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ON WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

HUSEYIN BUDAK*, MEHMET ZEKI SARIKAYA

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey

Abstract. In this paper, we obtain new weighted Ostrowski type inequalities for functions of two independent variables with bounded variation. Applications for qubature formulae are also given.

Keywords. Bounded variation; Ostrowski type inequality; Riemann-Stieltjes integral.

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1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. From [19], we have the inequality

$$\left| f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty, \forall x\in[a,b]. \tag{1.1}$$

The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [15], Dragomir proved following Ostrowski type inequalities related functions of bounded variation.

*Corresponding author.

E-mail addresses: hsyn.budak@gmail.com (H. Budak), sarikayamz@gmail.com (M.Z. Sarikaya).

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Theorem 1.1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [22], Tseng *et al.* gave the following weighted Ostrowski type inequalities for functions of bounded variation.

Theorem 1.2. *Let us have $0\leq\alpha\leq1$, let $w:[a,b]\rightarrow[0,\infty)$ continuous and positive on (a,b) and $h:[a,b]\rightarrow\mathbb{R}$ be differentiable such that $h'(t)=w(t)$ on $[a,b]$. Let $a_1=h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$, $b_1=h^{-1}\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right)$. If $f:[a,b]\rightarrow\mathbb{R}$ be mapping of bounded variation on $[a,b]$, then for all $x\in[a_1,b_1]$, we have the inequality*

$$\left|\int_a^b w(t)f(t)dt-\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]\int_a^b w(t)dt\right|\leq K\bigvee_a^b(f), \tag{1.2}$$

where

$$K:=\begin{cases}\frac{1-\alpha}{2}\int_a^b w(t)dt+\left|h(x)-\frac{h(a)+h(b)}{2}\right|, & \text{if } 0\leq\alpha\leq\frac{1}{2},\\ \max\left\{\frac{1-\alpha}{2}\int_a^b w(t)dt+\left|h(x)-\frac{h(a)+h(b)}{2}\right|,\frac{\alpha}{2}\int_a^b w(t)dt\right\}, & \text{if } \frac{1}{2}<\alpha<\frac{2}{3},\\ \frac{\alpha}{2}\int_a^b w(t)dt, & \text{if } \frac{2}{3}\leq\alpha\leq1\end{cases}$$

and $\bigvee_a^b(f)$ denotes the total variation of f on interval $[a,b]$. In (1.2), the constant $\frac{1-\alpha}{2}$ for $0\leq\alpha\leq\frac{1}{2}$ and the constant $\frac{\alpha}{2}$ for $\frac{2}{3}\leq\alpha\leq1$ are the best possible.

2. Preliminaries

In 1910, Fréchet [17] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $g(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$,

$$a=x_0<x_1<...<x_n=b, \text{ and } c=y_0<y_1<...<y_m=d;$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $(i=1,2,...,n;\ j=1,2,...,m)$; and for all i,j, let $\Delta_{11}g(x_i,y_j)=g(x_{i-1},y_{j-1})-g(x_{i-1},y_j)-g(x_i,y_{j-1})+g(x_i,y_j)$. Then if the sum $S=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}f\left(\xi_i,\eta_j\right)\Delta_{11}g(x_i,y_j)$ tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to g is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_xg(x,y). \tag{2.1}$$

If S is replaced by the sum $S^*=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}g(x_i,y_j)$, where ξ_{ij},η_{ij} are numbers satisfying $\xi_{ij}\in[x_{i-1},x_i]$, $\eta_{ij}\in\left[y_{j-1},y_j\right]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$(*)\int_a^b\int_c^d f(x,y)d_yd_xg(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson [13] has shown that the existence of (2.1) does not imply the existence of (2.2).

In [12], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables.

The function $f(x,y)$ is assumed to be defined in rectangle $R(a\le x\le b,\ c\le y\le d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i(i=0,1,2,...,m),\ a=x_0<x_1<...<x_m=b; \\ y &= y_j(j=0,1,2,...,n),\ c=y_0<y_1<...<y_n=d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11}f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_{i+1},y_j)-f(x_i,y_{j+1})+f(x_i,y_j)$$

$$\Delta f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_i,y_j)$$

The total variation function, $\phi(\bar{x})$ $[\psi(\bar{y})]$, is defined as the total variation of $f(\bar{x},y)$ $[f(x,\bar{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$,or as $+\infty$ if $f(\bar{x},y)$ $[f(x,\bar{y})]$ is of unbounded variation.

Definition 2.1. (Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \left|\Delta_{11} f(x_i,y_j)\right|$$

is bounded for all nets.

Definition 2.2. (Fréchet). The function $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \varepsilon_i \overline{\varepsilon_j} \left|\Delta_{11} f(x_i,y_j)\right|$$

is bounded for all nets and all possible choices of $\varepsilon_i = \pm 1$ and $\overline{\varepsilon_j} = \pm 1$.

Definition 2.3. (Hardy-Krause). The function $f(x,y)$ is said tobe of bounded variation if it satisfies the condition of Definition 2.1 and if in addition $f(\overline{x},y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x,\overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.

