstrate the application of this approach by deriving center conditions for some quasi quadratic systems.

Keywords. Algorithm; Common zero; Quasi analytic system; Singular point quantity.

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1. Introduction and main results

The center problem to distinguish between a focus and a center is a difficult one and is another of the challenges of the qualitative theory of differential equations with a long history and an extensive literature. The definition of *center* goes back to Poincaré in [1]; i.e. a singular point of a vector field on the real plane surrounded by a neighborhood fulfilled of periodic orbits with the unique exception of the singular point.

It is well known that the center problem can be reduced to the study of Poincaré return map, or equivalently to the computation of infinitely many real numbers $v_{2m+1}, m \geq 1$, called *Lyapunov*

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constants. In fact, we have that if for some k, $v_3 = v_5 = \cdots = v_{2k-1} = 0$ and $v_{2k+1} \neq 0$ the origin is a focus, while if all v_{2m+1} are zero the origin is a center.

The progress in solving the center problem is pretty slow since to do it requires a good knowledge, not only of the common zeros of the Lyapunov constants, but also of the finite generated ideal that they generate in the ring of polynomials taking as variables the coefficients of the polynomial differential system. Furthermore in general the calculation of the Lyapunov constants is not easy, and the computational complexity of finding their common zeros grows very quickly. A number of algorithms have been developed to compute them automatically up to a certain order (see [2, 3, 4] and the references therein). We also want to mention that even if we are able to obtain the Lyapunov constants it is in general extremely difficult to decompose the resulting variety into irreducible components. If this can be done we have necessary conditions to have a center at the origin. Usually, the sufficiency conditions will follow either from proving the existence of a first integral defined in a neighborhood of the origin, or from the existence of a symmetry through the origin.

The use of computer algebra has led to significant progress in the investigation of the properties of planar dynamical systems. The calculations involved in the derivation of center conditions are extremely heavy and are quite impossible to accomplish by hand except in the simplest cases. The expressions arising can be very large, and the difficulties are exacerbated by intermediate expression swell. Inevitably systems are considered in which the limits of available computing capacity are reached. At that stage every effort to reduce the burden of the computations in terms of computing time, but more often of space, is rewarded. Considerable effort has consequently been devoted to improving the algorithms used, thereby extending the range of their applicability.

Motivated by the center problem of analytic systems, increasing interests have been attracted to the same problem of planar quasi analytic systems. Consider the quasi analytic systems defined in [5]

$$
\begin{aligned}
\frac{dx}{dt} &= -y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}} X_k(x,y), \\
\frac{dy}{dt} &= x + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}} Y_k(x,y),
\end{aligned}
\tag{1}
$$

where

$$X_k(x,y) = \sum_{\alpha+\beta=k} A_{\alpha\beta} x^{\alpha} y^{\beta},\ Y_k(x,y) = \sum_{\alpha+\beta=k} B_{\alpha\beta} x^{\alpha} y^{\beta} \tag{2}$$

are k-degree homogeneous polynomials with respect to x and $y, k = 2,3,\cdots$. λ is a real constant and $\lambda \neq 0$. Clearly, when $\lambda = 1$, system (1) becomes analytic. Generally, for $\lambda \neq 2s+1$ where s is an integer, the functions on the right hand of system (1) are non-analytic.

For $\lambda > 0$ (or <0), the linear terms of system (1) are the lowest (or highest) degree terms on the right hand. Hence, when $\lambda > 0$, the origin of system (1) is a center or a focus. When $\lambda < 0$, system (1) has no real singular point in the equator of Poincaré compactification. The point at infinity is a center or a focus. Therefore, it is necessary to determine whether the origin (or the point at infinity) is a center or a weak focus for all $\lambda \neq 0$.

There are limited literatures with regard to quasi analytic systems, For quasi quadratic and quasi cubic homogeneous systems, the problems of center-focus determination and bifurcation of limit cycles for elementary critical point, higher critical point and infinite point were uniformly solved in [6] and [7], respectively. Further in [5] and [8], the isochronous center problems for quasi quadratic and quasi cubic homogeneous systems were completely solved, respectively. Meanwhile, two recursive formulas of computing focal values and period constants were given for quasi analytic systems with arbitrary degree in [5].

