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ASYMPTOTIC STABILITY IN NONLINEAR NEUTRAL LEVIN-NOHEL INTEGRO-DIFFERENTIAL EQUATIONS

KARIMA BESSIOUD[1], ABDELOUAHEB ARDJOUNI[2,*], AHCENE DJOUDI[1]

[1]Department of Mathematics, Annaba University, Algeria

[2]Department of Mathematics and Informatics, Souk Ahras University, Algeria

Abstract. In this paper we use the contraction mapping theorem to obtain asymptotic stability results about the zero solution for the following nonlinear neutral Levin-Nohel integro-differential equation $\frac{d}{dt}x(t)+\int_{t-\tau(t)}^{t}a(t,s)x(s)ds+\frac{d}{dt}g(t,x(t-\tau(t)))=0$. An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, the case of the equation with several delays is studied.

Keywords. Asymptotic stability; Contraction mapping theorem; Fixed point; Neutral integro-differential equation.

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1. Introduction

The Lyapunov direct method has been very effective in establishing stability results and the existence of periodic solutions for wide variety of ordinary, functional and partial differential equations. Nevertheless, in the application of Lyapunov's direct method to problems of stability in delay differential equations, serious difficulties occur if the delay is unbounded or if the equation has unbounded terms. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang and others began a study in

*Corresponding author.

E-mail addresses: karima_bess@yahoo.fr (K. Bessioud), abd_ardjouni@yahoo.fr (A. Ardjouni), adjoudi@yahoo.com (A. Djoudi).

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which they noticed that some of this difficulties vanish or might be overcome by means of fixed point theory; see [1]-[20] and the references therein. The fixed point theory does not only solve the problems on stability but have other significant advantage over Lyapunov's direct method. The conditions of the former are often average but those of the latter are usually pointwise; see [7] and the references therein.

In paper, we consider the following nonlinear neutral Levin-Nohel integro-differential equation with variable delay

$$\frac{d}{dt}x(t)+\int_{t-\tau(t)}^{t}a(t,s)x(s)ds+\frac{d}{dt}g\left(t,x(t-\tau(t))\right)=0,\ t\geq t_0, \tag{1}$$

with an assumed initial condition

$$x(t)=\phi(t),\ t\in[m(t_0),t_0],$$

where $\phi\in C([m(t_0),t_0],\mathbb{R})$ and

$$m(t_0)=\inf\{t-\tau(t):t\in[t_0,\infty)\}.$$

Throughout this paper, we assume that $a\in C([t_0,\infty)\times[m(t_0),\infty),\mathbb{R})$ and $\tau\in C^2([t_0,\infty),\mathbb{R}^+)$ with $t-\tau(t)$ as $t\to\infty$. The function $g(t,x)$ is globally Lipschitz continuous in x. That is, there is positive constant E such that

$$|g(t,x)-g(t,y)|\leq E\,\|x-y\|,\ g(t,0)=0. \tag{2}$$

In this paper, our purpose is to use the contraction mapping theorem [21] to show the asymptotic stability of the zero solution for Eq.(1). An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, A study of the general form of (1) with several delays is given. In the special case $g(t,x)=0$, Dung [14] proved the zero solution of (1) is asymptotically stable with a necessary and sufficient condition by using the contraction mapping theorem. The results presented in this paper extend the main results in [14].

2. Main results

For the convenience of the reader, let us recall the definition of asymptotic stability. For each t_0, we denote $C(t_0)$ the space of continuous functions on $[m(t_0),t_0]$ with the supremum norm

$\|.\|_{t_0}$. For each $(t_0,\phi)\in[0,\infty)\times C(t_0)$, denoted by $x(t)=x(t,t_0,\phi)$ the unique solution of Eq. (1).

Definition 2.1. *The zero solution of Eq. (1) is called*

(i) *stable if for each* $\varepsilon>0$ *there exists a* $\delta>0$ *such that* $|x(t,t_0,\phi)|<\varepsilon$ *for all* $t\geq t_0$ *if* $\|\phi\|_{t_0}<\delta$,

(ii) *asymptotically stable if it is stable and* $\lim_{t\to\infty}|x(t,t_0,\phi)|=0$.

