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INEQUALITIES VIA STRONGLY p-HARMONIC LOG-CONVEX FUNCTIONS

MUHAMMAD ASLAM NOOR[1,2,*], KHALIDA INAYAT NOOR[2], SABAH IFTIKHAR[2]

[1]Mathematics Department, King Saud University, Riyadh, Saudi Arabia

[2]Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan

Abstract. In this paper, we consider and investigate a new class of harmonic convex functions, which is called the strongly p-harmonic log-convex function. We establish some new integral inequalities of Hermite-Hadamard type for the product of strongly p-harmonic log-convex functions and related convex functions. Results obtained in this paper may be starting point for further research.

Keywords. Harmonic p-convex function; Strongly harmonic convex function; Strongly harmonic log-convex function.

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1. Introduction

The theory of convexity has been subject to extensive research during the past few years due to its utility in various branches of pure and applied mathematics. The concept of convexity has been extended and generalized in several directions. The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities. For useful details and generalization of Hermite-Hadamard inequalities, see [1, 2, 3, 4, 5, 6, 7, 8]. The harmonic convex function, was introduced and studied by Anderson *et al.* [9] and Iscan [1]. Iscan [3] introduced the concept of harmonic s-convex function in second sense. A

*Corresponding author.

E-mail addresses: noormaslam@gmail.com (M.A. Noor), khalidan@gmail.com (K.I. Noor), sabah.iftikhar22@gmail.com (S. Iftikhar).

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significant class of convex functions is that of strongly convex functions introduced by Polyak [10]. Motivated by the work of Polyak [10], Noor *et al.* [11, 12] considered the strongly harmonic convex functions. We would like to emphasize that strongly convex functions and strongly harmonic convex functions are two different extensions and generalizations of convex functions, introduced by *Noor et al.*; see [11, 12]. They obtained several Hadamard type integral inequalities. Noor *et al.* [11] have shown that a function is strongly harmonic convex function, if and only if, it satisfies the inequality

$$\begin{aligned}
f\left(\frac{2ab}{a+b}\right)+\frac{c}{12}\|\frac{a-b}{ab}\|^2 &\leq \frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right)+f\left(\frac{4ab}{3a+b}\right)\right]+\frac{c}{48}\|\frac{a-b}{ab}\|^2\\
&\leq \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\\
&\leq \frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{c}{24}\|\frac{a-b}{ab}\|^2\\
&\leq \frac{f(a)+f(b)}{2}-\frac{c}{6}\|\frac{a-b}{ab}\|^2 \quad \forall a,b\in[a,b].
\end{aligned}$$

which is called Hermite-Hadamard type inequality for strongly harmonic convex functions. Several integral inequalities for various types of strongly harmonic convex functions have been obtained in recent years, see [11, 13, 14] and references therein.

In this paper, we introduce and consider a new class of strongly p-harmonic log-convex function with modulus c. We obtain some new integral inequalities for product of this new class with other harmonic p-convex functions. We believe that our findings are new, novel and better than those already exist. The technique and ideas of this paper may open new ways for further research in this field.

2. Preliminaries

Definition 2.1. [12]. A set $I=[a,b]\subseteq\mathbb{R}\setminus\{0\}$ is said to be a harmonic p-convex set, where $p\neq 0$, if

$$\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\in I, \qquad \forall x,y\in I, t\in[0,1].$$

Definition 2.2. *Let $h: J \subset [0,1] \to \mathbb{R}$ be a non negative function. A function $f: I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be strongly harmonic (p,h)-convex function on I, if*

$$f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq h(1-t)f(x)+h(t)f(y)-ct(1-t)\left(\frac{x^p-y^p}{x^p y^p}\right)^2, \qquad \forall x,y \in I, t \in [0,1].$$

A function f is strongly harmonic (p,h)-concave, if $-f$ is strongly harmonic (p,h)-convex function. With $p=1$, this class reduces to the class of strongly harmonic h-convex function, introduced by Noor *et al.* [14]. This shows that this new class is more general and includes all previously know class as a special case. If $t=\frac{1}{2}$, then strongly harmonic (p,h)-convex function reduces to

$$f\left(\left[\frac{2x^p y^p}{x^p+y^p}\right]^{\frac{1}{p}}\right) \leq h\left(\frac{1}{2}\right)[f(x)+f(y)]-\frac{c}{4}\left(\frac{x^p-y^p}{x^p y^p}\right)^2, \qquad \forall x,y \in I,$$

which is called Jensen type strongly harmonic (p,h)-convex function.

