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SPLIT EQUALITY FIXED POINT PROBLEM FOR TWO QUASI-ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In this paper, we consider a split equality fixed point problem for two quasi-asymptotically pseudocontractive mappings. Some properties for quasi-asymptotically pseudocontractive operators are presented. An iterative algorithm for solving the split common fixed point problem for two quasi-pseudocontractive operators is constructed. Weak and strong convergence theorems are established in Hilbert spaces.

Keywords. Split equality fixed point problem; Quasi-asymptotically pseudocontractive mapping; Quasi-asymptotically nonexpansive mapping; Strong convergence theorem.

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1. Introduction

Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. We use $Fix(T)$ to denote the fixed point set of mapping T, i.e., $Fix(T) = \{x \in C : x = Tx\}$.

Recall that the split feasibility problem (SFP) is formulated as to find a point $q \in H_1$ such that:

$$q \in C \ \ and \ \ Aq \in Q. \tag{1.1}$$

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It is easy to see that $q \in C$ solves equation (1.1) if and only if it solves the following fixed point equation $q = P_C(I - \gamma A^*(I - P_Q)A)q$, where P_C (resp. P_Q) is the (orthogonal) projection from H_1 (resp. H_2) onto C (resp. Q), $\gamma > 0$, and A^* is the adjoint of A.

In 1994, Censor and Elfving [1] first introduced the (SFP) in finite-dimensional Hilbert s-paces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been found that the (SFP) can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [3-5]. The (SFP) in an infinite dimensional real Hilbert space can be found in [6-10].

Recently, Moudafi [11, 12] and Moudafi and Al-Shemas [13] introduced the following split equality feasibility problem (SEFP):

$$find \quad x \in C, \quad y \in Q \quad such\ that \quad Ax = By, \tag{1.2}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Obviously, if $B = I$ (identity mapping on H_2) and $H_3 = H_2$, then (1.2) reduces to (1.1).

In order to solve split equality feasibility problem (1.2), Moudafi [11] introduced the following simultaneous iterative algorithm:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.3}$$

and under suitable conditions he proved the weak convergence of the sequence $\{(x_n, y_n)\}$ to a solution of (1.2) in Hilbert spaces.

In order to avoid using the projection, recently, Moudafi and Al-Shemas [13] introduced and studied the following problem: Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear operators such that $Fix(T) \neq \emptyset$ and $Fix(S) \neq \emptyset$. If $C = Fix(T)$ and $Q = Fix(S)$, then the split equality feasibility problem (1.2) reduces to:

$$find\ \ x \in Fix(T)\ \ and\ \ y \in Fix(S)\ \ such\ that\ Ax = By, \tag{1.4}$$

which is called *split equality fixed point problem (in short, (SEFPP))*. Denote by Γ the solution set of split equality fixed point problem (1.4).

Recently Moudafi [12] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = S(y_n + \beta_n B^*(Ax_{n+1} - By_n)). \end{cases} \tag{1.5}$$

He also studied the weak convergence of the sequences generated by scheme (1.5) under the condition that T and S are firmly quasi-nonexpansive mappings. In 2015, Che and Li [14] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tu_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)Sv_n. \end{cases} \tag{1.6}$$

They also established the weak convergence of scheme (1.6) under the condition that both T and S are quasi-nonexpansive mappings.

Very recently, Chang, Wang and Qin [15] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)((1-\xi_n)I + \xi_n T((1-\eta_n)I + \eta_n T))u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)((1-\xi_n)I + \xi_n S((1-\eta_n)I + \eta_n S))v_n. \end{cases} \tag{1.7}$$

They established the weak convergence of scheme (1.7) under the condition that both T and S are quasi-pseudocontractive mappings which is more general than the classes of quasi-nonexpansive mappings, directed mappings and demi-contractive mappings.

Motived by above results, the purpose of this paper is to consider split equality fixed point problem (1.4) for the class of quasi-asymptotically pseudocontractive mappings which is more general than the classes of quasi-nonexpansive mappings and quasi-pseudocontractive mappings. Under suitable conditions, some weak and strong convergence theorems are proved.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $||\cdot||$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an operator.