Definition 2.4. (Arzelà). Let (x_i,y_i) $(i = 0,1,2,...,m)$ be any set of points satisfiying the conditions

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x,y)$ is said tobe of bounded variation if the sum $\sum\limits_{i=1}^{m} \left|\Delta f(x_i,y_i)\right|$ is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows: Let f be of bounded variation on $Q = [a,b] \times [c,d]$, and let $\sum(P)$ denote the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \left|\Delta_{11} f(x_i,y_j)\right|$$

corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f) := \bigvee_{c}^{d} \bigvee_{a}^{b}(f) := \sup\left\{\sum(P) : P \in P(Q)\right\},$$

is called the total variation of f on Q.

In [18], authors proved the following Lemmas related double Riemann-Stieltjes integral.

Lemma 2.5. (Integrating by parts) *If $f \in RS(g)$ on Q, then $g \in RS(f)$ on Q, and we have*

$$\begin{aligned}&\int_{c}^{d}\int_{a}^{b} f(t,s)d_t d_s g(t,s) + \int_{c}^{d}\int_{a}^{b} g(t,s)d_t d_s f(t,s)\\&= f(b,d)g(b,d) - f(b,c)g(b,c) - f(a,d)g(a,d) + f(a,c)g(a,c).\end{aligned}\tag{2.3}$$

Lemma 2.6. *Assume that $\Omega \in RS(g)$ on Q and g is of bounded variation on Q. Then*

$$\left|\int_{c}^{d}\int_{a}^{b} \Omega(x,y)d_x d_y g(x,y)\right| \le \sup_{(x,y)\in Q} |\Omega(x,y)| \bigvee_{Q}(g).\tag{2.4}$$

In [18], Jawarneh and Noorani obtained the following Ostrowski type inequality for functions of two variables with bounded variation.

Theorem 2.7. *Let $f : Q \to\to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$\begin{aligned}&\left|(b-a)(d-c)f(x,y) - \int_{c}^{d}\int_{a}^{b} f(t,s)dtds\right|\\&\le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right]\bigvee_{Q}(f),\end{aligned}\tag{2.5}$$

where $\bigvee_{Q}(f)$ denotes the total (double) variation of f on Q.

In [6], Budak and Sarikaya proved the following generalization of the inequality (2.5).

Theorem 2.8. *Let $f : Q \to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$\begin{aligned}&\Big|(b-a)(d-c)\Big[(1-\lambda)(1-\eta)f(x,y)\\&\quad + \frac{(1-\lambda)\eta}{2}[f(a,y)+f(b,y)] + \frac{\lambda(1-\eta)}{2}[f(x,c)+f(x,d)]\\&\quad + \frac{\lambda\eta}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\Big] - \int_{a}^{b}\int_{c}^{d} f(t,s)dsdt\Big|\\&\le \max\left\{\lambda\frac{b-a}{2}, \left(x - \frac{(2-\lambda)a+\lambda b}{2}\right), \left(\frac{(2-\lambda)b+\lambda a}{2} - x\right)\right\}\\&\times \max\left\{\eta\frac{d-c}{2}, \left(y - \frac{(2-\eta)c+\eta d}{2}\right), \left(\frac{(2-\eta)d+\eta c}{2} - y\right)\right\}\bigvee_{a}^{b}\bigvee_{c}^{d}(f)\end{aligned}\tag{2.6}$$

for any $\lambda,\eta\in[0,1]$ *and* $a+\lambda\frac{b-a}{2}\leq x\leq b-\lambda\frac{b-a}{2}$, $c+\eta\frac{d-c}{2}\leq y\leq d-\eta\frac{d-c}{2}$, *where* $\bigvee_a^b\bigvee_c^d(f)$ *denotes he total variation of* f *on* Q.

For more information and recent developments on inequalities for mappings of bounded variation, we refer authors to [1]-[11], [14]-[16], [18], [20]-[25] and the references therein. The aim of this paper is to establish new weighted Ostrowski type inequalities for functions of two independent variables with bounded variation.

3. Main results

Let us have $0\leq\alpha,\beta\leq1$ and let $w_1:[a,b]\to[0,\infty)$ continuous and positive on (a,b) and $h_1:[a,b]\to\mathbb{R}$ be differentiable such that $h_1'(t)=w_1(t)$ on $[a,b]$. Similarly, let $w_2:[c,d]\to[0,\infty)$ continuous and positive on (c,d) and $h_2:[c,d]\to\mathbb{R}$ be differentiable such that $h_2'(t)=w_2(t)$ on $[c,d]$. Let $a_1=h_1^{-1}\left(\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right)$, $b_1=h_1^{-1}\left(\frac{\alpha}{2}h_1(a)+\left(1-\frac{\alpha}{2}\right)h_1(b)\right)$, $c_1=h_2^{-1}\left(\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right)$ and $d_1=h_2^{-1}\left(\frac{\beta}{2}h_2(c)+\left(1-\frac{\beta}{2}\right)h_2(d)\right)$.