By means of transformation

$$z = x+iy,\ w = x-iy,\ T = it,\ i = \sqrt{-1}, \tag{3}$$

system (1) can be transferred into the following system

$$\begin{aligned} \frac{dz}{dT} &= z + \sum_{k=2}^{\infty} (zw)^{\frac{(k-1)(\lambda-1)}{2}} Z_k(z,w), \\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} (zw)^{\frac{(k-1)(\lambda-1)}{2}} W_k(z,w), \end{aligned} \tag{4}$$

where for each integer $k > 1$, we have

$$\begin{aligned} Z_k(x+iy, x-iy) &= Y_k(x,y) - iX_k(x,y), \\ W_k(x+iy, x-iy) &= Y_k(x,y) + iX_k(x,y). \end{aligned} \tag{5}$$

We denote that

$$Z_k(z,w)=\sum_{\alpha+\beta=k} a_{\alpha\beta}z^{\alpha}w^{\beta},\ W_k(z,w)=\sum_{\alpha+\beta=k} b_{\alpha\beta}w^{\alpha}z^{\beta}. \tag{6}$$

As in [9], we call that systems (1) and (4) are associated. Clearly, one can check that the coefficients of system (4) satisfy the conjugate condition, i.e.,

$$\overline{a_{\alpha\beta}}=b_{\alpha\beta},\ \alpha\geq 0,\ \beta\geq 0,\ \alpha+\beta\geq 2. \tag{7}$$

From [5], we know that making the transformation

$$\xi=x(x^2+y^2)^{\frac{\lambda-3}{6}},\ \eta=y(x^2+y^2)^{\frac{\lambda-3}{6}}, \tag{8}$$

for $\lambda>0\,(<0)$, (8) makes the origin (the singular point at infinity) to a new origin in $(\xi,\eta)-$plane and system (1) becomes

$$\begin{aligned}\frac{d\xi}{dt}&=-\eta+\frac{1}{3}\sum_{k=2}^{\infty}(\xi^2+\eta^2)^{k-2}[(\lambda\xi^2+3\eta^2)X_k(\xi,\eta)+(\lambda-3)\xi\eta Y_k(\xi,\eta)],\\ \frac{d\eta}{dt}&=\xi+\frac{1}{3}\sum_{k=2}^{\infty}(\xi^2+\eta^2)^{k-2}[(\lambda\eta^2+3\xi^2)Y_k(\xi,\eta)+(\lambda-3)\xi\eta X_k(\xi,\eta)].\end{aligned} \tag{9}$$

Accordingly, the study for system (1) is reduced to the discussion for system (9). Clearly, for $\lambda>0\,(<0)$, the origin (the singular point at infinity) of system (1) is a center if and only if the origin of system (9) is a center.

Using the transformation

$$z=\xi+i\eta,\ w=\xi-i\eta,\ T=it,\ i=\sqrt{-1}, \tag{10}$$

system (9) becomes its concomitant complex system

$$\begin{aligned}\frac{dz}{dT}&=z+\frac{z}{6}\sum_{k=2}^{\infty}(zw)^{k-2}[(\lambda+3)wZ_k(z,w)-(\lambda-3)zW_k(z,w)]=Z(z,w),\\ \frac{dw}{dT}&=-w-\frac{w}{6}\sum_{k=2}^{\infty}(zw)^{k-2}[(\lambda+3)zW_k(z,w)-(\lambda-3)wZ_k(z,w)]=-W(z,w).\end{aligned} \tag{11}$$

In addition, after the transformation

$$z=z_1^{\frac{\lambda+3}{2\lambda}}w_1^{\frac{3-\lambda}{2\lambda}},\ w=w_1^{\frac{\lambda+3}{2\lambda}}z_1^{\frac{3-\lambda}{2\lambda}}, \tag{12}$$

system (4) can be converted to system (11) (Renaming (z_1,w_1) by (z,w)).