Definition 2.1. An operator $T : C \to C$ is said to be pseudocontractive if

$$\langle Tx - Ty, x - y \rangle \le ||x - y||^2, \ \ \forall x, y \in C.$$

It is easy to know that T is pseudocontractive if and only if

$$||Tx - Ty||^2 \le ||x - y||^2 + ||(I - T)x - (I - T)y||^2, \ \ \forall x, y \in C.$$

Definition 2.2. $T : C \to C$ is said to be quasi-pseudocontractive if $Fix(T) \neq \emptyset$ and

$$||Tx - p||^2 \le ||x - p||^2 + ||Tx - x||^2, \ \ \forall x \in C, \forall p \in Fix(T).$$

Definition 2.3. An operator $T : C \to C$ is said to be asymptotically pseudocontractive if there exists a sequence $\{l_n\} \subset [1, +\infty)$ with $l_n \to 1$ such that

$$||T^n x - T^n y||^2 \le l_n ||x - y||^2 + ||(I - T^n)x - (I - T^n)y||^2,$$

for all $x, y \in C$ and for all $n \ge 1$.

It is well known that T is asymptotically pseudocontractive if and only if

$$\langle T^n x - T^n y, x - y \rangle \le \frac{l_n + 1}{2} ||x - y||^2,$$

for all $x, y \in C$ and $n \ge 1$.

Definition 2.4. An operator $T : C \to C$ is said to be quasi-asymptotically pseudocontractive if $Fix(T) \neq \emptyset$ and there exists a sequence $\{l_n\} \subset [1, +\infty)$ with $l_n \to 1$ such that

$$||T^n x - p||^2 \le l_n ||x - p||^2 + ||T^n x - x||^2, \tag{2.1}$$

for all $x \in C, p \in Fix(T)$ and for all $n \ge 1$.

Example 2.5. [16] *Let C be a unit ball in a real Hilbert space l^2 and let $S : C \to C$ be a mapping defined by*

$$S : (x_1, x_2, \cdots) \to (0, x_1^2, a_2 x_2, a_3 x_3, \cdots).$$

It is proved in [16] that

(i) $||Sx-Sy||^2 \leq (2||x-y||)^2,\ \forall x,y \in C;$

(ii) $||S^n x - S^n y||^2 \leq (2\prod_{j=2}^{n} a_j ||x-y||)^2,\ \forall x,y \in C,\ \forall n \geq 2.$

Taking $a_j = 2^{-\frac{1}{2^{j-1}}},\ j \geq 2$, *it is easy to see that* $\prod_{j=2}^{n} a_j = \frac{1}{2}$. *So we can take* $l_1 = 4,\ \ l_n = (2\prod_{j=2}^{n} a_j)^2,\ \forall n \geq 2$. *Then we have*

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} (2\prod_{j=2}^{n} 2^{-\frac{1}{2^{j-1}}})^2 = 1.$$

Therefore $S : C \to C$ *is an asymptotically pseudocontractive mapping with* $Fix(S) = \{(0,0,0,\cdots)\}$, *so it is also a quasi-asymptotically pseudocontractive mapping.*

Definition 2.6. An operator $T : C \to C$ is said to be uniformly L-Lipschtzian if there exists some $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x-y||,$$

for all $x,y \in C$ and for all $n \geq 1$.

Definition 2.7. (1) An operator $T : C \to C$ is said to be demiclosed at 0 if, for any sequence $\{x_n\} \subset C$ which converges weakly to x and $||x_n - T(x_n)|| \to 0$, then $Tx = x$.

(2) An operator $T : H \to H$ is said to be semi-compact if, for any bounded sequence $\{x_n\} \subset H$ with $||x_n - Tx_n|| \to 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x \in H$.

Lemma 2.8. *Let H be a real Hilbert space. For any $x,y \in H$, the following conclusions hold:*

$$||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2,\ t \in [0,1], \tag{2.2}$$

$$||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle. \tag{2.3}$$

Recall that a Banach space X is said to satisfy Opial's condition [17], if for any sequence $\{x_n\}$ in X which converges weakly to x^*, then

$$\limsup_{n\to\infty} ||x_n - x^*|| < \limsup_{n\to\infty} ||x_n - y||,\ \forall y \in X \text{with } y \neq x^*.$$

Lemma 2.9. [18] *Let* $\{p_n\},\{q_n\}$ *and* $\{r_n\}$ *be the nonnegative real sequences satisfying the following conditions*

$$p_{n+1} \le (1+q_n)p_n + r_n,\ n \ge 0,\ \sum_{n=1}^{\infty} q_n < \infty,\ \text{and}\ \sum_{n=1}^{\infty} r_n < \infty.$$

Then (i) $\lim_{n\to\infty} p_n$ *exists; (ii) if* $\liminf_{n\to\infty} p_n = 0$*, then* $\lim_{n\to\infty} p_n = 0$.