Now we discuss some special cases of strongly harmonic (p,h)-convex functions:

I. If $h(t)=t^s$ and $c=0$, then Definition 2.2 reduces to:

Definition 2.3. A function $f: I=[a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic s-convex function, where $s \in [-1,1]$, if

$$f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq (1-t)^s f(x)+t^s f(y), \qquad \forall x,y \in I,\ t \in (0,1).$$

II. If $h(t)=1$ and $c=0$, then Definition 2.2 reduces to:

Definition 2.4. A function $f: I=[a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be p-harmonic P-function, if

$$f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq f(x)+f(y), \qquad \forall x,y \in I.$$

Definition 2.5. A function $f: I=[a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic (p,m)-convex function, where $m \in (0,1]$, if

$$f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq m(1-t)f(xm)+tf(y), \qquad \forall x,y \in I,\ t \in [0,1].$$

Definition 2.6. A function $f: I=[a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be p-harmonic quasi convex function, where $m \in (0,1]$, if

$$f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq \max\{f(x),f(y)\}, \qquad \forall x,y \in I.$$

Definition 2.7. A function $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be strongly p-harmonic log-convex function on I, if

$$f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \le \big(f(x)\big)^{1-t}\big(f(y)\big)^{t} - ct(1-t)\Big(\frac{x^p-y^p}{x^p y^p}\Big)^2, \qquad \forall x,y \in I, t\in[0,1]. \tag{2.1}$$

A function f is strongly p-harmonic log-concave, if $-f$ is strongly harmonic log-convex function. If $t=\frac{1}{2}$, then strongly p-harmonic log-convex function reduces to

$$f\left(\left[\frac{2x^p y^p}{x^p+y^p}\right]^{\frac{1}{p}}\right) \le \sqrt{f(x)f(y)} - \frac{c}{4}\Big(\frac{x^p-y^p}{x^p y^p}\Big)^2, \qquad \forall x,y\in I,$$

which is called Jensen type strongly p-harmonic log-convex function. From (2.1), we have

$$\begin{aligned} f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) &\le \big(f(x)\big)^{1-t}\big(f(y)\big)^{t} - ct(1-t)\Big(\frac{x^p-y^p}{x^p y^p}\Big)^2 \\ &\le (1-t)f(x)+tf(y) - ct(1-t)\Big(\frac{x^p-y^p}{x^p y^p}\Big)^2 \\ &\le \max\{f(x),f(y)\} - ct(1-t)\Big(\frac{x^p-y^p}{x^p y^p}\Big)^2. \end{aligned}$$

Remark 2.8.

(1) For $p=1$, Definition 2.7 reduces to the definition of strongly harmonic log-convex function introduced by Noor *et al.* [11].

(2) For $p=-1$, Definition 2.7 reduces to the definition of strongly log-convex function, see [15].

3. Main results

Now we establish some new Hermite-Hadamard type inequalities involving the product of strongly p-harmonic log-convex and other harmonic p-convex functions.

Theorem 3.1. *Let $f,g : I=[a,b]\subseteq \mathbb{R}\setminus\{0\}\to\mathbb{R}_+$ be strongly p-harmonic log-convex functions, respectively with modulus $c>0$. If g^q is strongly p-harmonic log-convex function and $q\ge 1$,* *then*

$$\begin{aligned}&\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x\\&\leq\left\{\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right\}^{1-\frac{1}{q}}\times\left\{\frac{f(b)g^q(b)-f(a)g^q(a)}{\ln\left(\frac{f(b)}{f(a)}\right)+q\ln\left(\frac{g(b)}{g(a)}\right)}\right.\\&\left.-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left[\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}+\frac{g^q(b)-g^q(a)}{q[\ln g(b)-\ln g(a)]}\right]+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\right\}^{\frac{1}{q}}.\end{aligned}\tag{3.1}$$