Theorem 3.1. *If* $f:[a,b]\times[c,d]\to\mathbb{R}$ *is a mapping of bounded variation on* $[a,b]\times[c,d]$, *then we have the following inequality for all* $(x,y)\in[a_1,b_1]\times[c_1,d_1]$,

$$\begin{aligned}
&\left|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\left[(1-\alpha)(1-\beta)f(x,y)\right.\right.\\
&+(1-\alpha)\beta\frac{f(x,c)+f(x,d)}{2}+\alpha(1-\beta)\frac{f(a,y)+f(b,y)}{2}\Big]\\
&\left.+\alpha\beta\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\right|\\
&\leq KL\bigvee_a^b\bigvee_c^d(f),
\end{aligned}\tag{3.1}$$

where

$$K=\begin{cases}\frac{1-\alpha}{2}\int_a^b w_1(t)dt+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|, & \text{if } 0\leq\alpha\leq\frac{1}{2},\\ \max\left\{\frac{1-\alpha}{2}\int_a^b w_1(t)dt+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|,\frac{\alpha}{2}\int_a^b w_1(t)dt\right\}, & \text{if } \frac{1}{2}<\alpha<\frac{2}{3},\\ \frac{\alpha}{2}\int_a^b w_1(t)dt, & \text{if } \frac{2}{3}\leq\alpha\leq1,\end{cases}$$

and

$$L=\begin{cases}\frac{1-\beta}{2}\int\limits_c^d w_2(t)dt+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|, & \text{if } 0\le\beta\le\frac{1}{2},\\ \max\left\{\frac{1-\beta}{2}\int\limits_c^d w_2(t)dt+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|,\frac{\beta}{2}\int\limits_c^d w_2(t)dt\right\}, & \text{if } \frac{1}{2}<\beta<\frac{2}{3},\\ \frac{\beta}{2}\int\limits_c^d w_2(t)dt, & \text{if } \frac{2}{3}\le\beta\le 1,\end{cases}$$

and $\bigvee_a^b\bigvee_c^d(f)$ *denotes the total variation of* f *on interval* $[a,b]\times[c,d]$.

In (3.1), the constant $\frac{(1-\alpha)(1-\beta)}{4}$ for $\alpha,\beta\in\left[0,\frac{1}{2}\right]$ and the constant $\frac{\alpha\beta}{4}$ for $\alpha,\beta\in\left[\frac{2}{3},1\right]$ are the best possible.

Proof. For $(x,y)\in[a_1,b_1]\times[c_1,d_1]$, we define the following mappings q,p by

$$q(t)=\begin{cases}h_1(t)-\left[\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right], & t\in[a,x),\\ h_1(t)-\left[\frac{\alpha}{2}h_1(a)+\left(1-\frac{\alpha}{2}\right)h_1(b)\right], & t\in[x,b],\end{cases}$$

$$p(s)=\begin{cases}h_2(s)-\left[\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right], & s\in[c,y),\\ h_2(s)-\left[\frac{\beta}{2}h_2(c)+\left(1-\frac{\beta}{2}\right)h_2(d)\right], & s\in[y,d].\end{cases}$$

Using the $q(t)$ and $p(s)$ kernels, we have

$$\begin{aligned}&\int\limits_a^b\int\limits_c^d q(t)p(s)d_sd_tf(t,s)\\ &=\int\limits_a^x\int\limits_c^y\left(h_1(t)-\left[\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right]\right)\left(h_2(s)-\left[\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right]\right)d_sd_tf(t,s)\\ &+\int\limits_a^x\int\limits_y^d\left(h_1(t)-\left[\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right]\right)\left(h_2(s)-\left[\frac{\beta}{2}h_2(c)+\left(1-\frac{\beta}{2}\right)h_2(d)\right]\right)d_sd_tf(t,s)\\ &+\int\limits_x^b\int\limits_c^y\left(h_1(t)-\left[\frac{\alpha}{2}h_1(a)+\left(1-\frac{\alpha}{2}\right)h_1(b)\right]\right)\left(h_2(s)-\left[\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right]\right)d_sd_tf(t,s)\\ &+\int\limits_x^b\int\limits_y^d\left(h_1(t)-\left[\frac{\alpha}{2}h_1(a)+\left(1-\frac{\alpha}{2}\right)h_1(b)\right]\right)\left(h_2(s)-\left[\frac{\beta}{2}h_2(c)+\left(1-\frac{\beta}{2}\right)h_2(d)\right]\right)d_sd_tf(t,s)\\ &=I_1+I_2+I_3+I_4.\end{aligned}$$

By integrating by parts, we get

$$\begin{aligned}
I_1 &= \int_a^x \int_c^y \left(h_1(t) - \left[\left(1-\frac{\alpha}{2}\right)h_1(a) + \frac{\alpha}{2}h_1(b)\right]\right) \\
&\quad \times \left(h_2(s) - \left[\left(1-\frac{\beta}{2}\right)h_2(c) + \frac{\beta}{2}h_2(d)\right]\right) d_s d_t f(t,s) \\
&= \left[h_1(x) - \left(1-\frac{\alpha}{2}\right)h_1(a) - \frac{\alpha}{2}h_1(b)\right] \\
&\quad \times \left[h_2(y) - \left(1-\frac{\beta}{2}\right)h_2(c) - \frac{\beta}{2}h_2(d)\right] f(x,y) \\
&\quad + \left[h_1(x) - \left(1-\frac{\alpha}{2}\right)h_1(a) - \frac{\alpha}{2}h_1(b)\right] \left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(x,c) \\
&\quad + \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right) \left[h_2(y) - \left(1-\frac{\beta}{2}\right)h_2(c) - \frac{\beta}{2}h_2(d)\right] f(a,y) \\
&\quad + \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right) \left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(a,c) \\
&\quad - \int_a^x \int_c^y w_1(t)w_2(s)f(t,s)dsdt.
\end{aligned} \tag{3.2}$$