Lemma 2.10. *Let* H *be a real Hilbert space,* $T : H \to H$ *be a uniformly* L*-Lipschitzian and* $\{l_n\}$*-quasi-asymptotically pseudocontractive mapping with* $\{L_n\} \subset [1,\infty)$*,* $L \ge 1$ *and* $\lim_{n\to\infty} l_n = 1$*. Let* $\{K_n : H \to H\}$ *be a sequence of mappings defined by:*

$$K_n := (1-\xi)I + \xi T^n((1-\eta)I + \eta T^n). \tag{2.4}$$

If $0 < \xi < \eta < \frac{1}{\frac{M+1}{2}+\sqrt{\frac{(M+1)^2}{4}+L^2}}$*, where* $M = \sup_{n\ge 1} l_n$*, then the following conclusions hold:*

(i) $Fix(T) = Fix(T^n((1-\eta)I+\eta T^n)) = Fix(K_n)$ *for all* $n \ge 1$*;*

(ii) If T *is demiclosed at 0, then* K_1 *is also demiclosed at 0;*

(iii) For all $n \ge 1$ *and for all* $x \in H, u^* \in Fix(T) = Fix(K_n)$*,*

$$||K_n x - u^*|| \le k_n ||x - u^*||,$$

where $k_n = 1 + \xi(l_n - 1)(1 + \eta l_n), \{k_n\} \subset [1,+\infty)$ *and* $\lim_{n\to\infty} k_n = 1$.

Proof. (i) If $x^* \in Fix(T)$, i.e., $x^* = Tx^*$, we have

$$T^n((1-\eta)I+\eta T^n)x^* = T^n((1-\eta)x^* + \eta T^n x^*) = T^n x^* = x^*.$$

This shows that $x^* \in Fix(T^n((1-\eta)I+\eta T^n))$. Conversely, if $x^* \in Fix(T^n((1-\eta)I+\eta T^n))$ for all $n \ge 1$, i.e., $x^* = T^n((1-\eta)I+\eta T^n)x^*$, letting $U^n = (1-\eta)I+\eta T^n$, we have $T^n U^n x^* = x^*$. Putting $U^n x^* = y^*$, we have $T^n y^* = x^*$. Now we prove that $x^* = y^*$. In fact, we have

$$\begin{aligned}||x^* - y^*|| =& ||x^* - U^n x^*|| = ||x^* - ((1-\eta)I + \eta T^n)x^*|| \\ =& \eta ||x^* - T^n x^*|| = \eta ||T^n y^* - T^n x^*|| \le L\eta ||x^* - y^*||.\end{aligned}$$

Since $0 < L\eta < 1$, we have $x^* = y^*$, i.e., $x^* \in Fix(T)$. This shows that $Fix(T) = Fix(T^n((1-\eta)I+\eta T^n))$ for all $n \ge 1$. It is obvious that $x \in Fix(K_n)$ if and only if $x \in Fix(T^n((1-\eta)I+\eta T^n))$. The conclusion (i) is proved.

(ii) For any sequence $\{x_n\} \subset H$ satisfying $x_n \rightharpoonup x^*$ and $||x_n - Kx_n|| \to 0$, we show that $x^* \in Fix(K)$. From conclusion (i), we only need to prove that $x^* \in Fix(T)$. In fact, since T is L-Lipschizian, we have

$$\begin{aligned}||x_n - Tx_n|| \leq & ||x_n - T((1-\eta)I + \eta T)x_n|| + ||T((1-\eta)I + \eta T)x_n - Tx_n|| \\ = & \frac{1}{\xi}||x_n - (1-\xi)x_n - \xi T((1-\eta)I + \eta T)x_n|| + L\eta||x_n - Tx_n|| \\ = & \frac{1}{\xi}||x_n - K_1x_n|| + L\eta||x_n - Tx_n||.\end{aligned}$$

Simplifying it, we have

$$||x_n - Tx_n|| \leq \frac{1}{\xi(1-L\eta)}||x_n - K_1x_n|| \to 0. \tag{2.5}$$

Since T is demiclosed at 0, we have $x^* \in F(T) = F(K)$. The conclusion (ii) is proved.