The result that we get are the following.

Theorem 1.1. *For system* (11)*, we can derive successively the terms of the following formal series:*

$$F(z,w) = \sum_{\alpha+\beta=2}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta}, \tag{13}$$

such that

$$\frac{dF}{dT} = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}, \tag{14}$$

with $c_{kk} \in \mathbb{R}, k = 1,2,\cdots$ *arbitrarily valued. Besides, if we choose* c_{kk} *ahead (e.g.,* $c_{kk} = 0$*) for* $k \geq 2$*, then the formal series* (13) *is uniquely determined. When* $\alpha+\beta > 2$ *and* $\alpha \neq \beta$*, for every positive integer m,* μ_m *is determined by the following recursive formulae:*

$$\begin{aligned} c_{\alpha\beta} = \frac{1}{6(\beta-\alpha)} \sum_{k+j=3}^{\left[\frac{\alpha+\beta+4}{3}\right]} &\{[(\lambda+3)(\alpha-2k-j+3)+(\lambda-3)(\beta-2j-k+3)]a_{k,j-1} \\ &-[(\lambda+3)(\beta-2j-k+3)+(\lambda-3)(\alpha-2k-j+3)]b_{j,k-1}\} \\ &\times c_{\alpha-2k-j+3,\beta-2j-k+3}, \\ \mu_m = \frac{1}{6} \sum_{k+j=3}^{\left[\frac{2m+8}{3}\right]} &\{[(\lambda+3)(m-2k-j+5)+(\lambda-3)(m-2j-k+5)]a_{k,j-1} \\ &-[(\lambda+3)(m-2j-k+5)+(\lambda-3)(m-2k-j+5)]b_{j,k-1}\} \\ &\times c_{m-2k-j+5,m-2j-k+5}. \end{aligned} \tag{15}$$

In the last expression, for $2 \leq \alpha+\beta \leq 4$*, we have defined*

$$\begin{cases} c_{22} = 1, \\ c_{\alpha\beta} = 0, \ \textit{for other } (\alpha,\beta), \end{cases} \tag{16}$$

and if $\alpha = \beta > 0$ *or* $\alpha < 0$ *or* $\beta < 0$*, set* $a_{\alpha\beta} = b_{\alpha\beta} = c_{\alpha\beta} = 0$.

In our computations we use complex notation for real planar polynomial differential systems for finding such new families of centers, so we will be interested in the expression of the Poincaré-Lyapunov constants in complex notation. The reason for using the complex notation is that it simplifies the computations and the expressions of these constants.

Remark 1.2. The singular point quantities computed by Theorem 1.1 differ from those computed by Lemmas 3.1 and 3.2 of [5] only for a constant factor.

The paper is structured as follows. In Section 2, we state some preliminary results. We obtain the proof of Theorem 1.1 in the following section. Finally we present an example which shows the effectiveness of our method.

2. Definitions and preliminary results

In preparation for the proof of the main result, we need to introduce some well known definitions, lemmas and theorems.

Performing the transformation

$$x = r^{\frac{1}{\lambda}} \cos\theta,\ y = r^{\frac{1}{\lambda}} \sin\theta, \tag{17}$$

system (1) becomes

$$\frac{dr}{d\theta} = \lambda r \frac{\delta + \sum\limits_{k=1}^{\infty} \varphi_{k+2}(\theta) r^k}{1 + \sum\limits_{k=1}^{\infty} \psi_{k+2}(\theta) r^k}, \tag{18}$$

where

$$\begin{aligned} \varphi_k(\theta) &= \cos\theta X_{k-1}(\cos\theta, \sin\theta) + \sin\theta Y_{k-1}(\cos\theta, \sin\theta), \\ \psi_k(\theta) &= \cos\theta Y_{k-1}(\cos\theta, \sin\theta) - \sin\theta X_{k-1}(\cos\theta, \sin\theta). \end{aligned} \tag{19}$$

