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OSCILLATION AND ASYMPTOTIC BEHAVIOR OF THIRD ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS WITH MIXED TYPE

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Abstract. This paper concerned with the oscillatory and asymptotic behavior of third order difference equation with mixed type neutral term $\Delta\left(a_n\Delta^2\left(x_n+b_nx_{n-\tau_1}+c_nx_{n+\tau_2}\right)^{\alpha}\right)+q_nx^{\beta}_{n-\sigma_1}+p_nx^{\gamma}_{n+\sigma_2}=0$. Using arithmetic-geometric mean inequality, we obtained oscillation criteria which generalize and extend some of the known results. Examples are also provided to illustrate the main results.

Keywords. Third order; Neutral difference equation; Oscillation; Asymptotic behavior; Mixed type.

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1. Introduction

In this paper, we are concerned with the following third order mixed type neutral difference equation of the form

$$\Delta\left(a_n\Delta^2\left(x_n+b_nx_{n-\tau_1}+c_nx_{n+\tau_2}\right)^{\alpha}\right)+q_nx^{\beta}_{n-\sigma_1}+p_nx^{\gamma}_{n+\sigma_2}=0, \tag{1.1}$$

where $n\in\mathbb{N}(n_0)=\{n_0,n_0+1,n_0+2,\ldots\}$, and n_0 is a nonnegative integer, subject to the following conditions:

(i) $\{a_n\}$ is a positive sequence for all $n\in\mathbb{N}(n_0)$, such that $\sum\limits_{n=n_0}^{\infty}\frac{1}{a_n}=\infty$;

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(ii) $\{b_n\}$, and $\{c_n\}$ are real sequences such that $0 \le b_n \le b < \infty$, and $0 \le c_n \le c < \infty$ for all $n \in \mathbb{N}(n_0)$;

(iii) $\{p_n\}$, and $\{q_n\}$ are positive real sequences for all $n \in \mathbb{N}(n_0)$;

(iv) α, β, and γ are ratios of odd positive integers, τ_1, τ_2, σ_1 and σ_2 are positive integers.

Let $\theta = \max\{\tau_1, \sigma_1\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \ge n_0 - \theta$, and satisfying equation (1.1) for all $n \in \mathbb{N}(n_0)$. A nontrivial solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

Recently many authors investigated the oscillatory and asymptotic behavior of third order neutral difference equations; see for example [1, 2, 3], [6]-[10], [12]-[24] and the references cited therein. Regarding the higher order mixed type neutral difference equations, Grace [10] considered the third order mixed type neutral difference equation

$$\Delta^3 (x_n + a\, x_{n-m} - b\, x_{n+k}) \pm (q\, x_{n-g} + p\, x_{n+h}) = 0, \tag{1.2}$$

and established some sufficient conditions for the oscillation of all solutions of equation (1.2).

Agarwal, Grace and Bohner [6] considered the m^{th} order mixed type neutral difference equation

$$\Delta^m (x_n + a\, x_{n-k} + b\, x_{n+\sigma}) + q\, x_{n-g} + p\, x_{n+h} = 0, \tag{1.3}$$

and obtained some sufficient conditions for the oscillation of all solutions of equation (1.3).

In [5], Agarwal and Grace considered several third order mixed typed neutral difference equations, and established sufficient conditions for the oscillation of all solutions. In [24], the authors considered equation (1.1) with $a_n \equiv 1$ or $\alpha = 1$, $\beta = \gamma$, and $\{a_n\}$ is a nondecreasing real sequence, respectively, and established some sufficient condition for the oscillation of all solutions of equation (1.1). Motivated by this observation, in this paper we investigate the oscillate behavior of solutions of equation (1.1) without assuming any monotonic condition on $\{a_n\}$ and using arithmetic-geometric mean inequality. Therefore, the results obtained in this paper complement and generalize the results established in [5, 6, 10, 24].

In Section 2, we present some sufficient conditions which ensure that every solution $\{x_n\}$ of equation (1.1) is either oscillatory or $x_n \to 0$ as $n \to \infty$. In Section 3, examples are presented to illustrate the main results.

2. Oscillation results

In this section, we present some new oscillation criteria for equation (1.1). We prove all results for the positive solutions only, since the proof for the opposite case is similar.

For simplicity, we use the following notations:

$$\begin{aligned} z_n &= (x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2})^{\alpha}, \\ Q_n &= \min\{q_n, q_{n-\tau_1}, q_{n+\tau_2}\},\ P_n = \min\{p_n, p_{n-\tau_1}, p_{n+\tau_2}\}, \\ A_n &= \sum_{s=N}^{n-1} \frac{1}{a_s},\ B_n = \frac{1}{A_n}\sum_{s=N}^{n-1} A_s, \text{ and } R_n = \sum_{s=N}^{n-1} \frac{(n-s-1)}{a_s}. \end{aligned}$$

Lemma 2.1. *Let $\{x_n\}$ be a positive solution of equation (1.1). Then the corresponding sequence $\{z_n\}$ satisfies the two cases for sufficiently large $n \ge N \in \mathbb{N}(n_0)$:*

(I) $z_n > 0,\ \Delta z_n > 0,\ \Delta^2 z_n > 0,$ *and* $\Delta\left(a_n \Delta^2 z_n\right) \le 0;$

(II) $z_n > 0,\ \Delta z_n < 0,\ \Delta^2 z_n > 0,$ *and* $\Delta\left(a_n \Delta^2 z_n\right) \le 0.$

Proof. The proof can be found in Lemma 2.2 of [24].

Lemma 2.2. *Let $\{x_n\}$ be a positive solution of equation (1.1), and the corresponding sequence $\{z_n\}$ satisfies case (I) of Lemma 2.1 for $n \ge N$. Then*

$$z_n \ge R_n a_n \Delta^2 z_n,\ n \ge N \in \mathbb{N}(n_0). \tag{2.1}$$

Proof. From case (I), we have $\{a_n \Delta^2 z_n\}$ is positive and decreasing, and hence we obtain

$$\Delta z_n \ge \sum_{s=N}^{n-1} \frac{a_s \Delta^2 z_s}{a_s} \ge a_n \Delta^2 z_n \sum_{s=N}^{n-1} \frac{1}{a_s}.$$

Summing the last inequality from N to $n-1$, we have

$$z_n \ge R_n a_n \Delta^2 z_n,\ n \ge N.$$

This completes the proof.