Proof. Using power mean inequality, strongly p-harmonic log-convexity of the functions f and g^q, respectively, we have

$$\begin{aligned}&\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x\\&=\int_0^1 f\left[\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\&\leq\left[\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]^{1-\frac{1}{q}}\\&\quad\times\left[\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g^q\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]^{\frac{1}{q}}\\&\leq\left\{\int_0^1\left[\big(f(a)\big)^{1-t}\big(f(b)\big)^t-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\mathrm{d}t\right\}^{1-\frac{1}{q}}\\&\quad\times\left\{\int_0^1\left[\big(f(a)\big)^{1-t}\big(f(b)\big)^t-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\right.\\&\quad\left.\times\left[\big(g(a)\big)^{q(1-t)}\big(g(b)\big)^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\mathrm{d}t\right\}^{\frac{1}{q}}\\&\leq\left\{f(a)\int_0^1\left[\frac{f(b)}{f(a)}\right]^t\mathrm{d}t-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\int_0^1 t(1-t)\mathrm{d}t\right\}^{1-\frac{1}{q}}\\&\quad\times\left\{f(a)g^q(a)\int_0^1\left(\frac{f(b)}{f(a)}\right)^t\left(\frac{g(b)}{g(a)}\right)^{qt}-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\int_0^1\left[f(a)\left(\frac{f(b)}{f(a)}\right)^t\right.\right.\\&\quad\left.\left.+g^q(a)\left(\frac{g(b)}{g(a)}\right)^{qt}\right]t(1-t)+c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4\int_0^1 t^2(1-t)^2\mathrm{d}t\right\}^{\frac{1}{q}}\\&\leq\left\{\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right\}^{1-\frac{1}{q}}\times\left\{\frac{\big(f(b)g^q(b)-f(a)g^q(a)}{\ln\left(\frac{f(b)}{f(a)}\right)+q\ln\left(\frac{g(b)}{g(a)}\right)}\right.\\&\quad\left.-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left[\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}+\frac{g^q(b)-g^q(a)}{q[\ln g(b)-\ln g(a)]}\right]+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\right\}^{\frac{1}{q}}.\end{aligned}$$

This completes the proof.

We know discuss some special cases.

(1) If $q=1$, then (3.1) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \Bigg\{\frac{f(b)g(b)-f(a)g(a)}{\ln\left(\frac{f(b)}{f(a)}\right)+\ln\left(\frac{g(b)}{g(a)}\right)} - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\Bigg[\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} + \frac{g(b)-g(a)}{\ln g(b)-\ln g(a)}\Bigg] + \frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\Bigg\}.$$

(2) If $c=0$, then (3.1) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left\{\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}\right\}^{1-\frac{1}{q}} \times \left\{\frac{\left(f(b)g^q(b)-f(a)g^q(a)\right)}{\ln\left(\frac{f(b)}{f(a)}\right)+q\ln\left(\frac{g(b)}{g(a)}\right)}\right\}^{\frac{1}{q}}.$$

(3) If $c=0$ and $q=1$, then (3.1) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left\{\frac{\left(f(b)g(b)-f(a)g(a)\right)}{\ln\left(\frac{f(b)}{f(a)}\right)+\ln\left(\frac{g(b)}{g(a)}\right)}\right\}.$$

Theorem 3.2. *Let $f,g : I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}_+$ be p-harmonic s-convex function and strongly p-harmonic log-convex function, respectively with modulus $c>0$. If g^q is strongly p-harmonic log-convex function and $q\geq 1$, then*

$$\begin{aligned}&\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x\\ &\leq \left[\frac{f(a)+f(b)}{s+1}\right]^{1-\frac{1}{q}} \times \Bigg\{\left[\frac{g^q(b)-g^q(a)}{(s+1)(s+2)}+\frac{g^q(a)}{s+1}\right]f(a)\\ &\quad+\left[\frac{g^q(b)-g^q(a)}{s+2}+\frac{g^q(a)}{s+1}\right]f(b)-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{(s+2)(s+3)}[f(a)+f(b)]\Bigg\}^{\frac{1}{q}}.\end{aligned} \tag{3.2}$$