In order to be able to construct a new fixed mapping, we transform the Levin-Nohel equation into an equivalent equation. For this, we use the variation of parameter formula and the integration by parts.

Lemma 2.2. *x is a solution of equation (1) if and only if*

$$\begin{aligned} x(t) &= (\phi(t_0)+g(t_0,\phi(t_0-\tau(t_0))))\,e^{-\int_{t_0}^t A(z)dz}-g(t,x(t-\tau(t))) \\ &\quad -\int_{t_0}^t [L_x(s)-A(s)g(t,x(s-\tau(s)))]\,e^{-\int_s^t A(z)dz}ds,\ t\ \geq\ t_0, \end{aligned} \tag{3}$$

where

$$\begin{aligned} L_x(t) &= \int_{t-\tau(t)}^t a(t,s)\left(\int_s^t\left(\int_{u-\tau(u)}^u a(u,v)x(v)dv\right)du\right. \\ &\quad \left.+g(t,x(t-\tau(t)))-g(s,x(s-\tau(s)))\right)ds, \end{aligned} \tag{4}$$

and

$$A(t)=\int_{t-\tau(t)}^t a(t,s)ds. \tag{5}$$

Proof. Obviously, we have

$$x(s)=x(t)-\int_s^t\frac{\partial}{\partial u}x(u)du.$$

Inserting this relation into (1), we get

$$\frac{d}{dt}x(t)+\int_{t-r(t)}^t a(t,s)\left(x(t)-\int_s^t\frac{\partial}{\partial u}x(u)du\right)ds+\frac{d}{dt}g(t,x(t-\tau(t)))=0,\ t\geq t_0,$$

or equivalently

$$\frac{d}{dt}x(t)+x(t)\int_{t-\tau(t)}^t a(t,s)ds-\int_{t-\tau(t)}^t a(t,s)\left(\int_s^t\frac{\partial}{\partial u}x(u)du\right)ds+\frac{d}{dt}g(t,x(t-\tau(t)))=0,\ t\geq t_0.$$

Substituting $\frac{\partial x}{\partial u}$ from (1), we obtain

$$\begin{aligned}
&\frac{d}{dt}x(t)+x(t)\int_{t-\tau(t)}^{t}a(t,s)ds \\
&+\int_{t-\tau(t)}^{t}a(t,s)\left(\int_{s}^{t}\left(\int_{u-\tau(u)}^{u}a(u,v)x(v)dv+\frac{\partial}{\partial u}g\left(u,x(u-\tau(u))\right)\right)du\right)ds \\
&+\frac{d}{dt}g\left(t,x(t-\tau(t))\right) \;=\; 0,\; t\;\geq\; t_0.
\end{aligned} \tag{6}$$

By performing the integration, we have

$$\int_{s}^{t}\frac{\partial}{\partial u}g\left(u,x(u-\tau(u))\right)du=g\left(t,x(t-\tau(t))\right)-g\left(s,x(s-\tau(s))\right). \tag{7}$$

Substituting (7) into (6), we have

$$\frac{d}{dt}x(t)+A(t)x(t)+L_x(t)+\frac{d}{dt}g\left(t,x(t-\tau(t))\right))=0,\; t\geq t_0,$$

where A and L_x are given by (5) and (4), respectively. By the variation of constants formula, we get

$$x(t)=\phi(t_0)e^{-\int_{t_0}^{t}A(z)dz}-\int_{t_0}^{t}\left[L_x(s)+\frac{\partial}{\partial s}g\left(s,x(s-\tau(s))\right)\right]e^{-\int_{s}^{t}A(z)dz}ds,\; t\geq t_0. \tag{8}$$