Lemma 2.3. *Let $\{x_n\}$ be a positive solution of equation (1.1), and the corresponding sequence $\{z_n\}$ satisfies the case (I) of Lemma 2.1 for $n \geq N$. Then $z_n \geq B_n \Delta z_n,\ n \geq N$.*

Proof. From the proof of Lemma 2.2, we have

$$\Delta z_n \geq A_n a_n \Delta^2 z_n.$$

From the last inequality, we obtain

$$\begin{aligned} \Delta\Big(\frac{\Delta z_n}{A_n}\Big) &= \frac{\Delta^2 z_n}{A_{n+1}} - \frac{\Delta z_n}{a_n A_n A_{n+1}} \\ &= \frac{A_n a_n \Delta^2 z_n - \Delta z_n}{a_n A_n A_{n+1}} \leq 0,\ n \geq N. \end{aligned}$$

Therefore $\{\frac{\Delta z_n}{A_n}\}$ is nonincreasing for all $n \geq N$. Further, we have

$$z_n \geq \sum_{s=N}^{n-1} \frac{A_s \Delta z_s}{A_s} \geq \frac{\Delta z_n}{A_n} \sum_{s=N}^{n-1} A_s.$$

This completes the proof.

Lemma 2.4. *Let $\{x_n\}$ be a positive solution of equation (1.1), and the corresponding $\{z_n\}$ satisfies the case(II) of Lemma 2.1. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left(\frac{1}{a_s} \sum_{t=s}^{\infty} (Q_t + P_t) \right) = \infty \tag{2.2}$$

holds, then $\lim_{n\to\infty} x_n = 0$.

Proof. The proof is similar to that of Lemma 2.4 of [19], and hence the details are omitted.

Lemma 2.5. *Assume $A \geq 0, B \geq 0$ and $\delta \geq 1$. Then*

$$(A+B)^{\delta} \leq 2^{\delta-1}(A^{\delta} + B^{\delta}).$$

Proof. The proof is quite elementary and hence it is omitted.

Next, we establish some oscillation results which ensure that every solution of equation (1.1) either oscillates or converges to zero. In the following theorems, we assume that $0 \leq b \leq 1$, and $0 \leq c \leq 2$ without further mention.

Theorem 2.6. *Assume that condition (2.2), $\sigma_1 \geq \tau_1$, $1 \leq \beta < \gamma$, and $\alpha \leq \beta$, hold. If there exists a positive, nondecreasing sequence $\{\rho_n\}$, and a $N_1 \in \mathbb{N}(n_0)$ with*

$$\lim_{n\to\infty} \sup \sum_{s=N_1}^{n-1} \left[\rho_s D_s B_{s-\sigma_1} - \frac{\left(1+b^{\alpha}+\frac{c^{\beta}}{2^{\beta-1}}\right)}{4} \frac{a_{s-\sigma_1}(\Delta\rho_s)^2}{\rho_{s+1}} \right] = \infty, \tag{2.3}$$

where $D_n = \left(\frac{Q_n}{4^{\beta-1}}\right)^{\eta_1} \eta_1^{-\eta_1} \left(\frac{P_n}{4^{\gamma-1}}\right)^{\eta_2} \eta_2^{-\eta_2}$, $\eta_1 = \frac{\beta-\alpha}{\gamma-\alpha}$, and $\eta_2 = \frac{\gamma-\beta}{\gamma-\alpha}$ holds, then every solution $\{x_n\}$ of equation (1.1) either oscillates or $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0$, and $x_{n-\theta} > 0$ for all $n \geq n_1$. Then we have $z_n > 0$ for all $n \geq n_1$. From equation (1.1), we have

$$\begin{aligned}
&\Delta\left(a_n\Delta^2 z_n\right) + q_n x_{n-\sigma_1}^{\beta} + p_n x_{n+\sigma_2}^{\gamma} + b^{\beta}\Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right) \\
&+ b^{\beta} q_{n-\tau_1} x_{n-\tau_1-\sigma_1}^{\beta} + b^{\beta} p_{n-\tau_1} x_{n-\tau_1+\sigma_2}^{\gamma} + \frac{c^{\beta}}{2^{\beta-1}}\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right) \\
&+ \frac{c^{\beta}}{2^{\beta-1}} q_{n+\tau_2} x_{n+\tau_2-\sigma_1}^{\beta} + \frac{c^{\beta}}{2^{\beta-1}} p_{n+\tau_2} x_{n+\tau_2+\sigma_2}^{\gamma} = 0.
\end{aligned} \tag{2.4}$$

Using Lemma 2.5 in (2.4), we have

$$\begin{aligned}
&\Delta\left(a_n\Delta^2 z_n\right) + b^{\beta}\Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right) + \frac{c^{\beta}}{2^{\beta-1}}\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right) \\
&+ \frac{Q_n}{4^{\beta-1}} z_{n-\sigma_1}^{\frac{\beta}{\alpha}} + \frac{P_n}{4^{\gamma-1}} z_{n+\sigma_2}^{\frac{\gamma}{\alpha}} \leq 0, \ n \geq n_1.
\end{aligned} \tag{2.5}$$

By Lemma 2.1, there are two cases for $\{z_n\}$ for all $n \geq n_1$. First assume that case (I) holds for all $n \geq N_1 \geq n_1$. From 2.5 and $\{z_n\}$ is increasing, we have

$$\begin{aligned}
&\Delta\left(a_n\Delta^2 z_n\right) + b^{\beta}\Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right) + \frac{c^{\beta}}{2^{\beta-1}}\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right) \\
&+ \frac{Q_n}{4^{\beta-1}} z_{n-\sigma_1}^{\frac{\beta}{\alpha}} + \frac{P_n}{4^{\gamma-1}} z_{n-\sigma_1}^{\frac{\gamma}{\alpha}} \leq 0, \ n \geq N_1.
\end{aligned}$$