Using a similar method, we have

$$\begin{aligned}
I_2 &= \int_a^x \int_y^d \left(h_1(t) - \left[\left(1-\frac{\alpha}{2}\right)h_1(a) + \frac{\alpha}{2}h_1(b)\right]\right) \\
&\quad \times \left(h_2(s) - \left[\frac{\beta}{2}h_2(c) + \left(1-\frac{\beta}{2}\right)h_2(d)\right]\right) d_s d_t f(t,s) \\
&= \left[h_1(x) - \left(1-\frac{\alpha}{2}\right)h_1(a) - \frac{\alpha}{2}h_1(b)\right] \left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(x,d) \\
&\quad - \left[h_1(x) - \left(1-\frac{\alpha}{2}\right)h_1(a) - \frac{\alpha}{2}h_1(b)\right] \left[h_2(y) - \frac{\beta}{2}h_2(c) - \left(1-\frac{\beta}{2}\right)h_2(d)\right] f(x,y) \\
&\quad + \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right) \left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(a,d) \\
&\quad - \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right) \left[h_2(y) - \frac{\beta}{2}h_2(c) - \left(1-\frac{\beta}{2}\right)h_2(d)\right] f(a,y) \\
&\quad - \int_a^x \int_y^d w_1(t)w_2(s)f(t,s)dsdt,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
I_3 = & \int_x^b \int_c^y \left(h_1(t) - \left[\frac{\alpha}{2}h_1(a) + \left(1-\frac{\alpha}{2}\right)h_1(b)\right]\right) \\
& \times \left(h_2(s) - \left[\left(1-\frac{\beta}{2}\right)h_2(c) + \frac{\beta}{2}h_2(d)\right]\right) d_s d_t f(t,s) \\
= & \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right)\left[h_2(y) - \left(1-\frac{\beta}{2}\right)h_2(c) - \frac{\beta}{2}h_2(d)\right] f(b,y) \\
& + \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right)\left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(b,c) \qquad (3.4) \\
& - \left[h_1(x) - \frac{\alpha}{2}h_1(a) - \left(1-\frac{\alpha}{2}\right)h_1(b)\right]\left[h_2(y) - \left(1-\frac{\beta}{2}\right)h_2(c) - \frac{\beta}{2}h_2(d)\right] f(x,y) \\
& - \left[h_1(x) - \frac{\alpha}{2}h_1(a) - \left(1-\frac{\alpha}{2}\right)h_1(b)\right]\left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(x,c) \\
& - \int_x^b \int_c^y w_1(t)w_2(s)f(t,s)\,ds\,dt,
\end{aligned}$$

and

$$\begin{aligned}
I_4 = & \int_x^b \int_y^d \left(h_1(t) - \left[\frac{\alpha}{2}h_1(a) + \left(1-\frac{\alpha}{2}\right)h_1(b)\right]\right) \\
& \times \left(h_2(s) - \left[\frac{\beta}{2}h_2(c) + \left(1-\frac{\beta}{2}\right)h_2(d)\right]\right) d_s d_t f(t,s) \\
= & \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right)\left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(b,d) \\
& - \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right)\left[h_2(y) - \frac{\beta}{2}h_2(c) - \left(1-\frac{\beta}{2}\right)h_2(d)\right] f(b,y) \qquad (3.5) \\
& - \left[h_1(x) - \frac{\alpha}{2}h_1(a) - \left(1-\frac{\alpha}{2}\right)h_1(b)\right]\left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(x,d) \\
& + \left[h_1(x) - \frac{\alpha}{2}h_1(a) - \left(1-\frac{\alpha}{2}\right)h_1(b)\right]\left[h_2(y) - \frac{\beta}{2}h_2(c) - \left(1-\frac{\beta}{2}\right)h_2(d)\right] f(x,y) \\
& - \int_x^b \int_y^d w_1(t)w_2(s)f(t,s)\,ds\,dt.
\end{aligned}$$

Adding (3.2)-(3.5), we have

$$\begin{aligned}
&\int_a^b\int_c^d q(t)p(s)d_sd_tf(t,s)\\
&=\left(\int_a^b w(t)dt\right)\left(\int_c^d g(t)dt\right)\left[(1-\alpha)(1-\beta)f(x,y)+(1-\alpha)\beta\frac{f(x,c)+f(x,d)}{2}\right.\\
&\left.+\alpha(1-\beta)\frac{f(a,y)+f(b,y)}{2}+\alpha\beta\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right]\\
&-\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt.
\end{aligned}$$