(iii) For all $u^* \in Fix(T)$, from (2.1) we have

$$\begin{aligned}& ||T^n((1-\eta)I + \eta T^n)x - u^*||^2 \\ \leq & l_n||(1-\eta)x + \eta T^n x - u^*||^2 + ||((1-\eta)I + \eta T^n)x - T^n((1-\eta)I + \eta T^n)x||^2 \\ = & l_n||(1-\eta)(x-u^*) + \eta(T^n x - u^*)||^2 + ||((1-\eta)I + \eta T^n)x - T^n((1-\eta)I + \eta T^n)x||^2\end{aligned} \tag{2.6}$$

and

$$||T^n x - u^*||^2 \leq l_n||x - u^*||^2 + ||x - T^n x||^2. \tag{2.7}$$

Since T is L-Lip and $x - ((1-\eta)x + \eta T^n x) = \eta(x - T^n x)$, we have

$$||T^n x - T^n((1-\eta)x + \eta T^n x)|| \leq L||x - ((1-\eta)x + \eta T^n x)|| = L\eta||x - T^n x||. \tag{2.8}$$

From (2.2) and (2.7), we have

$$\begin{aligned}& ||(1-\eta)(x-u^*) + \eta(T^n x - u^*)||^2 \\ = & (1-\eta)||x-u^*||^2 + \eta||T^n x - u^*||^2 - \eta(1-\eta)||x - T^n x||^2 \\ \leq & (1-\eta)||x-u^*||^2 + \eta(l_n||x-u^*||^2 + ||x - T^n x||^2) - \eta(1-\eta)||x - T^n x||^2 \\ = & (1 + \eta(l_n - 1))||x - u^*||^2 + \eta^2||x - T^n x||^2.\end{aligned} \tag{2.9}$$

From (2.2) and (2.8), we have

$$\begin{aligned}
&||((1-\eta)I+\eta T^n)x-T^n((1-\eta)I+\eta T^n)x||^2\\
=&||(1-\eta)(x-T^n((1-\eta)x+\eta T^nx))+\eta(T^nx-T^n((1-\eta)x+\eta T^nx))||^2\\
=&(1-\eta)||x-T^n((1-\eta)x+\eta T^nx)||^2+\eta||T^nx-T^n((1-\eta)x+\eta T^nx)||^2\\
&-\eta(1-\eta)||x-T^nx||^2\\
\le&(1-\eta)||x-T^n((1-\eta)x+\eta T^nx)||^2-\eta(1-\eta-\eta^2L^2)||x-T^nx||^2.
\end{aligned} \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.6), we obtain

$$\begin{aligned}
&||T^n((1-\eta)I+\eta T^n)x-u^*||^2\\
\le&l_n(1+\eta(l_n-1))||x-u^*||^2+l_n\eta^2||T^nx-x||^2\\
&+(1-\eta)||x-T^n((1-\eta)x+\eta T^nx)||^2-\eta(1-\eta-\eta^2L^2)||T^nx-x||^2\\
=&l_n(1+\eta(l_n-1))||x-u^*||^2+(1-\eta)||x-T^n((1-\eta)x+\eta T^nx)||^2\\
&-\eta(1-\eta-\eta^2L^2-l_n\eta)||T^nx-x||^2.
\end{aligned} \tag{2.11}$$

Since $\eta<\frac{1}{\frac{M+1}{2}+\sqrt{\frac{(M+1)^2}{4}+L^2}}$, we deduce $1-\eta-\eta^2L^2-l_n\eta>0$. From (2.11), we get

$$\begin{aligned}
&||T^n((1-\eta)x+\eta T^nx)-u^*||^2\\
\le&l_n(1+\eta(l_n-1))||x-u^*||^2+(1-\eta)||x-T^n((1-\eta)x+\eta T^nx)||^2.
\end{aligned} \tag{2.12}$$

