J. Nonlinear Funct. Anal. 2017 (2017), Article ID 27 https://doi.org/10.23952/jnfa.2017.27

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

A FAMILY OF BALANCED AND ABSORBING SETS WITH EMPTY INTERIOR

FRANCISCO JAVIER GARCÍA-PACHECO*, ENRIQUE NARANJO-GUERRA

Department of Mathematical Sciences, University of Cadiz, Puerto Real, Spain, 11519, EU

Abstract. A balanced and absorbing subset with empty interior has already been explicitly constructed in every normed space of dimension strictly greater than 1 (see [4]). However this construction varies depending whether the normed space is separable or not. In this note, we provide a unique construction by means of a family of balanced and absorbing sets with empty interior in every normed space of dimension strictly greater than 1.

Keywords. Absorbing; Balanced; Convex; Empty interior; LCH.

2010 Mathematics Subject Classification. 54C05, 46A03, 46A55.

1. Introduction

Throughout this paper, all locally convex Hausdorff topological vector spaces (LCH) will be considered to have dimension strictly greater than 1. If X is an LCH, then by $\mathscr{A}_X$, $\mathscr{B}_X$, $\mathscr{C}_X$ and $\mathscr{F}_X$ we denote the set of absorbing, balanced, convex, and closed subsets of X, respectively. The filter of neighborhoods of 0 will be denoted by $\mathscr{U}_0$. It is trivial that $\mathscr{U}_0 \subseteq \mathscr{A}_X$.

Recall that every vector space can be endowed, for instance, with a norm or with the finest locally convex vector topology. A vector space endowed with the finest locally convex vector topology is called a finest LCH. Note that

*Corresponding author.

E-mail addresses: garcia.pacheco@uca.es (F. J. García-Pacheco), enrique.naranjo@alum.uca.es (E. Naranjo-Guerra).

Received March 8, 2017; Accepted March 26, 2017.

- If X is finest, then $\mathscr{U}_0 \cap \mathscr{C}_X = \mathscr{A}_X \cap \mathscr{C}_X$ (see [2, TVS II.26], [1, Page 142, Exercise 4.1.22] and [3, Theorem 2.1]).
- If X is Baire, then $\mathscr{U}_0 \cap \mathscr{F}_X = \mathscr{A}_X \cap \mathscr{F}_X$ (it is a trivial exercise).
- If X is barrelled, then $\mathscr{U}_0 \cap \mathscr{B}_X \cap \mathscr{C}_X \cap \mathscr{F}_X = \mathscr{A}_X \cap \mathscr{B}_X \cap \mathscr{C}_X \cap \mathscr{F}_X$ (by definition of barrelledness).
- If X is normed and separable, then $\mathscr{U}_0 \cap \mathscr{B}_X \subsetneq \mathscr{A}_X \cap \mathscr{B}_X$ (see [4, Theorem 3.2]).
- If X is normed and infinite dimensional, then $\mathscr{U}_0 \cap \mathscr{B}_X \cap \mathscr{C}_X \subsetneq \mathscr{A}_X \cap \mathscr{B}_X \cap \mathscr{C}_X$ (see [4, Corollary 2.2]).

In [4, Theorem 3.2] it was explicitly constructed an example of a balanced and absorbing bounded set with empty interior in every separable normed space. By relying on this fact, the strict inclusion $\mathscr{U}_0 \cap \mathscr{B}_X \subsetneq \mathscr{A}_X \cap \mathscr{B}_X$ can be easily shown for X an LCH.

Theorem 1.1. *If X is an LCH, then X has a balanced and absorbing subset with empty interior. In particular, $\mathscr{U}_0 \cap \mathscr{B}_X \subsetneq \mathscr{A}_X \cap \mathscr{B}_X$.*

Proof By [4, Theorem 3.2] we may assume that X is infinite dimensional. Let Y be a finite dimensional subspace of X and let Z be a topological complement for Y in X. In accordance with [4, Theorem 3.2], we can find a balanced and absorbing subset W_Y of Y with empty interior. Now fix an arbitrary balanced and absorbing neighborhood U_Z of 0 in Z. We will show that $W := W_Y + U_Z$ verifies the desired properties. It is a straight forward exercise to check that W is balanced and absorbing. Finally, since linear projections are open mappings, if W has non-empty interior in X, then W_Y has non-empty interior in Y, which contradicts the choice of W_Y. As a consequence, W has empty interior in X.

Notice that, according to the items remarked at the beginning of the introduction, the best strict inclusion concerning the sets $\mathscr{A}_X$, $\mathscr{B}_X$, $\mathscr{C}_X$, $\mathscr{F}_X$ and $\mathscr{U}_0$ is given by Theorem 1.1.