By using the integration by parts, we obtain

$$\begin{aligned}
&\int_{t_0}^{t}\frac{\partial}{\partial s}g\left(s,x(s-\tau(s))\right)e^{-\int_{s}^{t}A(z)dz}ds \\
&=g\left(t,x(t-\tau(t))\right)-g\left(t_0,x(t_0-\tau(t_0))\right)e^{-\int_{t_0}^{t}A(z)dz}-\int_{t_0}^{t}A(s)g\left(s,x(s-\tau(s))\right)e^{-\int_{s}^{t}A(z)dz}ds.
\end{aligned} \tag{9}$$

Finally, we obtain (3) by substituting (9) in (8). Since each step is reversible, the converse follows easily. This completes the proof.

Theorem 2.3. *Let (2) holds and suppose that the following two conditions hold:*

$$\liminf_{t\to\infty}\int_{0}^{t}A(z)dz>-\infty, \tag{10}$$

$$\sup_{t\geq 0}\left(E+\int_{0}^{t}\omega(s)e^{-\int_{s}^{t}A(z)dz}ds\right)=\alpha<1, \tag{11}$$

where

$$\omega(s)=\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,dv\right)du+2E\right)dw+E\,|A(s)|\,.$$

Then the zero solution of (1) is asymptotically stable if and only if

$$\int_{0}^{t}A(z)dz\to\infty \text{ as } t\to\infty. \tag{12}$$

Proof. Sufficient condition. Suppose that (12) holds. Denoted by C the space of continuous bounded functions $x:[m(t_0),\infty)\to\mathbb{R}$ such that $x(t)=\phi(t)$, $t\in[m(t_0),t_0]$. It is known that C is a complete metric space endowed with a metric $\|x\|=\sup_{t\geq m(t_0)}|x(t)|$. Define the operator P on C by $(Px)(t)=\phi(t)$, $t\in[m(t_0),t_0]$ and

$$\begin{aligned}(Px)(t) &= \left(\phi(t_0)+g\left(t_0,\phi(t_0-\tau(t_0))\right)\right)e^{-\int_{t_0}^{t}A(z)dz}-g\left(t,x(t-\tau(t))\right)\\ &\quad -\int_{t_0}^{t}\left[L_x(s)-A(s)g\left(s,x(s-\tau(s))\right)\right]e^{-\int_{s}^{t}A(z)dz}ds,\ t\ \geq\ t_0.\end{aligned}$$

Obviously, Px is continuous for each $x\in C$. Moreover, it is a contraction operator. Indeed, let $x,y\in C$

$$\begin{aligned}&|(Px)(t)-(Py)(t)|\\ &\leq|g\left(t,x(t-\tau(t))\right)-g\left(t,y(t-\tau(t))\right)|\\ &+\int_{t_0}^{t}\left[\left|L_x(s)-L_y(s)\right|+|A(s)|\left|g\left(s,x(s-\tau(s))\right)-g\left(s,y(s-\tau(s))\right)\right|\right]e^{-\int_{s}^{t}A(z)dz}ds.\end{aligned}$$

Since $x(t)=y(t)=\phi(t)$ for all $t\in[m(t_0),t_0]$, this implies that

$$\begin{aligned}&\left|L_x(s)-L_y(s)\right|\\ &\leq\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,dv\right)du+2E\right)dw\right)\|x-y\|\,.\end{aligned}$$

Consequently, it holds for all $t\geq t_0$ that

$$\begin{aligned}&|(Px)(t)-(Py)(t)|\\ &\leq\left[E+\int_{t_0}^{t}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,dv\right)du+2E\right)dw\right.\right.\\ &\left.\left.+E\,|A(s)|\right)e^{-\int_{s}^{t}A(z)dz}ds\right]\|x-y\|\,.\end{aligned}$$