Proof. Using power mean inequality, p-harmonic s-convexity and strongly p-harmonic log-convexity of f and g^q, respectively where $u=[\frac{g(b)}{g(a)}]$, we have

$$\begin{aligned}
&\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x=\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
\leq\ & \left[\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]^{1-\frac{1}{q}}\\
&\times\left[\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g^q\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]^{\frac{1}{q}}\\
\leq\ & \left\{\int_0^1\left[(1-t)^sf(a)+t^sf(b)\right]\mathrm{d}t\right\}^{1-\frac{1}{q}}\times\left\{\int_0^1\left[(1-t)^sf(a)+t^sf(b)\right]\right.\\
&\left.\times\left[\big(g(a)\big)^{q(1-t)}\big(g(b)\big)^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\mathrm{d}t\right\}^{\frac{1}{q}}\\
\leq\ & \left[\frac{f(a)+f(b)}{s+1}\right]^{1-\frac{1}{q}}\times\left\{g^q(a)\int_0^1\left[f(a)(1-t)^s\left(\frac{g(b)}{g(a)}\right)^{qt}+f(b)t^s\left(\frac{g(b)}{g(a)}\right)^{qt}\right]\mathrm{d}t\right.\\
&\left.-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{\frac{1}{q}}\\
\leq\ & \left[\frac{f(a)+f(b)}{s+1}\right]^{1-\frac{1}{q}}\times\left\{g^q(a)\int_0^1\left[f(a)(1-t)^su^{qt}+f(b)t^su^{qt}\right]\mathrm{d}t\right.\\
&\left.-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{\frac{1}{q}}\\
\leq\ & \left[\frac{f(a)+f(b)}{s+1}\right]^{1-\frac{1}{q}}\times\left\{g^q(a)\left[\left(\frac{u^q-1}{(s+1)(s+2)}+\frac{1}{s+1}\right)f(a)\right.\right.\\
&\left.\left.+\left(\frac{u^q-1}{s+2}+\frac{1}{s+1}\right)f(b)\right]-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{\frac{1}{q}}\\
=\ & \left[\frac{f(a)+f(b)}{s+1}\right]^{1-\frac{1}{q}}\times\left\{\left[\frac{g^q(b)-g^q(a)}{(s+1)(s+2)}+\frac{g^q(a)}{s+1}\right]f(a)\right.\\
&\left.+\left[\frac{g^q(b)-g^q(a)}{s+2}+\frac{g^q(a)}{s+1}\right]f(b)-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{\frac{1}{q}}.
\end{aligned}$$

Since $u^{qt}\leq(u^q-1)t+1$ for all $0\leq t\leq 1$, we obtain

$$\int_0^1(1-t)^su^{qt}\mathrm{d}t\leq\int_0^1(1-t)^s[(u^q-1)t+1]\mathrm{d}t=\frac{u^q-1}{(s+1)(s+2)}+\frac{1}{s+1}.$$

$$\int_0^1 t^su^{qt}\mathrm{d}t\leq\int_0^1 t^s[(u^q-1)t+1]\mathrm{d}t=\frac{u^q-1}{s+2}+\frac{1}{s+1}.$$

This completes the proof.

We now discuss some special cases.

(1) If $q=1$, then (3.2) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \Big\{\Big[\frac{g(b)-g(a)}{(s+1)(s+2)}+\frac{g(a)}{s+1}\Big]f(a)+\Big[\frac{g(b)-g(a)}{s+2} +\frac{g(a)}{s+1}\Big]f(b)-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{(s+2)(s+3)}[f(a)+f(b)]\Big\}.$$

(2) If $c=0$, then (3.2) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \Big[\frac{f(a)+f(b)}{s+1}\Big]^{1-\frac{1}{q}}\times\Big\{\Big[\frac{g^q(b)-g^q(a)}{(s+1)(s+2)}+\frac{g^q(a)}{s+1}\Big]f(a) +\Big[\frac{g^q(b)-g^q(a)}{s+2}+\frac{g^q(a)}{s+1}\Big]f(b)\Big\}^{\frac{1}{q}}.$$

(3) If $c=0$ and $q=1$, then (3.2) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \Big[\frac{g(b)-g(a)}{(s+1)(s+2)}+\frac{g(a)}{s+1}\Big]f(a)+\Big[\frac{g(b)-g(a)}{s+2}+\frac{g(a)}{s+1}\Big]f(b).$$