Let $u_1\eta_1 = \frac{Q_n}{4^{\beta-1}} z_{n-\sigma_1}^{\frac{\beta}{\alpha}}$, and $u_2\eta_2 = \frac{P_n}{4^{\gamma-1}} z_{n-\sigma_1}^{\frac{\gamma}{\alpha}}$. Using the arithmetic-gemetric mean inequality $\frac{u_1\eta_1+u_2\eta_2}{\eta_1+\eta_2} \geq (u_1^{\eta_1} u_2^{\eta_2})^{\frac{1}{\eta_1+\eta_2}}$, and using the fact $\eta_1+\eta_2 = 1$ the last inequality becomes

$$\Delta\left(a_n\Delta^2 z_n\right) + b^{\beta}\Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right) + \frac{c^{\beta}}{2^{\beta-1}}\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right) + D_n z_{n-\sigma_1} \leq 0, \ n \geq N_1. \tag{2.6}$$

Define

$$w_1(n) = \rho_n \frac{a_n \Delta^2 z_n}{\Delta z_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.7}$$

Then $w_1(n) > 0$ for $n \geq N_1$. From (2.7), we see that

$$\Delta w_1(n) = \frac{\Delta \rho_n}{\rho_{n+1}} w_1(n+1) + \rho_n \frac{\Delta\left(a_n \Delta^2 z_n\right)}{\Delta z_{n-\sigma_1}} - w_1(n+1)\left(\frac{\Delta^2 z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}}\right). \tag{2.8}$$

Using the monotonicity of $a_n \Delta^2 z_n$, we have $a_{n-\sigma_1} \Delta^2 z_{n-\sigma_1} \geq a_{n+1} \Delta^2 z_{n+1}$. In view of the monotonicity of Δz_n and from (2.8), we obtain

$$\Delta w_1(n) \leq \frac{\Delta \rho_n}{\rho_{n+1}} w_1(n+1) + \rho_n \frac{\Delta\left(a_n \Delta^2 z_n\right)}{\Delta z_{n-\sigma_1}} - \frac{w_1^2(n+1)}{\rho_{n+1} a_{n-\sigma_1}}, \ n \geq N_1. \tag{2.9}$$

Next, we define

$$w_2(n) = \rho_n \frac{a_{n-\tau_1} \Delta^2 z_{n-\tau_1}}{\Delta z_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.10}$$

Then $w_2(n) > 0$ for $n \geq N_1$. From (2.10), we obtain

$$\Delta w_2(n) = \frac{\Delta \rho_n}{\rho_{n+1}} w_2(n+1) + \rho_n \frac{\Delta\left(a_{n-\tau_1} \Delta^2 z_{n-\tau_1}\right)}{\Delta z_{n-\sigma_1}} - w_2(n+1) \frac{\Delta^2 z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}}. \tag{2.11}$$

Since $a_n \Delta^2 z_n$ is decreasing and $\sigma_1 \geq \tau_1$, we find $a_{n-\sigma_1} \Delta^2 z_{n-\sigma_1} \geq a_{n+1-\tau_1} \Delta^2 z_{n+1-\tau_1}$. Hence by (2.11), we obtain

$$\Delta w_2(n) \leq \frac{\Delta \rho_n}{\rho_{n+1}} w_2(n+1) + \rho_n \frac{\Delta\left(a_{n-\tau_1} \Delta^2 z_{n-\tau_1}\right)}{\Delta z_{n-\sigma_1}} - \frac{w_2^2(n+1)}{\rho_{n+1} a_{n-\sigma_1}}, \ n \geq N_1. \tag{2.12}$$

In the following, we define

$$w_3(n) = \rho_n \frac{a_{n+\tau_2} \Delta^2 z_{n+\tau_2}}{\Delta z_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.13}$$

Then $w_3(n) > 0$ for $n \geq N_1$. From (2.13), we have

$$\Delta w_3(n) = \frac{\Delta \rho_n}{\rho_{n+1}} w_3(n+1) + \rho_n \frac{\Delta\left(a_{n+\tau_2} \Delta^2 z_{n+\tau_2}\right)}{\Delta z_{n-\sigma_1}} - w_3(n+1) \frac{\Delta^2 z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}}.$$

Since $a_{n-\sigma_1} \Delta^2 z_{n-\sigma_1} \geq a_{n+1+\tau_2} \Delta^2 z_{n+1+\tau_2}$, and from the last inequality we obtain

$$\Delta w_3(n) \leq \frac{\Delta \rho_n}{\rho_{n+1}} w_3(n+1) + \rho_n \frac{\Delta\left(a_{n+\tau_2} \Delta^2 z_{n+\tau_2}\right)}{\Delta z_{n-\sigma_1}} - \frac{w_3^2(n+1)}{\rho_{n+1} a_{n-\sigma_1}}, \ n \geq N_1. \tag{2.14}$$

Combining (2.6), (2.9), (2.12) and (2.14), we obtain

$$\Delta w_1(n) + b^{\beta} \Delta w_2(n) + \frac{c^{\beta}}{2^{\beta-1}} \Delta w_3(n) \leq -\rho_n D_n \frac{z_{n+1-\sigma_1}}{\Delta z_{n-\sigma_1}} + \frac{\Delta \rho_n}{\rho_{n+1}} w_1(n+1)$$

$$-\frac{w_1^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}}+b^{\beta}\left(\frac{\Delta\rho_n w_2(n+1)}{\rho_{n+1}}-\frac{w_2^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}}\right)$$
$$+\frac{c^{\beta}}{2^{\beta-1}}\left(\frac{\Delta\rho_n w_3(n+1)}{\rho_{n+1}}-\frac{w_3^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}}\right),\ n\geq N_1. \tag{2.15}$$

On the other hand, by Lemma 2.3, we have

$$\frac{z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}}\geq B_{n-\sigma_1}. \tag{2.16}$$