Hence, it follows from (11) that

$$|(Px)(t)-(Py)(t)| \leq \alpha \|x-y\|, \ t \geq t_0.$$

Thus P is a contraction operator on C. We now consider a closed subspace S of C that is defined by

$$S = \{x \in C : |x(t)| \to 0 \text{ as } t \to \infty\}.$$

We will show that $P(S) \subset S$. To do this, we need to point out that for each $x \in S$, $|(Px)(t)| \to 0$ as $t \to \infty$. Let $x \in S$, by the definition of P we have

$$\begin{aligned}
(Px)(t) &= (\phi(t_0) + g(t_0, \phi(t_0 - \tau(t_0))))\, e^{-\int_{t_0}^{t} A(z)dz} - g(t, x(t - \tau(t))) \\
&\quad - \int_{t_0}^{t} [L_x(s) - A(s) g(s, x(s - \tau(s)))]\, e^{-\int_{s}^{t} A(z)dz} ds, \\
&= I_1 + I_2 + I_3, \ t \geq t_0.
\end{aligned}$$

The first term I_1 tends to 0 by (12) and I_2 tends to 0 by (2) and $t-\tau(t)\to\infty$ as $t\to\infty$. For any $T\in(t_0,t)$, we have the following estimate for the third term

$$
\begin{aligned}
I_3 &\le \left|\int_{t_0}^{T}\left[L_x(s)-A(s)g\left(s,x(s-\tau(s))\right)\right]e^{-\int_s^t A(z)dz}ds\right| \\
&+\left|\int_{T}^{t}\left[L_x(s)-A(s)g\left(s,x(s-\tau(s))\right)\right]e^{-\int_s^t A(z)dz}ds\right| \\
&\le \int_{t_0}^{T}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_w^s\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,\|x\|\,dv\right)du+2E\,\|\phi\|_{t_0}\right)dw\right. \\
&\left.+E\,|A(s)|\,\|\phi\|_{t_0}\right)e^{-\int_s^t A(z)dz}ds \\
&+\int_T^t\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_w^s\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,|x(v)|\,dv\right)du\right.\right. \\
&\left.+E\,|x(s-\tau(s))|+E\,|x(w-\tau(w))|\right)dw+E\,|A(s)|\,|x(s-\tau(s))|\Big)e^{-\int_s^t A(z)dz}ds \\
&\le \left[\int_{t_0}^{T}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_w^s\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,dv\right)du+2E\right)dw\right.\right. \\
&\left.\left.+E\,|A(s)|\right)e^{-\int_s^t A(z)dz}ds\right]\left(\|x\|+\|\phi\|_{t_0}\right) \\
&+\int_T^t\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_w^s\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,|x(v)|\,dv\right)du\right.\right. \\
&\left.+E\,|x(s-\tau(s))|+E\,|x(w-\tau(w))|\right)dw+E\,|A(s)|\,|x(s-\tau(s))|\Big)e^{-\int_s^t A(z)dz}ds \\
&= I_{31}+I_{32}.
\end{aligned}
$$

Since $t-\tau(t)\to\infty$ as $t\to\infty$, this implies that $u-\tau(u)\to\infty$ as $T\to\infty$. Thus, from the fact $|x(v)|\to 0,\ v\to\infty$ we can infer that for any $\varepsilon>0$ there exists $T_1=T>t_0$ such that

$$
\begin{aligned}
I_{32} &< \frac{\varepsilon}{2\alpha}\int_{T_1}^{t}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_w^s\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,dv\right)du++2E\right)dw\right. \\
&\quad\left.+E\,|A(s)|\right)e^{-\int_s^t A(z)dz}ds,
\end{aligned}
$$

and hence, $I_{32}<\frac{\varepsilon}{2}$ for all $t\ge T_1$. On the other hand, $\|x\|<\infty$ because $x\in S$. This combined with (12) yields $I_{31}\to 0$ as $t\to\infty$. As a consequence, there exists $T_2\ge T_1$ such that $I_{31}<\frac{\varepsilon}{2}$ for all $t\ge T_2$. Thus, $I_3<\varepsilon$ for all $t\ge T_2$, that is, $I_3\to 0$ as $t\to\infty$. So $P(S)\subset S$. By the Contraction Mapping Principle, P has a unique fixed point x in S which is a solution of (1) with $x(t)=\phi(t)$ on $[m(t_0),t_0]$ and $x(t)=x(t,t_0,\phi)\to 0$ as $t\to\infty$. To obtain the asymptotic stability, we need to show that the zero solution of (1) is stable. By condition (10), we can define