(4) If $c=0$, $s=1$ and $q=1$, then (3.2) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \frac{f(a)g(b)+2f(a)g(a)+2f(b)g(b)+f(b)g(a)}{6}.$$

Theorem 3.3. *Let $f,g:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}_+$ be p-harmonic tgs-convex function and strongly harmonic* log*-convex function, respectively with modulus $c>0$. If g^q is strongly p-harmonic log-convex function and $q\geq 1$, then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \Big[\frac{f(a)+f(b)}{6}\Big]^{1-\frac{1}{q}}\times\Big\{\frac{[g^q(b)+g^q(a)][f(a)+f(b)]}{12} -\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{30}[f(a)+f(b)]\Big\}^{\frac{1}{q}}. \tag{3.3}$$

Proof. Using power mean inequality, p-harmonic tgs-convexity and strongly p-harmonic log-convexity of f and g^q, respectively where $u=[\frac{g(b)}{g(a)}]$, we have

$$\begin{aligned}
&\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x=\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
\leq\;& \left[\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]^{1-\frac{1}{q}}\\
&\times\left[\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g^q\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]^{\frac{1}{q}}\\
\leq\;& \left\{\int_0^1 t(1-t)\big[f(a)+f(b)\big]\mathrm{d}t\right\}^{1-\frac{1}{q}}\times\left\{\int_0^1 t(1-t)\big[f(a)+f(b)\big]\right.\\
&\left.\times\left[\big(g(a)\big)^{q(1-t)}\big(g(b)\big)^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\mathrm{d}t\right\}^{\frac{1}{q}}\\
\leq\;& \left[\frac{f(a)+f(b)}{6}\right]^{1-\frac{1}{q}}\times\left\{g^q(a)[f(a)+f(b)]\int_0^1 t(1-t)\left(\frac{g(b)}{g(a)}\right)^{qt}\mathrm{d}t\right.\\
&\left.-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{30}[f(a)+f(b)]\right\}^{\frac{1}{q}}\\
\leq\;& \left[\frac{f(a)+f(b)}{s+1}\right]^{1-\frac{1}{q}}\times\left\{g^q(a)[f(a)+f(b)]\int_0^1 t(1-t)u^{qt}\mathrm{d}t\right.\\
&\left.-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{\frac{1}{q}}\\
\leq\;& \left[\frac{f(a)+f(b)}{6}\right]^{1-\frac{1}{q}}\times\left\{\frac{[g^q(b)+g^q(a)][f(a)+f(b)]}{12}-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{30}[f(a)+f(b)]\right\}^{\frac{1}{q}},
\end{aligned}$$

where

$$\int_0^1 t(1-t)u^{qt}\mathrm{d}t\leq\int_0^1 t(1-t)[(u^q-1)t+1]\mathrm{d}t=\frac{u^q+1}{12}.$$

This completes the proof.

We now discuss some special cases.

(1) If $q=1$, then (3.3) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \;\leq\; \left\{\frac{[g(b)+g(a)][f(a)+f(b)]}{12}-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{30}[f(a)+f(b)]\right\}.$$

(2) If $c=0$, then (3.3) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left[\frac{f(a)+f(b)}{6}\right]^{1-\frac{1}{q}}\times\left\{\frac{[g^q(b)+g^q(a)][f(a)+f(b)]}{12}\right\}^{\frac{1}{q}}.$$

(3) If $c=0$ and $q=1$, then (3.3) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left\{\frac{[g(b)+g(a)][f(a)+f(b)]}{12}\right\}.$$

Theorem 3.4. *Let $f,g: I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}_+$ be p-harmonic m-convex function and strongly p-harmonic log-convex function, respectively with modulus $c>0$. If g^q is strongly p-harmonic log-convex function and $q\geq 1$, then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left[\frac{mf(am)+f(b)}{2}\right]^{1-\frac{1}{q}}\times\left\{\left[\frac{g^q(b)+2g^q(a)}{6}\right]mf(am) + \left[\frac{g^q(b)-g^q(a)}{3}+\frac{g^q(a)}{2}\right]f(b)-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{12}[mf(am)+f(b)]\right\}^{\frac{1}{q}}. \tag{3.4}$$