Suppose that a solution of (18) with the initial condition $r|_{\theta=0} = r_0$ has the form

$$r = \tilde{r}(\theta, r_0, \delta) = \sum_{k=1}^{\infty} \nu_k(\theta, \delta) r_0^k, \tag{20}$$

where r_0 is small and

$$\nu_1(\theta, \delta) = e^{\delta\lambda\theta},\ \nu_k(0, \delta) = 0, k = 2, 3, \cdots. \tag{21}$$

Definition 2.1. (See [5]). Suppose that $\lambda > 0$(or < 0). If $\nu_1(2\pi, \delta) \neq 1$, the origin (or the singular point at infinity) of system (1) is called a generalized focus. If $\nu_1(2\pi, \delta) = 1$ and there exists a positive integer k, such that $\nu_2(2\pi, 0) = \nu_3(2\pi, 0) = \cdots = \nu_{2k-1}(2\pi, 0) = 0$, and $\nu_{2k+1}(2\pi, 0) \neq 0$, then the origin (or the singular point at infinity) of system (1) is called the k-order generalized fine focus. The number $\nu_{2k+1}(2\pi, 0)$ is called the k-order generalized focal value. If $v_1(2\pi,\delta) = 1$ and for all k, $v_{2k+1}(2\pi,0) = 0$ holds, then the origin (or the singular point at infinity) of system (1) is called a center.

It is easy to see the following conclusions hold.

Theorem 2.2. (See [5]). *For $\lambda > 0$, if the origin is a generalized focus of system* (1)*, then when $v_1(2\pi) - 1 < 0$ (> 0), it is stable (unstable); if the origin is a k-order generalized fine focus of system* (1)*, then when $v_{2k+1}(2\pi,0) < 0$ (> 0), the origin is stable (unstable); if the origin is a center, then there exists a family of closed orbits of system* (1) *enclosing the origin.*

Theorem 2.3. (See [5]). *For $\lambda < 0$, if the singular point at infinity is a generalized focus of system* (1)*, then when $v_1(2\pi) - 1 < 0$ (> 0), it is stable (unstable); if the singular point at infinity is a k-order generalized fine focus of system* (1)*, then when $v_{2k+1}(2\pi,0) < 0$ (> 0), the singular point at infinity is stable (unstable); if the singular point at infinity is a center, then there exists a family of closed orbits of system* (1) *which lies in the inner neighborhood of the equator in Poincaré compactification.*

Consider the real polynomial differential system

$$\begin{aligned} \frac{dx}{dt} &= -y + \sum_{k=2}^{\infty} X_k(x,y), \\ \frac{dy}{dt} &= x + \sum_{k=2}^{\infty} Y_k(x,y), \end{aligned} \tag{22}$$

where

$$X_k(x,y) = \sum_{\alpha+\beta=k} A_{\alpha\beta} x^{\alpha} y^{\beta},\ Y_k(x,y) = \sum_{\alpha+\beta=k} B_{\alpha\beta} x^{\alpha} y^{\beta}. \tag{23}$$

Obviously, under $x = r\cos\theta, y = r\sin\theta$, the polar coordinate form of system (1) differs from (18) only in a constant factor λ.

By means of transformation (3), system (22) becomes the following complex system:

$$\begin{aligned} \frac{dz}{dT} &= z + \sum_{k=2}^{\infty} Z_k(z,w), \\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z,w), \end{aligned} \tag{24}$$

where

$$(25)\quad \begin{aligned} X_k(x,y) &= \tfrac{1}{2i}\left[W_k(x+iy,x-iy)-Z_k(x+iy,x-iy)\right], \\ Y_k(x,y) &= \tfrac{1}{2}\left[W_k(x+iy,x-iy)+Z_k(x+iy,x-iy)\right], \end{aligned}$$

or equivalently

$$(26)\quad \begin{aligned} Z_k(z,w) &= Y_k\left(\tfrac{z+w}{2},\tfrac{z-w}{2i}\right)-iX_k\left(\tfrac{z+w}{2},\tfrac{z-w}{2i}\right), \\ W_k(z,w) &= Y_k\left(\tfrac{z+w}{2},\tfrac{z-w}{2i}\right)+iX_k\left(\tfrac{z+w}{2},\tfrac{z-w}{2i}\right). \end{aligned}$$