Using (2.16), with (2.15), and then using the completing the square, we obtain

$$\Delta w_1(n)+b^{\beta}\Delta w_2(n)+\frac{c^{\beta}}{2^{\beta-1}}\Delta w_3(n)\leq-\rho_n D_n B_{n-\sigma_1}$$
$$+\frac{\left(1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}}\right)a_{n-\sigma_1}\left(\Delta\rho_n\right)^2}{4\rho_{n+1}},\ n\geq N_1. \tag{2.17}$$

Summing the last inequality from $N_2\geq N_1$to $n-1$, we obtain

$$\sum_{s=N_2}^{n-1}\left(\rho_s D_s B_{s-\sigma_1}-\frac{\left(1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}}\right)a_{s-\sigma_1}\left(\Delta\rho_s\right)^2}{4\rho_{s+1}}\right)\ \leq\ w_1(N_2)+b^{\beta}w_2(N_2)$$
$$+\ \frac{c^{\beta}}{2^{\beta-1}}w_3(N_2).$$

Taking lim sup in the last inequality, we obtain a contradiction to (2.3).

Assume that case (II) of Lemma 2.1 holds. Then as in Lemma 2.4, we prove that $\lim_{n\to\infty}x_n=0$. This completes the proof.

Theorem 2.7. *Assume that condition (2.2),* $\sigma_1\leq\tau_1$, $1\leq\beta<\gamma$, *and* $\alpha\leq\beta$, *hold. If there exists a positive nondecreasing sequence* $\{\rho_n\}$, *and a* $N_1\in\mathbb{N}(n_0)$ *with*

$$\limsup_{n\to\infty}\sum_{s=N_1}^{n-1}\left(\rho_s D_s B_{s-\sigma_1}-\frac{\left(1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}}\right)a_{s-\tau_1}\left(\Delta\rho_s\right)^2}{4\rho_{s+1}}\right)=\infty \tag{2.18}$$

holds, then every solution $\{x_n\}$ *of equation (1.1) either oscillates or* $\lim_{n\to\infty}x_n=0$.

Proof. Proceeding as in the proof of Theorem 2.6, we obtain (2.5). By Lemma 2.1, there are two cases for $\{z_n\}$. Assume that case (I) holds for all $n\geq N_1$. Then, we obtain (2.6). Using the following transformations

$$w_1(n)=\rho_n\frac{a_n\Delta^2 z_n}{\Delta z_{n-\tau_1}},\quad n\geq N_1,$$

$$w_2(n) = \rho_n \frac{a_{n-\tau_1}\Delta^2 z_{n-\tau_1}}{\Delta z_{n-\tau_1}}, \quad n \geq N_1,$$

and

$$w_3(n) = \rho_n \frac{a_{n+\tau_2}\Delta^2 z_{n+\tau_2}}{\Delta z_{n-\tau_1}}, \quad n \geq N_1,$$

and similar to the proof of Theorem 2.6, we obtain

$$\Delta w_1(n) + b^\beta \Delta w_2(n) + \frac{c^\beta}{2^{\beta-1}}\Delta w_3(n) \leq -\rho_n D_n \frac{z_{n+1-\sigma_1}}{\Delta z_{n-\tau_1}} + \frac{\Delta\rho_n}{\rho_{n+1}} w_1(n+1)$$

$$-\frac{w_1^2(n+1)}{\rho_{n+1}a_{n-\tau_1}} + b^\beta\left(\frac{\Delta\rho_n}{\rho_{n+1}} - \frac{w_2^2(n+1)}{\rho_{n+1}a_{n-\tau_1}}\right)$$

$$+\frac{c^\beta}{2^{\beta-1}}\left(\frac{\Delta\rho_n}{\rho_{n+1}} - \frac{w_3^2(n+1)}{\rho_{n+1}a_{n-\tau_1}}\right). \tag{2.19}$$

On the other hand, we find from Lemma 2.3 that

$$\frac{z_{n-\sigma_1}}{\Delta z_{n-\tau_1}} = \frac{z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}} . \frac{\Delta z_{n-\sigma_1}}{\Delta z_{n-\tau_1}} \geq B_{n-\sigma_1},$$

due to $\tau_1 \geq \sigma_1$, and $\Delta^2 z_n \geq 0$ for all $n \geq N_1$. Combining the above inequality with (2.19), and then using the completing the square, we obtain

$$\Delta w_1(n) + b^\beta \Delta w_2(n) + \frac{c^\beta}{2^{\beta-1}}\Delta w_3(n) \leq -\rho_n D_n B_{n-\sigma_1}$$

$$+\frac{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)a_{n-\tau_1}(\Delta\rho_s)^2}{4\rho_{n+1}}, \; n \geq N_1. \tag{2.20}$$

Summing the last inequality from $N_2 \geq N_1$ to $n-1$, we obtain

$$\sum_{s=N_2}^{n-1}\left(\rho_s D_s B_{s-\sigma_1} - \frac{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)a_{s-\tau_1}(\Delta\rho_s)^2}{4\rho_{s+1}}\right) < w_1(N_2) + b^\beta w_2(N_2)$$

$$+\frac{c^\beta}{2^{\beta-1}} w_3(N_2).$$

Taking lim sup on both sides of the last inequality, we obtain a contradiction with (2.18).

Assume that case (II) holds. Then by Lemma 2.4, we proved that $\lim_{n\to\infty} x_n = 0$. The proof is now complete.