Proof. Using power mean inequality, p-harmonic m-convexity and strongly p-harmonic log-convexity of f and g^q, respectively where $u=[\frac{g(b)}{g(a)}]$, we have

$$\begin{aligned}
\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x &= \int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
&\leq \left\{\int_0^1\left[m(1-t)f(am)+tf(b)\right]\mathrm{d}t\right\}^{1-\frac{1}{q}}\times\left\{\int_0^1\left[m(1-t)f(am)+tf(b)\right]\right.\\
&\quad\left.\times\left[\left(g(a)\right)^{q(1-t)}\left(g(b)\right)^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\mathrm{d}t\right\}^{\frac{1}{q}}\\
&\leq \left[\frac{mf(am)+f(b)}{2}\right]^{1-\frac{1}{q}}\times\left\{g^q(a)\int_0^1\left[mf(am)(1-t)\left(\frac{g(b)}{g(a)}\right)^{qt}+tf(b)\left(\frac{g(b)}{g(a)}\right)^{qt}\right]\mathrm{d}t\right.\\
&\quad\left.-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{12}[mf(ma)+f(b)]\right\}^{\frac{1}{q}}\\
&\leq \left[\frac{mf(am)+f(b)}{2}\right]^{1-\frac{1}{q}}\times\left\{g^q(a)\int_0^1\left[mf(am)(1-t)u^{qt}+tf(b)u^{qt}\right]\mathrm{d}t\right.\\
&\quad\left.-\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{12}[mf(am)+f(b)]\right\}^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{mf(am)+f(b)}{2}\right]^{1-\frac{1}{q}} \times \left\{ g^q(a)\left[\left(\frac{u^q+2}{6}\right)mf(am)+\left(\frac{u^q-1}{3}+\frac{1}{2}\right)f(b)\right] \right.\\
&\quad \left. -\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{12}[mf(am)+f(b)]\right\}^{\frac{1}{q}}\\
&= \left[\frac{mf(am)+f(b)}{2}\right]^{1-\frac{1}{q}} \times \left\{ \left[\frac{g^q(b)+2g^q(a)}{6}\right]mf(am)+\left[\frac{g^q(b)-g^q(a)}{3}+\frac{g^q(a)}{2}\right]f(b) \right.\\
&\quad \left. -\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{12}[mf(am)+f(b)]\right\}^{\frac{1}{q}},
\end{aligned}$$

where

$$\int_0^1 (1-t)u^{qt}\mathrm{d}t \leq \int_0^1 (1-t)[(u^q-1)t+1]\mathrm{d}t = \frac{u^q+2}{6},$$
$$\int_0^1 tu^{qt}\mathrm{d}t \leq \int_0^1 t[(u^q-1)t+1]\mathrm{d}t = \frac{u^q-1}{3}+\frac{1}{2}.$$

This completes the proof.

We now discuss some special cases.

(1) If $q=1$, then (3.4) reduces to

$$\begin{aligned}
\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x &\leq \left\{ \left[\frac{g(b)+2g(a)}{6}\right]mf(am)+\left[\frac{g(b)-g(a)}{3}+\frac{g(a)}{2}\right]f(b)\right.\\
&\quad \left. -\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{12}[mf(am)+f(b)]\right\}.
\end{aligned}$$

(2) If $c=0$, then (3.4) reduces to

$$\begin{aligned}
\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x &\leq \left[\frac{mf(am)+f(b)}{2}\right]^{1-\frac{1}{q}} \times \left\{ \left[\frac{g^q(b)+2g^q(a)}{6}\right]mf(am)\right.\\
&\quad \left. +\left[\frac{g^q(b)-g^q(a)}{3}+\frac{g^q(a)}{2}\right]f(b)\right\}^{\frac{1}{q}}.
\end{aligned}$$

(3) If $c=0$ and $q=1$, then (3.4) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left[\frac{g(b)+2g(a)}{6}\right]mf(am)+\left[\frac{2g(b)+g(a)}{6}\right]f(b).$$

Theorem 3.5. *Let $f,g : I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}_+$ be p-harmonic P-convex and strongly p-harmonic log-convex function, respectively with modulus $c>0$ on $[a,b]$. If g^q is strongly p-harmonic log-convex function and $q\geq 1$, then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x\leq\big[f(a)+(f(b)\big]\left[\frac{g^q(b)-g^q(a)}{q[\ln g(b)-\ln g(a)]}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right]^{\frac{1}{q}}. \qquad (3.5)$$