$$K = \sup_{t \geq 0} e^{-\int_0^t A(z)dz} < \infty. \tag{13}$$

Using the formula (3) and condition (11), we can obtain

$$|x(t)| \leq K(1+E)\|\phi\|_{t_0} e^{\int_0^{t_0} A(z)dz} + \alpha(\|x\| + \|\phi\|_{t_0}),\ t \geq t_0,$$

which leads us to

$$\|x\| \leq \frac{K(1+E)e^{\int_0^{t_0} A(z)dz} + \alpha}{1-\alpha} \|\phi\|_{t_0}. \tag{14}$$

Thus for every, $\varepsilon > 0$, we can find $\delta > 0$ such that $\|\phi\|_{t_0} < \delta$ implies that $\|x\| < \varepsilon$. This shows that the zero solution of (1) is stable and hence, it is asymptotically stable.

Necessary condition. Suppose that the zero solution of (1) is asymptotically stable and that the condition (12) fails. It follows from (10) that there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \int_0^{t_n} A(z)dz \text{ exists and is finite.}$$

Hence, we can choose a positive constant L satisfying

$$-L \leq \lim_{n\to\infty} \int_0^{t_n} A(z)dz \leq L,\ \forall n \geq 1. \tag{15}$$

Then, condition (11) gives us

$$c_n = \int_0^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \leq \alpha e^{\int_0^{t_n} A(z)dz} < e^L.$$

The sequence $\{c_n\}$ is increasing and bounded, so it has a finite limit. For any $\delta_0 > 0$, there exists $n_0 > 0$ such that

$$\int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds < \frac{\delta_0}{2K},\ \forall n \geq n_0, \tag{16}$$

where K is as in (13). We choose δ_0 such that $\delta_0 < \frac{1-\alpha}{K(1+E)e^L+1}$ and consider the solution $x(t) = x(t, t_n, \phi)$ of (1) with the initial data $\phi(t_{n_0}) = \delta_0$ and $|\phi(s)| \leq \delta_0$, $s \leq t_{n_0}$. It follows from (14) that

$$|x(t)| \leq 1 - \delta_0,\ \forall t \geq t_{n_0}. \tag{17}$$

Applying the fundamental inequality $|a-b| \geq |a|-|b|$ and then using (17), (16) and (15), we get

$$\begin{aligned}
&|x(t_n)+g(t_n,x(t_n-\tau(t_n)))| \\
&\geq \delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} - \int_{t_{n_0}}^{t_n} \omega(s) e^{-\int_s^{t_n} A(z)dz} ds \\
&\geq e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - e^{-\int_0^{t_{n_0}} A(z)dz} \int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \right) \\
&\geq e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - K \int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \right) \\
&\geq \frac{1}{2}\delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \geq \frac{1}{2}\delta_0 e^{-2L} > 0,
\end{aligned}$$

which is a contradiction because $x(t_n)+g(t_n,x(t_n-\tau(t_n))) \to 0$ as $t_n \to \infty$. The proof is complete.

Letting $g(t,x)=0$, we get the following result.