Combining (2.2) and (2.12), we have

$$\begin{aligned}
&||K_nx-u^*||^2=||(1-\xi)x+\xi T^n((1-\eta)x+\eta T^nx)-u^*||^2\\
=&(1-\xi)||x-u^*||^2+\xi||T^n((1-\eta)x+\eta T^nx)-u^*||^2-\xi(1-\xi)||x-T^n((1-\eta)x+\eta T^nx)||^2\\
\le&(1-\xi)||x-u^*||^2+\xi l_n(1+\eta(l_n-1))||x-u^*||^2\\
&+(\xi(1-\eta)-\xi(1-\xi))||x-T^n((1-\eta)x+\eta T^nx)||^2\\
=&(1+\xi(l_n-1)(1+\eta l_n))||x-u^*||^2-\xi(\eta-\xi)||x-T^n((1-\eta)x+\eta T^nx)||^2.
\end{aligned}$$

This together with $\xi<\eta$ implies that $||K_nx-u^*||^2\le k_n||x-u^*||^2$ for all $x\in H, u^*\in Fix(K_n)$ and $n\ge 1$, where $k_n=1+\xi(l_n-1)(1+\eta l_n)$. In view of that $\{l_n\}\subset[1,+\infty)$ and $\lim_{n\to\infty}l_n=1$ we have $\{k_n\}\subset[1,+\infty)$ and $\lim_{n\to\infty}k_n=1$. The conclusion (iii) is proved.

2. Main results

Throughout this section, we assume that:

(1) H_1, H_2 and H_3 are three real Hilbert spaces, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators and A^* and B^* are the adjoint operators of A and B, respectively;

(2) $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are two uniformly L-Lipschitzian and $\{l_n\}$-quasi-asymptotically pseudocontractive mappings with $L \geq 1,\ l_n \in [1,\infty),\ l_n \to 1$ and $\Sigma_{n=1}^{\infty}(l_n^2 - 1) < \infty$, $Fix(T) \neq \emptyset$, and $Fix(S) \neq \emptyset$.

In the sequel, we denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$, respectively.

Our object is to solve the following split equality fixed point problem:

$$find\ x \in Fix(T), y \in Fix(S)\ such\ that\ Ax^* = By^*. \tag{3.1}$$

In the sequel we use Γ to denote the set of solutions of (3.1), that is,

$$\Gamma = \{(x^*, y^*) \in Fix(T) \times Fix(S)\ such\ that\ Ax^* = By^*\}, \tag{3.2}$$

and we assume that $\Gamma \neq \emptyset$.

Now, we present our theorem for finding $(x^*, y^*) \in \Gamma$.

Theorem 3.1. *Let H_1, H_2, H_3, A, B, S, T, Γ, $\{l_n\}$ be the same as above. Choose $\{\alpha_{n,i}\} \subset (0,1)$ such that for each $n \geq 1$, $\sum_{i=0}^{+\infty} \alpha_{n,i} = 1$ and for each $i \geq 0$, $\liminf_{n\to\infty} \alpha_{n,i} > 0$. Taking arbitrary $x_0 \in H_1, y_0 \in H_2$, the sequence $\{(x_n, y_n)\}$ is generated by:*

$$\begin{cases} (a)\ u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ (b)\ x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^{+\infty} \alpha_{n,i} K_i u_n, \\ (c)\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ (d)\ y_{n+1} = \alpha_{n,0}y_n + \sum_{i=1}^{+\infty} \alpha_{n,i} G_i v_n, \end{cases} \tag{3.3}$$

where $K_i = (1-\xi)I + \xi T^i((1-\eta)I + \eta T^i)$ and $G_i = (1-\xi)I + \xi S^i((1-\eta)I + \eta S^i)$.

If T and S are demiclosed at 0 and the following conditions are satisfied:

(i) $\gamma_n \in (0, \min(\frac{1}{||A||^2}, \frac{1}{||B||^2}))$, $\forall n \geq 1$ with $\liminf_{n\to\infty} \gamma_n > 0$;

(ii) $0 < a < \xi < \eta < b < \frac{1}{\frac{M+1}{2}+\sqrt{\frac{(M+1)^2}{4}+L^2}}$, $\forall n \geq 1$, *where* $M = \sup_{n\geq 1} l_n$.
Then the following conclusions hold: (I) $\{(x_n, y_n)\}$ *converges weakly to a solution of problem (3.1); (II) In addition, if both* S *and* T *are also semi-compact, then* $\{(x_n, y_n)\}$ *converges strongly to a solution of problem (3.1).*

Proof. First we prove the conclusion (I).