On the other hand, using Lemma 2.2, we have

$$\begin{aligned}
&\left|\int_a^b\int_c^d q(t)p(s)d_sd_tf(t,s)\right|\\
&\leq \sup_{t\in[a,b]}|q(t)|\sup_{s\in[c,d]}|p(s)|\bigvee_a^b\bigvee_c^d(f)\\
&=\max\{h_1(x)-[(1-\frac{\alpha}{2})h_1(a)+\frac{\alpha}{2}h_1(b)],[\frac{\alpha}{2}h_1(a)+(1-\frac{\alpha}{2})h_1(b)]-h_1(x),\\
&\frac{\alpha}{2}[h_1(b)-h_1(a)]\}\\
&\times\max\{h_2(y)-[(1-\frac{\beta}{2})h_2(c)+\frac{\beta}{2}h_2(d)],[\frac{\beta}{2}h_2(c)+(1-\frac{\beta}{2})h_2(d)]-h_2(y),\\
&\frac{\beta}{2}[h_2(d)-h_2(c)]\}\bigvee_a^b\bigvee_c^d(f)\\
&=\max\left\{\frac{1-\alpha}{2}[h_1(b)-h_1(a)]+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|,\frac{\alpha}{2}[h_1(b)-h_1(a)]\right\}\\
&\times\max\left\{\frac{1-\beta}{2}[h_2(d)-h_2(c)]+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|,\frac{\beta}{2}[h_2(d)-h_2(c)]\right\}\bigvee_a^b\bigvee_c^d(f)\\
&=\max\left\{\frac{1-\alpha}{2}\int_a^b w_1(t)dt+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|,\frac{\alpha}{2}\int_a^b w_1(t)dt\right\}\\
&\times\max\left\{\frac{1-\beta}{2}\int_c^d w_2(t)dt+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|,\frac{\beta}{2}\int_c^d w_2(t)dt\right\}\bigvee_a^b\bigvee_c^d(f)\\
&=KL\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

This completes the proof of inequality (3.1). Assume $(\alpha,\beta)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]$. Suppose (3.1) holds with a constant $A=A_1.A_2$, $A_1,A_2>0$, that is,

$$\begin{aligned}
&\left|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\left[(1-\alpha)(1-\beta)f(x,y)\right.\right.\\
&\left.+(1-\alpha)\beta\frac{f(x,c)+f(x,d)}{2}+\alpha(1-\beta)\frac{f(a,y)+f(b,y)}{2}\right]\\
&\left.+\alpha\beta\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\right| \qquad (3.6)\\
&\le\left[A_1\int_a^b w_1(t)dt+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|\right]\\
&\times\left[A_2\int_c^d w_2(t)dt+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

If we choose $f:Q\to\mathbb{R}$ with

$$f(t,s)=\begin{cases}1, \text{ if } & (t,s)=\left(h_1\left(\frac{h_1(a)+h_1(b)}{2}\right),h_2\left(\frac{h_2(c)+h_2(d)}{2}\right)\right),\\ 0, \text{ if } & (t,s)\in[a,b]\times[c,d]\setminus\left\{\left(h_1\left(\frac{h_1(a)+h_1(b)}{2}\right),h_2\left(\frac{h_2(c)+h_2(d)}{2}\right)\right)\right\},\end{cases}$$

then f is of bounded variation on Q. For $(x,y)=\left(h_1\left(\frac{h_1(a)+h_1(b)}{2}\right),h_2\left(\frac{h_2(c)+h_2(d)}{2}\right)\right)$, we have

$$\beta\frac{f(x,c)+f(x,d)}{2}=0,\ \frac{f(a,y)+f(b,y)}{2}=0,\ \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}=0$$

$$\int_a^b\int_c^d w_1(t)w_2f(t,s)dsdt=0, \text{ and } \bigvee_Q(f)=4.$$

Putting this equalities in (3.6), we get

$$\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)(1-\alpha)(1-\beta)\le 4\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)A_1A_2.$$

It follows that $A\ge\frac{(1-\alpha)(1-\beta)}{4}$ which implies $\frac{(1-\alpha)(1-\beta)}{4}$ is the best possible.

The sharpness of inequality (3.1) for $\alpha,\beta\in\left[\frac{2}{3},1\right]$ can be easily proved by choosing the function f

$$f(t,s)=\begin{cases}1, \text{ if } & (t,s)=(b,d),\\ 0, \text{ if } & (t,s)\in[a,b]\times[c,d]\setminus\{(b,d)\}.\end{cases}$$

This completes the proof.

Remark 3.2. If we choose $w(t) \equiv g(s) \equiv 1$ ($h_1(t) = t$ and $h_2(s) = s$), and $\alpha = \beta = 0$, then the inequality (3.1) reduces the inequality (2.5).

Remark 3.3. If we choose $w(t) \equiv g(s) \equiv 1$ ($h_1(t) = t$ and $h_2(s) = s$), $\alpha = \beta = \frac{1}{3}$, $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in inequality (3.1), then we have the Simpson's inequality

$$\begin{aligned}
&\left| \frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{36} \right. \\
&+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{9} \\
&\left. +\frac{4}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{1}{9}\bigvee_a^b\bigvee_c^d(f),
\end{aligned}$$

which is proved by Jawarneh and Noorani in [18].