Denote that

$$(27)\quad Z_k(z,w)=\sum_{\alpha+\beta=k} a_{\alpha\beta}z^\alpha w^\beta,\ W_k(z,w)=\sum_{\alpha+\beta=k} b_{\alpha\beta}w^\alpha z^\beta.$$

The coefficients of system (24) satisfy the conjugate condition, i.e.,

$$(28)\quad \overline{a_{\alpha\beta}}=b_{\alpha\beta},\ \alpha\geq 0,\ \beta\geq 0,\ \alpha+\beta\geq 2.$$

We call that systems (22) and (24) are concomitant.

Lemma 2.4. (See [10]). *For system* (24)*, we can derive uniquely the following formal series:*

$$(29)\quad \xi=z+\sum_{k+j=2}^{\infty} c_{kj}z^k w^j,\ \eta=w+\sum_{k+j=2}^{\infty} d_{kj}w^k z^j,$$

where $c_{k+1,k}=d_{k+1,k}=0, k=1,2,\cdots$, *such that*

$$(30)\quad \frac{d\xi}{dT}=\xi+\sum_{j=1}^{\infty} p_j\xi^{j+1}\eta^j,\ \frac{d\eta}{dT}=-\eta-\sum_{j=1}^{\infty} q_j\eta^{j+1}\xi^j.$$

Definition 2.5. (See [9]). (i) $\mu_k=p_k-q_k, k=0,1,2,\cdots$ is called the k-th singular point quantity at the origin of system (24).

(ii) If $\mu_0=\mu_1=\cdots=\mu_{k-1}=0$, and $\mu_k\neq 0$, then the origin of system (24) is called a fine singular point of order k.

(iii) If for all k, $\mu_k=0$, then the origin of system (24) is called a complex center.

3. Proof of the main results

Proof. Write $Z(z,w), W(z,w)$ as

$$Z(z,w) = z + \frac{1}{6}\sum_{k+j=3}^{\infty}[(\lambda+3)a_{k,j-1} - (\lambda-3)b_{j,k-1}]z^{2k+j-2}w^{2j+k-3},$$
$$W(z,w) = w + \frac{1}{6}\sum_{k+j=3}^{\infty}[(\lambda+3)b_{j,k-1} - (\lambda-3)a_{k,j-1}]z^{2k+j-3}w^{2j+k-2}. \tag{31}$$

From (31) and (13), differentiating F with respect to T and grouping like terms give

$$\begin{aligned}
\frac{dF}{dT} &= \frac{\partial F}{\partial z}Z - \frac{\partial F}{\partial w}W \\
&= \sum_{\alpha+\beta=2}^{\infty}\alpha c_{\alpha\beta}z^{\alpha-1}w^{\beta}\Big(z + \frac{1}{6}\sum_{k+j=3}^{\infty}\big((\lambda+3)a_{k,j-1} - (\lambda-3)b_{j,k-1}\big)z^{2k+j-2}w^{2j+k-3}\Big) \\
&\quad - \sum_{\alpha+\beta=2}^{\infty}\beta c_{\alpha\beta}z^{\alpha}w^{\beta-1}\Big(w + \frac{1}{6}\sum_{k+j=3}^{\infty}\big((\lambda+3)b_{j,k-1} - (\lambda-3)a_{k,j-1}\big)z^{2k+j-3}w^{2j+k-2}\Big) \\
&= \sum_{\alpha+\beta=2}^{\infty}(\alpha-\beta)c_{\alpha\beta}z^{\alpha}w^{\beta} + \frac{1}{6}\sum_{\alpha+\beta=2}^{\infty}\sum_{k+j=3}^{\infty}\Big(\big((\lambda+3)\alpha + (\lambda-3)\beta\big)a_{k,j-1} \\
&\quad - \big((\lambda+3)\beta + (\lambda-3)\alpha\big)b_{j,k-1}\Big)c_{\alpha\beta}z^{\alpha+2k+j-3}w^{\beta+2j+k-3}.
\end{aligned} \tag{32}$$