For any given $(p,q) \in \Gamma$, we have $p \in Fix(T), q \in Fix(S)$ and $Ap = Bq$. From (3.3)(a), we have

$$\begin{aligned}||u_n - p||^2 =&||x_n - \gamma_n A^*(Ax_n - By_n) - p||^2\\ =&||x_n - p||^2 + \gamma_n^2||A^*(Ax_n - By_n)||^2 - 2\gamma_n\langle x_n - p, A^*(Ax_n - By_n)\rangle\\ \leq&||x_n - p||^2 + \gamma_n^2||A||^2||Ax_n - By_n||^2 - 2\gamma_n\langle Ax_n - Ap, Ax_n - By_n\rangle.\end{aligned} \tag{3.4}$$

Similarly, from (3.3)(c), we have

$$||v_n - q||^2 \leq ||y_n - q||^2 + \gamma_n^2||B||^2||Ax_n - By_n||^2 + 2\gamma_n\langle By_n - Bq, Ax_n - By_n\rangle. \tag{3.5}$$

By condition (ii) and Lemma 2.9, sequences $\{K_i\}$ and $\{G_i\}$ have the following properties:

(i) $Fix(T) = Fix(K_i)$ and $Fix(S) = Fix(G_i)$ for all $i \geq 1$;

(ii) K_1 and G_1 are demiclosed at 0;

(iii) For all $i \geq 1$ and for all $x \in H_1$, $y \in H_2$, $u^* \in Fix(T) = Fix(K_i), v^* \in Fix(S) = Fix(G_i)$, $||K_i x - u^*|| \leq k_i||x - u^*||$, $||G_i y - v^*|| \leq k_i||y - v^*||$, where $k_i = 1 + \xi(l_i - 1)(1 + \eta l_i), \{k_i\} \subset [1, +\infty)$ and $\lim_{i\to\infty} k_i = 1$. By the assumption that $\sum_{i=1}^{\infty}(l_i^2 - 1) < \infty$, therefore we have

$$\sum_{i=1}^{\infty}(k_i - 1) \leq \sum_{i=1}^{\infty}\xi(l_i - 1)(l_i + 1) \leq \sum_{i=1}^{\infty}(l_i^2 - 1) < \infty. \tag{3.6}$$

Hence from (3.3)(b) and (2.1), for any positive integer $l \geq 1$ we have

$$\begin{aligned}||x_{n+1} - p||^2 =&||\alpha_{n,0}(x_n - p) + \sum_{i=1}^{+\infty}\alpha_{n,i}(K_i u_n - p)||^2\\ \leq&\alpha_{n,0}||x_n - p||^2 + \sum_{i=1}^{+\infty}\alpha_{n,i}||K_i u_n - p||^2 - \alpha_{n,0}\alpha_{n,l}||K_l u_n - x_n||^2\\ \leq&\alpha_{n,0}||x_n - p||^2 + \sum_{i=1}^{+\infty}\alpha_{n,i}k_i^2||u_n - p||^2 - \alpha_{n,0}\alpha_{n,l}||K_l u_n - x_n||^2.\end{aligned} \tag{3.7}$$

Similarly from (3.3)(c) and (2.1), we have

$$||y_{n+1}-q||^2 \leq \alpha_{n,0}||y_n-q||^2+\sum_{i=1}^{+\infty}\alpha_{n,i}k_i^2||v_n-q||^2-\alpha_{n,0}\alpha_{n,l}||G_lv_n-y_n||^2. \tag{3.8}$$