Remark 3.4. If we choose $w_1(t) \equiv w_2(s) \equiv 1$ ($h_1(t) = t$ and $h_2(s) = s$), $\alpha = 1$, $\beta = 0$, $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in inequality (3.1), then we have

$$\left| \frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)}{2} - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \leq \frac{1}{4}\bigvee_Q(f),$$

which is given by Budak and Sarikaya in [10]. The constant $\frac{1}{4}$ is the best possible.

Remark 3.5. If we choose $w_1(t) \equiv w_2(s) \equiv 1$ ($h_1(t) = t$ and $h_2(s) = s$), $\alpha = 0$, $\beta = 1$, $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in inequality (3.1), then we have

$$\left| \frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{2} - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \leq \frac{1}{4}\bigvee_Q(f),$$

which is proved by Budak and Sarikaya in [10]. The constant $\frac{1}{4}$ is the best possible.

Remark 3.6. If we choose $w_1(t) \equiv w_2(s) \equiv 1$ $(h_1(t) = t$ and $h_2(s) = s)$, $\alpha = \beta = \frac{1}{2}$, $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in the inequality (3.1), then we have

$$\begin{aligned} &\left| \frac{1}{4}\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\right.\right. \\ &\frac{1}{2}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right] \\ &\left.\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\ &\leq \frac{1}{16}\bigvee_Q(f), \end{aligned} \tag{3.7}$$

which is given by Budak and Sarikaya in [6]. The constant $\frac{1}{16}$ is the best possible.

Corollary 3.7. (Weighted Ostrowski Inequality) *Under the assumption Theorem 3.1, if we choose* $\alpha = \beta = 0$, *for all* $(x,y) \in [a,b] \times [c,d]$, *then we have the following weighted Ostrowski inequality*

$$\begin{aligned} &\left| \left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right) f(x,y) - \int_a^b\int_c^d f(t,s)dsdt \right| \\ &\leq \left[\frac{1}{2}\int_a^b w_1(t)dt + \left|h_1(x) - \frac{h_1(a)+h_1(b)}{2}\right|\right]\left[\frac{1}{2}\int_c^d w_2(t)dt + \left|h_2(y) - \frac{h_2(c)+h_2(d)}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f). \end{aligned}$$

Corollary 3.8. (Weighted Trapezoid Inequality) *Under the assumption Theorem 3.1, if we choose* $\alpha = \beta = 1$, *then we have the following weighted trapezoid inequality;*

$$\begin{aligned} &\left| \frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right) - \int_a^b\int_c^d f(t,s)dsdt \right| \\ &\leq \ \frac{1}{4}\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\bigvee_a^b\bigvee_c^d(f). \end{aligned}$$

Corollary 3.9. (Weighted Simpson's Inequality) *Under assumption Theorem 3.1, if we choose* $\alpha=\beta=\frac{1}{3}$, $x=h_1^{-1}\left(\frac{h_1(a)+h_1(b)}{2}\right)$ *and* $y=h_2^{-1}\left(\frac{h_2(c)+h_2(d)}{2}\right)$, *then we have the weighted Simpson's inequality*

$$\left|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{36}\right.\right.$$
$$\left.\left.+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{9}+\frac{4}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{1}{9}\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\bigvee_a^b\bigvee_c^d(f).$$

4. Applications for qubature formulae

Now, we apply the results presented previous section to qubature formulae.

Let us consider the arbitrary division $I_n: a=x_0<x_1<...<x_n=b$, $J_m: c=y_0<y_1<...<y_m=d, l_1^i:=x_{i+1}-x_i$, and $l_2^j:=y_{j+1}-y_j$,

$$\upsilon(l_1):=\max\left\{l_1^i\right|\ i=0,...,n-1\right\},$$

$$\upsilon(l_2):=\max\left\{l_2^j\right|\ j=0,...,m-1\right\},$$

$$\upsilon(W_1):=\max\left\{W_1^i\right|\ i=0,...,n-1\right\},\ W_1^i:=\int_{x_i}^{x_{i+1}} w_1(u)du=h_1(x_{i+1})-h_1(x_i),$$

$$\upsilon(W_2):=\max\left\{W_2^j\right|\ j=0,...,m-1\right\},\ W_2^j:=\int_{y_j}^{y_{j+1}} w_2(u)du=h_2(y_{j+1})-h_2(y_j).$$

Let us have α, β, w_1, h_1, w_2, and h_2 defined as in Theorem 3.1. Let $a_1^i=h_1^{-1}((1-\frac{\alpha}{2})h_1(x_i)+\frac{\alpha}{2}h_1(x_{i+1}))$, $b_1^i=h_1^{-1}\left(\frac{\alpha}{2}h_1(x_i)+\left(1-\frac{\alpha}{2}\right)h_1(x_{i+1})\right)$, $c_1^j=h_2^{-1}\left(\left(1-\frac{\beta}{2}\right)h_2(y_{j+1})+\frac{\beta}{2}h_2(y_{j+1})\right)$ and $d_1^j=h_2^{-1}\left(\frac{\beta}{2}h_2(y_j)+\left(1-\frac{\beta}{2}\right)h_2(y_{j+1})\right)$, $\xi_i\in\left[a_1^i,b_1^i\right]$, $(i=0,...,n-1)$ and $\eta_j\in\left[c_1^j,d_1^j\right]$ $(j=0,...,m-1)$.