Theorem 2.8. *Assume that condition (2.2), $\sigma_1 \geq \tau_1$, $1 \leq \gamma < \beta$, and $\alpha \leq \gamma$ hold. If there exists a positive, nondecreasing sequence $\{\rho_n\}$, and a $N_1 \in \mathbb{N}(n_0)$ with*

$$\limsup_{n\to\infty} \sum_{s=N_1}^{n-1} \left[\rho_s D_s B_{s-\sigma_1} - \frac{\left(1+b^\gamma+\frac{c^\gamma}{2^{\gamma-1}}\right)}{4} \frac{a_{s-\sigma_1}(\Delta\rho_s)^2}{\rho_{s+1}} \right] = \infty, \tag{2.21}$$

where $D_n = \left(\frac{Q_n}{4^{\beta-1}}\right)^{\eta_1} \eta_1^{-\eta_1} \left(\frac{P_n}{4^{\gamma-1}}\right)^{\eta_2} \eta_2^{-\eta_2}$, $\eta_1 = \frac{\gamma-\alpha}{\beta-\alpha}$, and $\eta_2 = \frac{\beta-\gamma}{\beta-\alpha}$ holds, then every solution $\{x_n\}$ of equation (1.1) either oscillates or $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a $N \in \mathbb{N}(n_0)$ such that $x_n > 0$, and $x_{n-\theta} > 0$ for all $n \geq N$. Then we have $z_n > 0$ for $n \geq N$. From equation (1.1), we have

$$\begin{aligned} &\Delta\left(a_n\Delta^2 z_n\right) + q_n x_{n-\sigma_1}^{\beta} + p_n x_{n+\sigma_2}^{\gamma} + b^\gamma \Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right) \\ &+ b^\gamma q_{n-\tau_1} x_{n-\tau_1-\sigma_1}^{\beta} + b^\gamma p_{n-\tau_1} x_{n-\tau_1+\sigma_2}^{\gamma} + \frac{c^\gamma}{2^{\gamma-1}}\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right) \\ &+ \frac{c^\gamma}{2^{\gamma-1}} q_{n+\tau_2} x_{n+\tau_2-\sigma_1}^{\beta} + \frac{c^\gamma}{2^{\gamma-1}} p_{n+\tau_2} x_{n+\tau_2+\sigma_2}^{\gamma} = 0,\ n \geq N. \end{aligned} \tag{2.22}$$

Using Lemma 2.1 in (2.22), we have

$$\begin{aligned} &\Delta\left(a_n\Delta^2 z_n\right) + b^\gamma \Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right) + \frac{c^\gamma}{2^{\gamma-1}}\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right) \\ &+ \frac{Q_n}{4^{\beta-1}} z_{n-\sigma_1}^{\frac{\beta}{\alpha}} + \frac{P_n}{4^{\gamma-1}} z_{n+\sigma_2}^{\frac{\gamma}{\alpha}} \leq 0,\ n \geq N. \end{aligned} \tag{2.23}$$

By Lemma 2.1, there are two cases for $\{z_n\}$ for all $n \geq N$. First assume that case (I) holds for all $n \geq N_1 \geq N$. From (2.23), and using arithmetic-geometric inequality we obtain

$$\begin{aligned} &\Delta\left(a_n\Delta^2 z_n\right) + b^\gamma \Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right) + \frac{c^\gamma}{2^{\gamma-1}}\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right) \\ &+ D_n z_{n-\sigma_1} \leq 0,\ n \geq N_1. \end{aligned} \tag{2.24}$$

Define

$$w_1(n) = \rho_n \frac{a_n\Delta^2 z_n}{\Delta z_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.25}$$

Then $w_1(n) > 0$ for $n \geq N_1$. From (2.25), we see that

$$\Delta w_1(n) = \frac{\Delta\rho_n}{\rho_{n+1}} w_1(n+1) + \rho_n \frac{\Delta\left(a_n\Delta^2 z_n\right)}{\Delta z_{n-\sigma_1}} - w_1(n+1)\left(\frac{\Delta^2 z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}}\right). \tag{2.26}$$

By case (I) in Lemma 2.1, we have $a_{n-\sigma_1}\Delta^2 z_{n-\sigma_1} \geq a_{n+1}\Delta^2 z_{n+1}$. Thus from (2.26), we obtain

$$\Delta w_1(n) \leq \frac{\Delta\rho_n}{\rho_{n+1}} w_1(n+1) + \rho_n \frac{\Delta\left(a_n\Delta^2 z_n\right)}{\Delta z_{n-\sigma_1}} - \frac{w_1^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}}. \tag{2.27}$$

Next, we define

$$w_2(n) = \rho_n \frac{a_{n-\tau_1}\Delta^2 z_{n-\tau_1}}{\Delta z_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.28}$$

Then $w_2(n) > 0$ for $n \geq N_1$. Then from (2.28), we obtain

$$\Delta w_2(n) = \frac{\Delta\rho_n}{\rho_{n+1}} w_2(n+1) + \rho_n \frac{\Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right)}{\Delta z_{n-\sigma_1}} - w_2(n+1)\frac{\Delta^2 z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}}. \tag{2.29}$$

Note that $\sigma_1 \geq \tau_1$, and from case (I) of Lemma 2.1, we find

$$a_{n-\sigma_1}\Delta^2 z_{n-\sigma_1} \geq a_{n+1-\tau_1}\Delta^2 z_{n+1-\tau_1}.$$

Hence by (2.29), we obtain

$$\Delta w_2(n) \leq \frac{\Delta\rho_n}{\rho_{n+1}} w_2(n+1) + \rho_n \frac{\Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right)}{\Delta z_{n-\sigma_1}} - \frac{w_2^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}},\ n \geq N_1. \tag{2.30}$$

In the following, we define

$$w_3(n) = \rho_n \frac{a_{n+\tau_2}\Delta^2 z_{n+\tau_2}}{\Delta z_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.31}$$

Then $w_3(n) > 0$ for $n \geq N_1$. From (2.31), we have

$$\Delta w_3(n) = \frac{\Delta\rho_n}{\rho_{n+1}} w_3(n+1) + \rho_n \frac{\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right)}{\Delta z_{n-\sigma_1}} - w_3(n+1)\frac{\Delta^2 z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}}. \tag{2.32}$$

By case (I) of Lemma 2.1, we obtain $a_{n-\sigma_1}\Delta^2 z_{n-\sigma_1} \geq a_{n+1+\tau_2}\Delta^2 z_{n+1+\tau_2}$. Then by (2.32), we obtain

$$\Delta w_3(n) \leq \frac{\Delta\rho_n}{\rho_{n+1}} w_3(n+1) + \rho_n \frac{\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right)}{\Delta z_{n-\sigma_1}} - \frac{w_3^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}},\ n \geq N_1. \tag{2.33}$$