Adding up (3.7) and (3.8) and by using (3.4) and (3.5), we have that

$$\begin{aligned}
&||x_{n+1}-p||^2+||y_{n+1}-q||^2 \\
&\leq \alpha_{n,0}||x_n-p||^2+\alpha_{n,0}||y_n-q||^2+\sum_{i=1}^{+\infty}\alpha_{n,i}k_i^2(||u_n-p||^2+||v_n-q||^2) \\
&-\alpha_{n,0}\alpha_{n,l}(||K_lu_n-x_n||^2+||G_lv_n-y_n||^2) \\
&\leq \alpha_{n,0}||x_n-p||^2+\sum_{i=1}^{+\infty}\alpha_{n,i}k_i^2\{||x_n-p||^2+\gamma_n^2||A||^2||Ax_n-By_n||^2 \\
&-2\gamma_n\langle Ax_n-Ap, Ax_n-By_n\rangle\}+\alpha_{n,0}||y_n-q||^2+\sum_{i=1}^{+\infty}\alpha_{n,i}k_i^2\{||y_n-q||^2+\gamma_n^2||B||^2||Ax_n-By_n||^2 \\
&+2\gamma_n\langle By_n-Bq, Ax_n-By_n\rangle\}-\alpha_{n,0}\alpha_{n,l}(||K_lu_n-x_n||^2+||G_lv_n-y_n||^2) \\
&=(1+\sum_{i=1}^{+\infty}\alpha_{n,i}(k_i^2-1))\{||x_n-p||^2+||y_n-q||^2\} \\
&\quad+\sum_{i=1}^{+\infty}\alpha_{n,i}k_i^2\gamma_n^2(||A||^2+||B||^2)||Ax_n-By_n||^2-2\sum_{i=1}^{+\infty}\alpha_{n,i}k_i^2\gamma_n||Ax_n-By_n||^2 \\
&-\alpha_{n,0}\alpha_{n,l}(||K_lu_n-x_n||^2+||G_lv_n-y_n||^2).
\end{aligned} \tag{3.9}$$

Since $\gamma_n \in (0, \min\{\frac{1}{||A||^2}, \frac{1}{||B||^2}\})$, $\gamma_n||A||^2<1$ and $\gamma_n||B||^2<1$, we get $0<\gamma_n(||A||^2+||B||^2)<2$. This implies that $\gamma_n(2-\gamma_n(||A||^2+||B||^2))>0$. Putting

$$X_n(p,q)=||x_n-p||^2+||y_n-q||^2, \tag{3.10}$$

hence (3.9) can be written as

$$\begin{aligned}
X_{n+1}(p,q) &\leq (1+\sum_{i=1}^{+\infty}\alpha_{n,i}(k_i^2-1))X_n(p,q)-\sum_{i=1}^{+\infty}\alpha_{n,i}k_i^2\gamma_n(2-\gamma_n(||A||^2+||B||^2))||Ax_n-By_n||^2 \\
&\quad-\alpha_{n,0}\alpha_{n,l}(||K_lu_n-x_n||^2+||G_lv_n-y_n||^2) \\
&\leq (1+\sum_{i=1}^{+\infty}\alpha_{n,i}(k_i^2-1))X_n(p,q) \\
&=(1+\sigma_n)X_n(p,q),
\end{aligned} \tag{3.11}$$

where $\sigma_n = (\sum_{i=1}^{+\infty} \alpha_{n,i}(k_i^2 - 1))$. Since $k_n \to 1$, and by (3.6), $\sum_{i=1}^{+\infty}(k_i - 1) < \infty$. This implies that $\sum_{i=1}^{+\infty}(k_i^2 - 1) < \infty$. Again since

$$\begin{aligned}\sum_{n=1}^{+\infty} \sigma_n &= \sum_{n=1}^{+\infty}\sum_{i=1}^{+\infty} \alpha_{n,i}(k_i^2 - 1)) \\ &= \sum_{i=1}^{+\infty}(k_i^2 - 1)\sum_{n=1}^{+\infty} \alpha_{n,i} \\ &\leq \sum_{i=1}^{+\infty}(k_i^2 - 1) < \infty,\end{aligned}$$

and $\sigma_n \to 0$, by virtue of Lemma 2.8, the limit $\lim_{n\to\infty} X_n(p,q)$ exists. Therefore the following limits exist:

$$\lim_{n\to\infty} ||x_n - p|| \ \ and \ \ \lim_{n\to\infty} ||y_n - q|| \ \forall\, (p,q) \in \Gamma. \tag{3.12}$$

Rewrite (3.11) as

$$\begin{aligned}&\sum_{i=1}^{+\infty} \alpha_{n,i} k_i^2 \gamma_n (2 - \gamma_n(||A||^2 + ||B||^2))||Ax_n - By_n||^2 \\ &+ \alpha_{n,0}\alpha_{n,l}(||K_l u_n - x_n||^2 + ||G_l v_n - y_n||^2) \\ &\qquad \leq (1+\sigma_n)X_n(p,q) - X_{n+1}(p,q).\end{aligned} \tag{3.13}$$

Letting $n \to \infty$ and taking limit in (3.13), we have for all $l = 1, 2, \cdots$

$$||Ax_n - By_n|| \to 0; \ \ ||K_l u_n - x_n|| \to 0; \ \ ||G_l v_n - y_n|| \to 0. \tag{3.14}$$