Define the sum

$$\begin{aligned}
&A(f,w_1,h_1,w_2,h_2,I_n,J_m)\\
&=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Big[(1-\alpha)(1-\beta)f(\xi_i,\eta_j)\\
&+(1-\alpha)\beta\frac{f(\xi_i,y_j)+f(\xi_i,y_{j+1})}{2}+\alpha(1-\beta)\frac{f(x_i,\eta_j)+f(x_{i+1},\eta_j)}{2}\\
&+\alpha\beta\frac{f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})}{4}\Big]W_1^iW_2^j.
\end{aligned}$$

Theorem 4.1. *Let f defined as in Theorem 3.1 and let*

$$\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt=A(f,w_1,h_1,w_2,h_2,I_n,J_m)+R(f,w_1,h_1,w_2,h_2,I_n,J_m).$$

The remainder term $R(f,w_1,h_1,w_2,h_2,I_n,J_m)$ satisfies

$$\begin{aligned}
|R(f,w_1,h_1,w_2,h_2,I_n,J_m)|&\le\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}K_iL_j\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)\le M_1N_1\bigvee_a^b\bigvee_c^d(f)\\
&\le M_2N_2\bigvee_a^b\bigvee_c^d(f)\le M_3N_3\bigvee_a^b\bigvee_c^d(f),
\end{aligned}\tag{4.1}$$

where

$$K_i=\begin{cases}\frac{1-\alpha}{2}W_1^i+\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|, & \text{if } 0\le\alpha\le\frac12,\\ \max\left\{\frac{1-\alpha}{2}W_1^i+\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|,\frac{\alpha}{2}W_1^i\right\}, & \text{if } \frac12<\alpha<\frac23\\ \frac{\alpha}{2}W_1^i, & \text{if } \frac23\le\alpha\le1,\end{cases}\quad(i=0,\dots,n-1),$$

$$M_1=\begin{cases}\max\limits_{i=0,\dots,n-1}\left\{\frac{1-\alpha}{2}W_1^i+\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|\right\}, & \text{if } 0\le\alpha\le\frac12,\\ \max\limits_{i=0,\dots,n-1}\left\{\max\left\{\frac{1-\alpha}{2}\upsilon(W_1)+\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|,\frac{\alpha}{2}\upsilon(W_1)\right\}\right\}, & \text{if } \frac12<\alpha<\frac23,\\ \frac{\alpha}{2}\upsilon(W_1), & \text{if } \frac23\le\alpha\le1,\end{cases}$$

$$M_2=\begin{cases}\frac{1-\alpha}{2}\upsilon(W_1)+\max\limits_{i=0,\dots,n-1}\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|, & \text{if } 0\le\alpha\le\frac12,\\ \max\limits_{i=0,\dots,n-1}\left\{\max\left\{\frac{1-\alpha}{2}\upsilon(W_1)dt+\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|,\frac{\alpha}{2}\upsilon(W_1)\right\}\right\}, & \text{if } \frac12<\alpha<\frac23,\\ \frac{\alpha}{2}\upsilon(W_1), & \text{if } \frac23\le\alpha\le1,\end{cases}$$

$$M_3=\begin{cases}(1-\alpha)\upsilon(W_1) & \text{if } 0\le\alpha\le\frac23,\\ \frac{\alpha}{2}\upsilon(W_1), & \text{if } \frac23\le\alpha\le1,\end{cases}$$

and similarly

$$L_i=\begin{cases}\frac{1-\beta}{2}W_2^j+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|, & if\, 0\le\beta\le\frac{1}{2},\\ \max\left\{\frac{1-\beta}{2}W_2^j+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|,\frac{\beta}{2}W_2^j\right\}, & if\, \frac{1}{2}<\beta<\frac{2}{3},\quad (j=0,...,m-1),\\ \frac{\beta}{2}W_2^j, & if\, \frac{2}{3}\le\beta\le 1.\end{cases}$$

$$N_1=\begin{cases}\max\limits_{j=0,...,m-1}\left\{\frac{1-\beta}{2}W_2^j+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|\right\}, & if\, 0\le\beta\le\frac{1}{2},\\ \max\limits_{j=0,...,m-1}\left\{\max\left\{\frac{1-\beta}{2}\upsilon(W_2)+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|,\frac{\beta}{2}\upsilon(W_2)\right\}\right\}, & if\, \frac{1}{2}<\beta<\frac{2}{3},\\ \frac{\beta}{2}\upsilon(W_2), & if\, \frac{2}{3}\le\beta\le 1,\end{cases}$$

$$N_2=\begin{cases}\frac{1-\beta}{2}\upsilon(W_2)+\max\limits_{j=0,...,m-1}\left\{\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|\right\}, & if\, 0\le\beta\le\frac{1}{2},\\ \max\limits_{j=0,...,m-1}\left\{\max\left\{\frac{1-\beta}{2}\upsilon(W_2)+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|,\frac{\beta}{2}\upsilon(W_2)\right\}\right\}, & if\, \frac{1}{2}<\beta<\frac{2}{3},\\ \frac{\beta}{2}\upsilon(W_2), & if\, \frac{2}{3}\le\beta\le 1,\end{cases}$$

$$N_3=\begin{cases}(1-\beta)\,\upsilon(W_2), & if\, 0\le\beta\le\frac{2}{3},\\ \frac{\beta}{2}\upsilon(W_2), & if\, \frac{2}{3}\le\beta\le 1.\end{cases}$$