Therefore by (2.27), (2.30) and (2.33), we obtain

$$\begin{aligned} \Delta w_1(n) + b^\gamma \Delta w_2(n) + \frac{c^\gamma}{2^{\gamma-1}}\Delta w_3(n) &\leq -\rho_n A_n \frac{z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}} + \frac{\Delta\rho_n}{\rho_{n+1}} w_1(n+1) \\ &- \frac{w_1^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}} + b^\gamma\left(\frac{\Delta\rho_n w_2(n+1)}{\rho_{n+1}} - \frac{w_2^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}}\right) \\ &+ \frac{c^\gamma}{2^{\gamma-1}}\left(\frac{\Delta\rho_n w_3(n+1)}{\rho_{n+1}} - \frac{w_3^2(n+1)}{\rho_{n+1}a_{n-\sigma_1}}\right),\ n \geq N_1. \end{aligned} \tag{2.34}$$

On the other hand, by Lemma 2.3, we have

$$\frac{z_{n-\sigma_1}}{\Delta z_{n-\sigma_1}} \geq B_{n-\sigma_1},\ n \geq N_1. \tag{2.35}$$

Combining (2.34) with (2.35), and then using the completing the square, we obtain

$$\Delta w_1(n) + b^\gamma \Delta w_2(n) + \frac{c^\gamma}{2^{\gamma-1}} \Delta w_3(n) \leq -\rho_n D_n B_{n-\sigma_1}$$

$$+\frac{\left(1+b^\gamma+\frac{c^\gamma}{2^{\gamma-1}}\right) a_{n-\sigma_1} (\Delta\rho_n)^2}{4\rho_{n+1}},\ n \geq N_1. \tag{1}$$

Summing the last inequality from $N_2 \geq N_1$to $n-1$, we obtain

$$\begin{aligned}\sum_{s=N_2}^{n-1}\left(\rho_s D_s B_{s-\sigma_1} - \frac{\left(1+b^\gamma+\frac{c^\gamma}{2^{\gamma-1}}\right) a_{s-\sigma_1} (\Delta\rho_s)^2}{4\rho_{s+1}}\right) &\leq w_1(N_2) + b^\gamma w_2(N_2) \\ &+ \frac{c^\gamma}{2^{\gamma-1}} w_3(N_2).\end{aligned}$$

Taking lim sup in the last inequality, we obtain a contradiction to (2.21).

Assume that case (II) of Lemma 2.1 holds. Then by Lemma 2.4, we proved that $\lim_{n\to\infty} x_n = 0$. This completes the proof.

Theorem 2.9. *Assume that condition (2.2),* $\sigma_1 \leq \tau_1$, $1 \leq \gamma < \beta$, *and* $\alpha \leq \gamma$ *hold. If there exists a positive, nondecreasing sequence* $\{\rho_n\}$, *and a* $N_1 \in \mathbb{N}(n_0)$ *with*

$$\limsup_{n\to\infty} \sum_{s=N_1}^{n-1}\left(\rho_s D_s B_{s-\sigma_1} - \frac{\left(1+b^\gamma+\frac{c^\gamma}{2^{\gamma-1}}\right) a_{s-\tau_1} (\Delta\rho_s)^2}{4\rho_{s+1}}\right) = \infty \tag{2.38}$$

holds, then every solution $\{x_n\}$ *of equation (1.1)either oscillates or* $\lim_{n\to\infty} x_n = 0$.

Proof. Proceeding as in the proof of Theorem 2.8, we obtain (2.23). By Lemma 2.1, there are two cases for $\{z_n\}$ for $n \geq N$ Assume that case (I) holds for all $n \geq N_1 \geq N$. Then, we obtain (2.24). Using the following transformations

$$w_1(n) = \rho_n \frac{a_n \Delta^2 z_n}{\Delta z_{n-\tau_1}}, \quad n \geq N_1,$$

$$w_2(n) = \rho_n \frac{a_{n-\tau_1} \Delta^2 z_{n-\tau_1}}{\Delta z_{n-\tau_1}}, \quad n \geq N_1,$$

and

$$w_3(n)=\rho_n\frac{a_{n+\tau_2}\Delta^2 z_{n+\tau_2}}{\Delta z_{n-\tau_1}},\quad n\geq N_1,$$

and similar to the proof of Theorem 2.8, we obtain

$$\begin{aligned}\Delta w_1(n)+b^\gamma\Delta w_2(n)+\frac{c^\gamma}{2^{\gamma-1}}\Delta w_3(n) &\leq -\rho_n D_n\frac{z_{n-\sigma_1}}{\Delta z_{n-\tau_1}}+\frac{\Delta\rho_n}{\rho_{n+1}}w_1(n+1)\\ &-\frac{w_1^2(n+1)}{\rho_{n+1}a_{n-\tau_1}}+b^\gamma\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{w_2^2(n+1)}{\rho_{n+1}a_{n-\tau_1}}\right)\\ &+\frac{c^\gamma}{2^{\gamma-1}}\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{w_3^2(n+1)}{\rho_{n+1}a_{n-\tau_1}}\right).\end{aligned}\tag{2.39}$$

On the other hand, we have by Lemma 2.3,

$$\frac{z_{n-\sigma_1}}{\Delta z_{n-\tau_1}}=\frac{z_{n+1-\sigma_1}}{\Delta z_{n-\sigma_1}}.\frac{\Delta z_{n-\sigma_1}}{\Delta z_{n-\tau_1}}\geq B_{n-\sigma_1},$$

due to $\tau_1\geq\sigma_1$, and $\Delta^2 z_n\geq 0$ for all $n\geq N_2\geq N_1$. Combining the above inequality with (2.39), and then using the completing the square, we obtain

$$\begin{aligned}\Delta w_1(n)+b^\gamma\Delta w_2(n)+\frac{c^\gamma}{2^{\gamma-1}}\Delta w_3(n) &\leq -\rho_n D_n B_{n-\sigma_1}\\ &+\frac{\left(1+b^\gamma+\frac{c^\gamma}{2^{\gamma-1}}\right)a_{n-\tau_1}(\Delta\rho_s)^2}{4\rho_{n+1}},\ n\geq N_2.\end{aligned}$$

Summing the last inequality from N_2 to $n-1$, we obtain

$$\sum_{s=N_2}^{n-1}\left(\rho_s D_s B_{s-\sigma_1}-\frac{\left(1+b^\gamma+\frac{c^\gamma}{2^{\gamma-1}}\right)a_{s-\tau_1}(\Delta\rho_s)^2}{4\rho_{s+1}}\right) < w_1(N_2)+b^\gamma w_2(N_2)+\frac{c^\gamma}{2^{\gamma-1}}w_3(N_2).$$

Taking lim sup on both sides of the last inequality, we obtain a contradiction with (2.38).