Performing the following subscript change

$$\alpha' = \alpha + 2k + j - 3,\ \beta' = \beta + 2j + k - 3, \tag{33}$$

and jointing both restrictions imposed on α, β, k, j we have that

$$\alpha' + \beta' = \alpha + \beta + 3(k+j) - 6 \geq 5. \tag{34}$$

We preserve the notation (α, β) for the new variables (α', β'), then

$$\begin{aligned}
&\sum_{\alpha+\beta=2}^{\infty}\sum_{k+j=3}^{\infty}\Big(\big((\lambda+3)\alpha + (\lambda-3)\beta\big)a_{k,j-1} \\
&\qquad - \big((\lambda+3)\beta + (\lambda-3)\alpha\big)b_{j,k-1}\Big)c_{\alpha\beta}z^{\alpha+2k+j-3}w^{\beta+2j+k-3} \\
&= \sum_{\alpha+\beta=5}^{\infty}\sum_{k+j=3}^{\left[\frac{\alpha+\beta+4}{3}\right]}\big\{[(\lambda+3)(\alpha-2k-j+3) + (\lambda-3)(\beta-2j-k+3)]a_{k,j-1} \\
&\qquad - [(\lambda+3)(\beta-2j-k+3) + (\lambda-3)(\alpha-2k-j+3)]b_{j,k-1}\big\} \\
&\qquad \times c_{\alpha-2k-j+3,\beta-2j-k+3}z^{\alpha}w^{\beta}.
\end{aligned} \tag{35}$$

For $2 \leq \alpha+\beta \leq 4$, set

$$\begin{cases} c_{22}=1, \\ c_{\alpha\beta}=0, \ for \ other \ (\alpha,\beta). \end{cases} \tag{36}$$

Hence, we have

$$\sum_{\alpha+\beta=2}^{\infty}(\alpha-\beta)c_{\alpha\beta}z^{\alpha}w^{\beta} = \sum_{\alpha+\beta=5}^{\infty}(\alpha-\beta)c_{\alpha\beta}z^{\alpha}w^{\beta}. \tag{37}$$

So we can write (32) in the form

$$\begin{aligned} \frac{dF}{dT} = \sum_{\alpha+\beta=5}^{\infty}\Big((\alpha-\beta)c_{\alpha\beta} + \frac{1}{6}\sum_{k+j=3}^{\left[\frac{\alpha+\beta+4}{3}\right]}\big(((\lambda+3)(\alpha-2k-j+3) \\ +(\lambda-3)(\beta-2j-k+3))a_{k,j-1}\big) - \big((\lambda+3)(\beta-2j-k+3) \\ +(\lambda-3)(\alpha-2k-j+3)\big)b_{j,k-1}\big)c_{\alpha-2k-j+3,\beta-2j-k+3}\Big)z^{\alpha}w^{\beta}. \end{aligned} \tag{38}$$

Write

(39)

$$\begin{aligned} \Delta_{\alpha\beta} = (\alpha-\beta)c_{\alpha\beta} + \frac{1}{6}\sum_{k+j=3}^{\left[\frac{\alpha+\beta+4}{3}\right]}\big\{[(\lambda+3)(\alpha-2k-j+3)+(\lambda-3)(\beta-2j-k+3)]a_{k,j-1}] \\ -[(\lambda+3)(\beta-2j-k+3)+(\lambda-3)(\alpha-2k-j+3)]b_{j,k-1}\big\} \\ \times c_{\alpha-2k-j+3,\beta-2j-k+3}, \end{aligned}$$

equating the coefficients of $(zw)^m$ in (14) and $z^{\alpha}w^{\beta}$ in (38), when $\alpha \neq \beta$, take $\Delta_{\alpha\beta}=0$; else take $\Delta_{m+2,m+2}=\mu_m$, then we get (15). This completes the proof.