Proof. Using power mean inequality, harmonic (p,P)-convexity and the strongly harmonic $(\log,p)$-convexity of the functions f and g^q, respectively, we have

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x=\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t$$

$$\leq \left[\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g^q\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]^{\frac{1}{q}}$$

$$\leq \left\{\int_0^1\big[f(a)+f(b)\big]\mathrm{d}t\right\}^{1-\frac{1}{q}}\times\left\{\int_0^1\big[f(a)+f(b)\big]\right.$$

$$\left.\times\left[\big(g(a)\big)^{q(1-t)}\big(g(b)\big)^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\mathrm{d}t\right\}^{\frac{1}{q}}$$

$$\leq \big[f(a)+f(b)\big]\left[g^q(a)\int_0^1\left(\frac{g(b)}{g(a)}\right)^{qt}\mathrm{d}t-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right]^{\frac{1}{q}}$$

$$\leq \big[f(a)+f(b)\big]\left[\frac{g^q(b)-g^q(a)}{q[\ln g(b)-\ln g(a)]}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right]^{\frac{1}{q}}.$$

This completes the proof.

We now discuss sone special cases.

(1) If $q=1$, the (3.5) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \big[f(a)+(f(b)\big]\left[\frac{g(b)-g(a)}{\ln g(b)-\ln g(a)}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right].$$

(2) If $c=0$, the (3.5) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \big[f(a)+f(b)\big]\left[\frac{g^q(b)-g^q(a)}{q[\ln g(b)-\ln g(a)]}\right]^{\frac{1}{q}}.$$

(3) when $c=0$ and $q=1$, the (3.5) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \big[f(a)+f(b)\big]\left[\frac{g(b)-g(a)}{\ln g(b)-\ln g(a)}\right].$$

Theorem 3.6. *Let $f,g : I = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}_+$ be p-harmonic quasi convex and strongly p-harmonic log-convex function, respectively with modulus $c > 0$ on $[a,b]$. If g^q is strongly p-harmonic log-convex function and $q \geq 1$, then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left[\max\{f(a)+f(b)\}\right]\left[\frac{g^q(b)-g^q(a)}{q[\ln g(b)-\ln g(a)]}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right]^{\frac{1}{q}}. \quad (3.6)$$

Proof. Using power mean inequality, p-harmonic quasi convexity and strongly p-harmonic log-convexity of the functions f and g^q, respectively, we have

$$\begin{aligned}
&\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x = \int_0^1 f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
\leq\ &\left\{\int_0^1\left[\max\{f(a)+f(b)\}\right]\mathrm{d}t\right\}^{1-\frac{1}{q}}\times\left\{\int_0^1\left[\max\{f(a)+f(b)\}\right]\right.\\
&\left.\times\left[\left(g(a)\right)^{q(1-t)}\left(g(b)\right)^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\mathrm{d}t\right\}^{\frac{1}{q}}\\
\leq\ &\left[\max\{f(a)+f(b)\}\right]\left[g^q(a)\int_0^1\left(\frac{g(b)}{g(a)}\right)^{qt}\mathrm{d}t-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right]^{\frac{1}{q}}\\
\leq\ &\left[\max\{f(a)+f(b)\}\right]\left[\frac{g^q(b)-g^q(a)}{q[\ln g(b)-\ln g(a)]}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof.

Now we discuss some special cases.

(1) If $q=1$, then (3.6) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left[\max\{f(a)+f(b)\}\right]\left[\frac{g(b)-g(a)}{\ln g(b)-\ln g(a)}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right].$$

(2) If $c=0$, then (3.6) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left[\max\{f(a)+f(b)\}\right]\left[\frac{g^q(b)-g^q(a)}{q[\ln g(b)-\ln g(a)]}\right]^{\frac{1}{q}}.$$

(3) when $c=0$ and $q=1$, then (3.6) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \left[\max\{f(a)+f(b)\}\right]\left[\frac{g(b)-g(a)}{\ln g(b)-\ln g(a)}\right].$$

Remark 3.7. For $p=-1$, our results reduces to [15]. This shows that the class of strongly harmonic p-convex functions is more general and unify one.

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