Assume that case (II) of Lemma 2.1 holds. Then by Lemma 2.4, we prove that $\lim_{n\to\infty}x_n=0$. The proof is now complete.

Theorem 2.10. *Assume that condition (2.2), $\sigma_1\geq\tau_1+1,\ 1\leq\beta<\gamma$ and $\alpha\leq\beta$ hold. If*

$$\liminf_{n\to\infty}\sum_{s=n+\tau_1-\sigma_1}^{n-1}D_s R_{s-\sigma_1}>\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)\left(\frac{\sigma_1-\tau_1}{\sigma_1-\tau_1+1}\right)^{\sigma_1-\tau_1+1},\tag{2.40}$$

where $D_n=\left(\frac{Q_n}{4^{\beta-1}}\right)^{\eta_1}\eta_1^{-\eta_1}\left(\frac{P_n}{4^{\gamma-1}}\right)^{\eta_2}\eta_2^{-\eta_2}$, $\eta_1=\frac{\beta-\alpha}{\gamma-\alpha}$, and $\eta_2=\frac{\gamma-\beta}{\gamma-\alpha}$ holds, then every solution $\{x_n\}$ of equation (1.1) either oscillates or $\lim_{n\to\infty}x_n=0$.

Proof. Proceeding as in the proof of Theorem 2.6, we obtain (2.6), for all $n \geq N_1$. Set

$$w_n = a_n \Delta^2 z_n + b^\beta a_{n-\tau_1} \Delta^2 z_{n-\tau_1} + \frac{c^\beta}{2^{\beta-1}} a_{n+\tau_2} \Delta^2 z_{n+\tau_2},$$

then $w_n > 0$, and

$$w_n \leq (1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}) a_{n-\tau_1} \Delta^2 z_{n-\tau_1},\ n \geq N_1. \tag{2.41}$$

In view of Lemma 2.2, and (2.41) we have

$$R_{n-\sigma_1} w_{n+\tau_1-\sigma_1} \leq (1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}) z_{n-\sigma_1},\ n \geq N_1. \tag{2.42}$$

Using (2.42) in (2.6), we obtain

$$\Delta w_n + \left(\frac{D_n R_{n-\sigma_1}}{1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}} \right) w_{n+\tau_1-\sigma_1} \leq 0,\ n \geq N_1.$$

In view of condition (2.40) and Theorem 7.5.1 of [9], we see that the last inequality has no positive solution, which is a contradiction. The proof of case (II) of Lemma 2.1 is similar to that of Theorem 2.6. The proof of the theorem is now complete.

Theorem 2.11. *Assume that condition (2.2),* $\sigma_1 \geq \tau_1 + 1,\ 1 \leq \gamma < \beta$, *and* $\alpha \leq \gamma$ *hold. If*

$$\liminf_{n\to\infty} \sum_{s=n+\tau_1-\sigma_1}^{n-1} D_s R_{s-\sigma_1} > \left(1 + b^\gamma + \frac{c^\gamma}{2^{\gamma-1}}\right) \left(\frac{\sigma_1 - \tau_1}{\sigma_1 - \tau_1 + 1}\right)^{\sigma_1 - \tau_1 + 1}, \tag{2.43}$$

where $D_n = \left(\frac{Q_n}{4^{\beta-1}}\right)^{\eta_1} \eta_1^{-\eta_1} \left(\frac{P_n}{4^{\gamma-1}}\right)^{\eta_2} \eta_2^{-\eta_2}$, $\eta_1 = \frac{\gamma-\alpha}{\beta-\alpha}$, *and* $\eta_2 = \frac{\beta-\gamma}{\beta-\alpha}$ *holds, then every solution* $\{x_n\}$ *of equation (1.1) either oscillates or* $\lim\limits_{n\to\infty} x_n = 0$.

Proof. Proceeding as in proof of Theorem 2.8, we arrive at (2.24).The rest of the proof is similar to that of Theorem 2.10, and hence the details are omitted.

In the final result of the paper, we assume $\alpha = \beta = \gamma$, and therefore no condition on b and c is required.

Theorem 2.12. *Assume that* $\alpha = \beta = \gamma = 1, \sigma_1 \geq \tau_1 + 1$, *and condition (2.2) hold. If*

$$\limsup_{n\to\infty} \sum_{s=n+\tau_1-\sigma_1}^{n-1} (Q_s + P_s) R_{s-\sigma_1} \geq (1 + b + c) \left(\frac{\sigma_1 - \tau_1}{\sigma_1 - \tau_1 + 1}\right)^{\sigma_1 - \tau_1 + 1} \tag{2.44}$$

holds, then every solution $\{x_n\}$ *of equation (1.1) either oscillates or* $\lim_{n\to\infty} x_n = 0$.