4. Examples

We end this paper with finding center conditions for a class of quasi quadratic systems about the practical implementation of Theorem 1.1.

We concretely consider the following class of systems

$$\begin{aligned} \frac{dx}{dt} &= -y+(x^2+y^2)^{\frac{\lambda-1}{2}}\left[-(B_{20}+B_{02})x^2+2(A_{02}-A_{20})xy+(B_{20}+B_{02})y^2\right], \\ \frac{dy}{dt} &= x+(x^2+y^2)^{\frac{\lambda-1}{2}}\left[(A_{20}+A_{02})x^2+2(B_{02}-B_{20})xy-(A_{20}+A_{02})y^2\right], \end{aligned} \tag{40}$$

the concomitant complex system of (40) has the form

$$\frac{dz}{dT} = z + (zw)^{\frac{\lambda-1}{2}} \left(a_{20}z^2 + a_{02}w^2\right), \quad \frac{dw}{dT} = -w - (zw)^{\frac{\lambda-1}{2}} \left(b_{20}w^2 + b_{02}z^2\right), \tag{41}$$

where

$$a_{20} = A_{20} + iB_{20},\ b_{20} = A_{20} - iB_{20},\ a_{02} = A_{02} + iB_{02},\ b_{02} = A_{02} - iB_{02}. \tag{42}$$

Computing the singular point quantities of system (41) using (15) we get the following.

Proposition 4.1. *For system* (41)*, the shortened expressions of its first twelve singular point quantities are given by*

$$\begin{aligned} &\mu_3 = 0, \\ &\mu_6 = -\tfrac{1}{9}(a_{20}^3a_{02} - b_{20}^3b_{02})\lambda(\lambda-1)(\lambda-3), \\ &\mu_9 \sim -\tfrac{1}{12}(6a_{02}b_{02} - 5a_{20}b_{20})(a_{20}^3a_{02} - b_{20}^3b_{02})\lambda(\lambda-1), \\ &\mu_{12} \sim \tfrac{67}{324}a_{20}^2b_{20}^2(a_{20}^3a_{02} - b_{20}^3b_{02})\lambda(\lambda-1), \end{aligned} \tag{43}$$

where $\mu_k = 0, k \neq 3i, i \leq 4, i \in \mathbb{N}$. In the above expression of μ_k, we have already let $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0, k = 2,3,\cdots,12$.

From Proposition 4.1, we have the following.

Lemma 4.2. *For system* (41)*, the first twelve singular point quantities vanish if and only if one of the three conditions holds:*

(i) $\lambda = 1$;

(ii) $a_{20}^3a_{02} = b_{20}^3b_{02}$;

(iii) $\lambda = 3,\ a_{20} = b_{20} = a_{02} = b_{02} = 0$.

From the technique employed in [9], we have the following.

Lemma 4.3. *All the elementary Lie invariants of system* (41) *are listed as follows:*

$$a_{20}b_{20},\ a_{02}b_{02},\ a_{20}^3a_{02},\ b_{20}^3b_{02}. \tag{44}$$

We next get the center conditions.

Theorem 4.4. *For system* (41)*, the origin is a complex center if and only if condition (i) or (iii) holds; when* $\lambda > 0$ (< 0)*, the origin (the singular point at infinity) is a complex center if and only if condition (ii) holds.*

Proof. When condition (i) holds, system (41) has the analytic integrating factor f^{-1}, where

(45)
$$\begin{aligned} f = & 1+2(a_{20}z+b_{20}w)+[(a_{20}^2+b_{20}b_{02})z^2+3(a_{20}b_{20}-a_{02}b_{02})zw+(b_{20}^2+a_{20}a_{02})w^2] \\ & +(a_{20}b_{20}-a_{02}b_{02})(b_{02}z^3+a_{20}z^2w+b_{20}zw^2+a_{02}w^3). \end{aligned}$$

When condition (ii) holds, system (41) satisfies the Extended Symmetric Principle. When condition (iii) holds, system (41) reduces to the trivial linear system $\frac{dz}{dT}=z$, $\frac{dw}{dT}=-w$.