From (3.14) and (3.3) we have that

$$\begin{cases} \lim_{n\to\infty} ||u_n - x_n|| = 0 \ \ and \ \ \lim_{n\to\infty} ||v_n - y_n|| = 0. \\ \lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} \Sigma_{i=1}^{\infty} \alpha_{n,i} ||K_i u_n - x_n|| \\ \qquad \leq \lim_{n\to\infty} \Sigma_{i=1}^{\infty} \alpha_{n,i} \sup_{i\in I} ||K_i u_n - x_n|| \\ \qquad \leq \lim_{n\to\infty} \sup_{i\in I} ||K_i u_n - x_n|| = 0. \\ \lim_{n\to\infty} ||y_{n+1} - y_n|| = \lim_{n\to\infty} \Sigma_{i=1}^{\infty} \alpha_{n,i} ||G_i v_n - y_n|| \\ \qquad \leq \lim_{n\to\infty} \Sigma_{i=1}^{\infty} \alpha_{n,i} \sup_{i\in I} ||G_i v_n - y_n|| \\ \qquad \leq \lim_{n\to\infty} \sup_{i\in I} ||G_i v_n - y_n|| = 0. \end{cases} \tag{3.15}$$

This together with (3.14) shows that

$$\begin{cases} ||K_1u_n - u_n|| \leq ||K_1u_n - x_n|| + ||x_n - u_n|| \to 0; \\ ||G_1v_n - v_n|| \leq ||G_1v_n - y_n|| + ||y_n - v_n|| \to 0. \end{cases} \tag{3.16}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded sequences, there exist some weakly convergent subsequences, say $\{x_{n_i}\} \subset \{x_n\}$ and $\{y_{n_i}\} \subset \{y_n\}$ such that $x_{n_i} \rightharpoonup x^*$ and $y_{n_i} \rightharpoonup y^*$. Since every Hilbert space has the Opial's property which guarantees that the weakly subsequential limit of $\{(x_n, y_n)\}$ is unique. Therefore we have $x_n \rightharpoonup x^*$, and $y_n \rightharpoonup y^*$.

On the other hand, from (3.15), it gets that $u_n \rightharpoonup x^*$ and $v_n \rightharpoonup y^*$. By (3.16) and the demiclosed property of K_1 and G_1, we have $K_1x^* = x^*$ and $G_1y^* = y^*$. This implies that $x^* \in Fix(T)$ and $y^* \in Fix(S)$.

Now we show that $Ax^* = By^*$. In fact, since $Ax_n - By_n \rightharpoonup Ax^* - By^*$, by using the weakly lower semi-continuity of norm, we have

$$||Ax^* - By^*||^2 \leq \liminf_{n\to\infty} ||Ax_n - By_n||^2 = \lim_{n\to\infty} ||Ax_n - By_n||^2 = 0.$$

Thus $Ax^* = By^*$. This completes the proof of the conclusion (I).

Now we prove the conclusion (II). In fact, since K_1 is uniformly continuous, we have $||K_1u_n - K_1x_n|| \to 0$ as $n \to \infty$. Hence from (3.15), we have

$$||x_n - K_1x_n|| \leq ||x_n - K_1u_n|| + ||K_1u_n - K_1x_n|| \to 0(n \to \infty). \tag{3.17}$$

Similarly, we can also prove that

$$||y_n - G_1y_n|| \to 0(n \to \infty). \tag{3.18}$$

By virtue of (2.5), (3.14),(3.17) and (3.18), we have

$$\begin{cases} ||x_n - Tx_n|| \leq \dfrac{1}{\xi(1-L\eta)}||x_n - K_1x_n|| \to 0(n \to \infty); \\ ||y_n - Sy_n|| \leq \dfrac{1}{\xi(1-L\eta)}||y_n - G_1y_n|| \to 0(n \to \infty). \end{cases} \tag{3.19}$$

Since S, T are semi-compact, it follows from (3.19) that there exist subsequences $\{x_{n_i}\} \subset \{x_n\}$ and $\{y_{n_j}\} \subset \{y_n\}$ such that $x_{n_i} \to x$ (some point in $Fix(T)$) and $y_{n_j} \to y$ (some point in $Fix(S)$).

It follows from (3.12), $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup y^*$ that $x_n \to x^*$ and $y_n \to y^*$ and $Ax^* = By^*$. This completes the proof.

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