Proof. Applying Theorem 3.1 to the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we have

$$\begin{aligned}&\Big|\Big[(1-\alpha)(1-\beta)f(\xi_i,\eta_j)\\ &+(1-\alpha)\beta\frac{f(\xi_i,y_j)+f(\xi_i,y_{j+1})}{2}\\ &+\alpha(1-\beta)\frac{f(x_i,\eta_j)+f(x_{i+1},\eta_j)}{2}\\ &+\alpha\beta\frac{f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})}{4}\Big]W_1^iW_2^j\\ &-\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}w_1(t)w_2(s)f(t,s)dsdt\Big|\\ &\le K_iL_j\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)\end{aligned}\tag{4.2}$$

for any $\xi_i\in[a_1^i,b_1^i]$, $(i=0,...,n-1)$ and $\eta_j\in\left[c_1^j,d_1^j\right]$ $(j=0,...,m-1)$. Summing inequality (4.2) over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality, we get

$$|R(f,w_1,h_1,w_2,h_2,I_n,J_m)| \leq \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} K_i L_j \bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)$$

$$\leq \left(\max_{i=0,\ldots,n-1} K_i\right)\left(\max_{j=0,\ldots,m-1} L_j\right)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)$$

$$= M_1 N_1 \bigvee_a^b \bigvee_c^d (f) \leq M_2 N_2 \bigvee_a^b \bigvee_c^d (f).$$

This completes the proof of the first three inequalities in (4.1). In inequality (4.3), we observe that $\left|h_1(\xi_i) - \frac{h_1(x_i)+h_1(x_{i+1})}{2}\right| \leq \frac{1-\alpha}{2} W_1^i$. Hence, we have

$$\max_{i=0,\ldots,n-1}\left|h_1(\xi_i) - \frac{h_1(x_i)+h_1(x_{i+1})}{2}\right| \leq \frac{1-\alpha}{2}\upsilon(W_1).$$

Similarly, we obtain

$$\max_{j=0,\ldots,m-1}\left|h_2(\eta_j) - \frac{h_2(y_j)+h_2(y_{j+1})}{2}\right| \leq \frac{1-\beta}{2}\upsilon(W_2).$$

These show that $M_2 \leq M_3$ and $N_2 \leq N_3$. This completes the proof.

Remark 4.2. If we choose $\alpha = \beta = 0$, $w_1(t) \equiv 1, h_1(t) = t$ on $[a,b]$ and $w_2(s) \equiv 1$, $h_2(s) = s$ on $[c,d]$ in Theorem 4.1, then inequalities (4.1) reduce to the inequality (4.2) in [5].

Remark 4.3. If we choose $\alpha = \beta = \frac{1}{3}$, $w(t) \equiv 1$, $h_1(t) = t$ on $[a,b]$ and $g(s) \equiv 1$, $h_2(s) = s$ on $[c,d]$, $\xi_i = \frac{x_i x_{i+1}}{2}$ $(i = 0,\ldots,n-1)$ and $\eta_j = \frac{y_j + y_{j+1}}{2}$ $(j = 0,\ldots,m-1)$ in Theorem 4.1, then we have the Simpson's sum

$$A_S(f,I_n,J_m) = \frac{4}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{x_i+x_{i+1}}{2}, \frac{y_j+y_{j+1}}{2}\right) l_1^i l_2^j$$

$$+ \frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(\frac{x_i+x_{i+1}}{2}, y_j\right) + f\left(\frac{x_i+x_{i+1}}{2}, y_{j+1}\right)\right] l_1^i l_2^j$$

$$+ \frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(x_i, \frac{y_j+y_{j+1}}{2}\right) + f\left(x_{i+1}, \frac{y_j+y_{j+1}}{2}\right)\right] l_1^i l_2^j$$

$$+ \frac{1}{36}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j) + f(x_i,y_{j+1}) + f(x_{i+1},y_j) + f(x_{i+1},y_{j+1})\right] l_1^i l_2^j$$

with

$$\int_a^b \int_c^d f(t,s)\,ds\,dt = A_S(f,I_n,J_m) + R_S(f,I_n,J_m)$$

and the remainder term $R_S(f,I_n,J_m)$ satisfies

$$|R_S(f,I_n,J_m)| \leq \frac{1}{9}\upsilon(l_1)\upsilon(l_2)\bigvee_a^b\bigvee_c^d(f),$$

which was given by Budak and Sarikaya in [10].

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