Appendix

The recursive formulae from Theorem 1.1 for the computation of singular point quantities of system (41) in *Mathematica* is as follows:

For $2 \le \alpha+\beta \le 4$,
$$\begin{cases} c_{2,2}=1, \\ c_{\alpha,\beta}=0,\ for\ other\ (\alpha,\beta), \end{cases}$$
when $\alpha=\beta>0$ or $\alpha<0$, or $\beta<0$,

$c_{\alpha,\beta}=0$;

else
$$\begin{aligned} c_{\alpha,\beta} = & \big(-b_{02}((-3+\alpha)(-3+\lambda)+\beta(3+\lambda))c_{-3+\alpha,\beta}+a_{20}((-1+\beta)(-3+\lambda) \\ & +(-2+\alpha)(3+\lambda))c_{-2+\alpha,-1+\beta}-b_{20}((-1+\alpha)(-3+\lambda) \\ & +(-2+\beta)(3+\lambda))c_{-1+\alpha,-2+\beta}+a_{02}((-3+\beta)(-3+\lambda) \\ & +\alpha(3+\lambda))c_{\alpha,-3+\beta}\big)/6/(\beta-\alpha), \end{aligned}$$
$$\begin{aligned} \mu_m = & (-b_{02}((-1+m)(-3+\lambda)+(2+m)(3+\lambda))c_{-1+m,2+m}+a_{20}((1+m)(-3+\lambda) \\ & +m(3+\lambda))c_{m,1+m}-b_{20}((1+m)(-3+\lambda)+m(3+\lambda))c_{1+m,m} \\ & +a_{02}((-1+m)(-3+\lambda)+(2+m)(3+\lambda))c_{2+m,-1+m})/6. \end{aligned}$$

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REFERENCES

[1] H. Poincaré, Mémoire sur les courbes définies par les équations différentielles, in: Oeuvreus de Henri Poincaré, vol. I, Gauthier-Villars, Paris, 1951, pp. 95-114.

[2] A. Cima, A. Gasull, V. Mañosa, F. Mañosas, Algebraic properties of the Liapunov and period constants, Rocky Mountain J. Math. 27 (1997), 471-501.

[3] N.G. Lloyd, J.M. Pearson, Bifurcation of limit cycles and integrability of planar dynamical systems in complex form, J. Phys. A 32 (1999), 1973-1984.

[4] J.M. Pearson, N.G. Lloyd, C.J. Christopher, Algorithmic derivation of centre conditions, SIAM Rev. 38 (1996), 619-636.

[5] Y. Liu, J. Li, Center and isochronous center problems for quasi analytic system, Acta Math. Sinica 23 (2007), 965-972.

[6] Y. Liu, The generalized focal values and bifurcation of limit cycles for quasi-quadratic system, Acta Math. Sinica 45 (2002), 671-682.

[7] P. Xiao, Critical Point Quantities and Integrability Conditions for Complex Resonant Polynomial Differential Systems, Ph.D. Thesis, School of Mathematics, Central South University, 2005.

[8] Y. Wu, F. Li, P. Li, Derivation of isochronicity conditions for quasi-cubic homogeneous analytic systems, Int. J. Bifurcation Chaos 23 (2013), Article ID 1350149.

[9] Y. Liu, J. Li, Theory of values of singular point in complex autonomous differential system, Sci. China (Series A) 3 (1990), 10-24.

[10] V.V. Amelkin, N.A. Lukashevich, A.P. Sadovskii, Nonlinear Oscillations in Second Order Systems, BSU, Minsk, 1982 (in Russian).