Corollary 2.4. *Suppose that the following two conditions hold*

$$\liminf_{t\to\infty} \int_0^t A_0(z)dz > -\infty, \tag{18}$$

$$\sup_{t\geq 0} \int_0^t \left(\int_{s-\tau(s)}^s |a(s,w)| \int_w^s \int_{u-\tau(u)}^u |a(u,v)|\, dvdudw \right) e^{-\int_s^t A_0(z)dz} ds = \alpha < 1, \tag{19}$$

where

$$A_0(z) = \int_{z-\tau(z)}^z a(z,s)ds.$$

Then the zero solution of

$$\frac{d}{dt}x(t) + \int_{t-\tau(t)}^t a(t,s)x(s)ds = 0,$$

is asymptotically stable if and only if

$$\int_0^t A_0(z)dz \to \infty \text{ as } t \to \infty. \tag{20}$$

Next we turn our attention to the following neutral Levin-Nohel integro-differential equations with several delays

$$\frac{d}{dt}x(t) + \sum_{k=1}^M \int_{t-\tau_k(t)}^t a_k(t,s)x(s)ds + \sum_{k=1}^M \frac{d}{dt} g_k(t,x(t-\tau_k(t))) = 0,\ t \geq t_0, \tag{21}$$

where $a_k \in C([t_0,\infty)\times[m(t_0),\infty),\mathbb{R})$ and $\tau_k \in C^2([t_0,\infty),\mathbb{R}^+)$ with $t-\tau_k(t)$ as $t\to\infty$, $1\le k\le M$. The function $g_k(t,x)$ is globally Lipschitz continuous in x. That is, there is positive constant E_k such that

$$|g_k(t,x)-g_k(t,y)| \le E_k\|x-y\|,\ g_k(t,0)=0,\ 1\le k\le M. \tag{22}$$

Lemma 2.5. *x is a solution of equation (21) if and only if*

$$\begin{aligned} x(t) &= \left(\phi(t_0)+\sum_{k=1}^{M} g_k(t_0,\phi(t_0-\tau_k(t_0)))\right)e^{-\int_{t_0}^{t}\overline{A}(z)dz} - \sum_{k=1}^{M} g_k(t,x(t-\tau_k(t))) \\ &\quad -\int_{t_0}^{t}\left[\overline{L}_x(s)-\sum_{k=1}^{M}\overline{A}(s)g_k(t,x(s-\tau_k(s)))\right]e^{-\int_s^t\overline{A}(z)dz}ds,\ t\ \ge\ t_0, \end{aligned}$$

where

$$\begin{aligned} \overline{L}_x(t) = \sum_{k=1}^{M}\int_{t-\tau_k(t)}^{t} a_k(t,s)&\left(\int_s^t\left(\sum_{i=1}^{M}\int_{u-\tau_i(u)}^{u} a_i(u,v)x(v)dv\right)du\right. \\ &\left.+\sum_{i=1}^{M} g_i(t,x(t-\tau_i(t)))-\sum_{i=1}^{M} g_i(s,x(s-\tau_i(s)))\right)ds, \end{aligned}$$

and

$$\overline{A}(t)=\sum_{k=1}^{M}\int_{t-\tau_k(t)}^{t} a_k(t,s)ds.$$

The proof follows along the lines of Lemma 2.2, and hence we omit it.

Theorem 2.6. *Let (22) holds and suppose that the following two conditions hold*

$$\liminf_{t\to\infty}\int_0^t \overline{A}(z)dz > -\infty,$$

and

$$\sup_{t\ge 0}\left(\sum_{k=1}^{M}E_k+\int_0^t\overline{\omega}(s)e^{-\int_s^t\overline{A}(z)dz}ds\right)=\alpha<1,$$

where

$$\begin{aligned} \overline{\omega}(s) \ &= \ \sum_{k=1}^{M}\int_{s-\tau_k(s)}^{s}|a_k(s,w)|\left(\int_w^s\left(\sum_{i=1}^{M}\int_{u-\tau_i(u)}^{u}|a_i(u,v)|dv\right)du\right. \\ &\quad\left.+2\sum_{k=1}^{M}E_k\right)dw+\sum_{k=1}^{M}E_k\left|\overline{A}(s)\right|. \end{aligned}$$

Then the zero solution of (21) is asymptotically stable if and only if

$$\int_0^t \overline{A}(z)dz \to \infty \text{ as } t \to \infty.$$

Since the proof is similar to that of Theorem 2.3, we, therefore, omit it.

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