Proof. Proceeding as in the proof of Theorem 2.6, we have

$$\Delta\left(a_n\Delta^2 z_n\right)+b\Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right)+c\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right)$$

$$+Q_n z_{n-\sigma_1}+P_n z_{n+\sigma_2}\leq 0,\ n\geq N_1.$$

By Lemma 2.1, there are two cases for $\{z_n\}$ for all $n\geq n_1$. First assume that case (I) holds for all $n\geq N_1\geq n_1$. From the last inequality we obtain

$$\Delta\left(a_n\Delta^2 z_n\right)+b\Delta\left(a_{n-\tau_1}\Delta^2 z_{n-\tau_1}\right)+c\Delta\left(a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right)$$

$$+(Q_n+P_n)z_{n-\sigma_1}\leq 0,\ n\geq N_1, \tag{2.45}$$

where we have used $\{z_n\}$ is positive and nondecreasing. From (2.45) and Lemma 2.2, we have

$$\Delta\left(a_n\Delta^2 z_n+ba_{n-\tau_1}\Delta^2 z_{n-\tau_1}+c\Delta a_{n+\tau_2}\Delta^2 z_{n+\tau_2}\right)$$

$$+(Q_n+P_n)R_{n-\sigma_1}a_{n-\sigma_1}\Delta^2 z_{n-\sigma_1}\leq 0,\ n\geq N_1. \tag{2.46}$$

Let $w_n=a_n\Delta^2 z_n+ba_{n-\tau_1}\Delta^2 z_{n-\tau_1}+c\Delta a_{n+\tau_2}\Delta^2 z_{n+\tau_2}$, then $w_n>0$ for all $n\geq N_1$, and since $\{a_n\Delta^2 z_n\}$ is positive and decreasing, we have

$$w_n \;\leq\; (1+b+c)a_{n-\tau_1}\Delta^2 z_{n-\tau_1},\ n\geq N_1,$$

or

$$w_{n+\tau_1-\sigma_1}\leq(1+b+c)a_{n-\tau_1}\Delta^2 z_{n-\sigma_1},\ n\geq N_1. \tag{2.47}$$

Using (2.47) in (2.46), we obtain

$$\Delta w_n+\frac{(Q_n+P_n)R_{n-\sigma_1}}{1+b+c}w_{n+\tau_1-\sigma_1}\leq 0,\ n\geq N_1. \tag{2.48}$$

By condition (2.44) and Theorem 7.5.1 of [9], we see that (2.48) has no positive solution, which is a contradiction. The proof of case (II) is similar to previous theorems. The proof is now complete.

3. Examples

In this section, we present three examples to illustrate the main results.

Example 3.1. *Consider the third order difference equation*

$$\Delta(\frac{1}{2^n}\Delta^2(x_n+\frac{1}{2}x_{n-1}+2x_{n+1}))+2^{n+2}x_{n-2}^3+\frac{2^{3n}}{8}x_{n+1}^5=0,\ n\geq 1. \tag{3.1}$$

Here $a_n=\frac{1}{2^n}$, $b_n=\frac{1}{2}, c_n=2$, $q_n=2^{n+2}$, $p_n=\frac{2^{3n}}{8}$, $\tau_1=1$, $\tau_2=1$, $\sigma_1=2$, $\sigma_2=1$, $\alpha=1$, $\beta=3$, and $\gamma=5$. Then $\eta_1=\frac{1}{2}$, $\eta_2=\frac{1}{2}$, $Q_n=2^{n+1}$, and $P_n=\frac{2^{3n}}{64}$. By taking $\rho_n=1$, it is easy to see that the conditions (2.2) and (2.3) hold. Hence by Theorem 2.6, every solution of equation (3.1) either oscillates or $\lim_{n\to\infty} x_n=0$. In fact, $\{x_n\}=\left\{\frac{1}{2^n}\right\}$ is one such nonoscillatory solution of equation (3.1), which tends to zero as $n\to\infty$.

Example 3.2. *Consider the third order difference equation*

$$\Delta(n\Delta^2(x_n+\frac{1}{2}x_{n-2}+\frac{3}{2}x_{n+1})^{5/3})+2^n x_{n-3}^5+2^n x_{n+2}^3=0,\ n\geq 1. \tag{3.2}$$

Here $a_n=n$, $b_n=\frac{1}{2}, c_n=\frac{3}{2}$, $q_n=2^n$, $p_n=2^n$, $\tau_1=2$, $\tau_2=1$, $\sigma_1=3$, $\sigma_2=2$, $\alpha=5/3$, $\beta=5$, and $\gamma=3$. Then $\eta_1=\frac{2}{5}$, $\eta_2=\frac{3}{5}$, $Q_n=2^{n-2}$, and $P_n=2^{n-2}$. By taking $\rho_n=1$, it is easy to verify that all conditions of Theorem 2.8 are satisfied, and hence every solution of equation (3.2) either oscillates or tends to zero as $n\to\infty$. In fact, $\{x_n\}=\{(-1)^n\}$ is one such oscillatory solution of equation (3.2).

Example 3.3. *Consider the third order difference equation*

$$\Delta(\frac{1}{n}\Delta^2(x_n+3x_{n-3}+2x_{n+2}))+(n+3)x_{n-4}+(n+3)x_{n+3}=0,\ n\geq 1. \tag{2}$$

Here $a_n=\frac{1}{n}$, $b_n=3, c_n=2$, $q_n=n+3$, $p_n=n+3$, $\tau_1=3$, $\tau_2=2$, $\sigma_1=4$, $\sigma_2=3$, and $\alpha=\beta=\gamma=1$. Then $Q_n=n$, $P_n=n$, and $R_n=\frac{n(n-1)(n-2)}{6}$. Now one can easily verify that all conditions of Theorem 2.12 are satisfied, and hence every solution of equation (3.3) either oscillates or tends to zero as $n\to\infty$. In fact, $\{x_n\}=\{(-1)^n\}$ is one such oscillatory solution of equation (3.3).

We conclude this paper with the following remark.

Remark 3.4. 1. The established results are presented in a form which is essentially new and include some of the existing results as special cases.

2. The existing results [5, 6, 24] cannot to be applied to equations (3.1), (3.2) and (3.3), since either $\{a_n\}$ is nonincreasing, or $\beta\neq\gamma$.

3. It is interesting to study the equation (1.1) when $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$.

4. The results of this paper may be extend to equation of the form

$$\Delta\left(a_n(\Delta^2\left(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2}\right))^{\alpha}\right) + q_n x_{n-\sigma_1}^{\beta} + p_n x_{n+\sigma_2}^{\gamma} = 0,$$

when $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} = \infty$ or $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} < \infty$, and the details are left to the reader.

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