The aim of this paper is to provide a unique construction of a balanced and absorbing set with empty interior that works for all normed spaces, not only for separable normed spaces like in [4, Theorem 3.2].

The standard notation used throughout this manuscript follows to conclude the introduction:

- $\mathbb{K}$ stands for the real or complex field.

- If Z is a topological space and A is a subset of Z, then $\operatorname{int}(A)$, $\operatorname{cl}(A)$, and $\operatorname{bd}(A)$ will denote the topological interior, the topological closure, and the topological boundary of A, respectively.
- Given a normed space X, the open unit ball of X will be denoted by U_X, the closed unit ball or simply the unit ball of X will be denoted by B_X, and the unit sphere of X is S_X. The open ball of center x and raduis s will be denoted by $\mathsf{U}_X(x,s)$, the closed ball of center x and radius s will be denoted by $\mathsf{B}_X(x,s)$, and the unit sphere of center x and radius s is $\mathsf{S}_X(x,s)$.

2. Main results

2.1. Construction of the set W_γ

We recall the reader that if A is an absorbing set, then $\mathbf{j}_A$ stands for the Minkowski functional of A. Here we will apply the inverse process, that is, we start off with a function and end up with an absorbing (and balanced) set.

Let us define the following equivalence relation on S_X

$$\mathscr{R} := \{(x,y) \in \mathsf{S}_X \times \mathsf{S}_X : x \in \mathbb{K}y\}.$$

A function $\gamma : \mathsf{S}_X \to (0,1]$ is said to be $\mathscr{R}$-invariant provided that $\gamma(x) = \gamma(y)$ whenever $(x,y) \in \mathscr{R}$.

Definition 2.1.1. Let X be a normed space and $\gamma : \mathsf{S}_X \to (0,1]$ $\mathscr{R}$-invariant. We define the following two balanced and absorbing sets generated by γ:

- $\mathsf{W}_\gamma := \bigcup_{x \in \mathsf{S}_X} \gamma(x)\mathsf{B}_{\mathbb{K}}x.$
- $\mathsf{V}_\gamma := \bigcup_{x \in \mathsf{S}_X} \gamma(x)\mathsf{U}_{\mathbb{K}}x.$

The following proposition highlights the main characteristics of the set W_γ.

Proposition 2.1.2. *Let X be a normed space and $\gamma : \mathsf{S}_X \to (0,1]$ $\mathscr{R}$-invariant.*

(1) $\mathsf{W}_\gamma = \left\{x \in X \setminus \{0\} : \|x\| \leq \gamma\left(\frac{x}{\|x\|}\right)\right\} \cup \{0\}.$
(2) $\mathsf{V}_\gamma = \left\{x \in X \setminus \{0\} : \|x\| < \gamma\left(\frac{x}{\|x\|}\right)\right\} \cup \{0\}.$
(3) $\mathbf{j}_{\mathsf{W}_\gamma}(x) = \frac{\|x\|}{\gamma\left(\frac{x}{\|x\|}\right)}$ *for all $x \neq 0$.*

(4) V_γ *is dense in* W_γ *and both* W_γ *and* V_γ *are balanced, absorbing and are contained in* B_X.

(5) If W *is a bounded, balanced and absorbing subset of* X*, then there exists an* $\mathscr{R}$*-invariant function* $\gamma : \mathsf{S}_X \to (0,1]$ *and* $\alpha > 0$ *such that* $\mathsf{V}_\gamma \subseteq \alpha W \subseteq \mathsf{W}_\gamma$.

Proof. We will only sketch the proofs of the first, third and last items, since the proofs of the other items are either similar or trivial.

(1) Let $x \in \mathsf{W}_\gamma \setminus \{0\}$ and write $x = \gamma(y)ty$ for some $t \in \mathsf{B}_\mathbb{K}$ and $y \in \mathsf{S}_X$. Note that $\|x\| = |t|\gamma(y) \leq \gamma(y) = \gamma\left(\frac{x}{\|x\|}\right)$ since $\left(y, \frac{x}{\|x\|}\right) \in \mathscr{R}$. Conversely, let $x \in X \setminus \{0\}$ with $\|x\| \leq \gamma\left(\frac{x}{\|x\|}\right)$. Observe that

$$x = \gamma\left(\frac{x}{\|x\|}\right) \frac{\|x\|}{\gamma\left(\frac{x}{\|x\|}\right)} \frac{x}{\|x\|} \in \gamma\left(\frac{x}{\|x\|}\right) \mathsf{B}_\mathbb{K} \frac{x}{\|x\|} \subseteq \mathsf{W}_\gamma.$$

(3) Recall that $\mathbf{j}_{\mathsf{W}_\gamma}(x) = \inf\{\lambda > 0 : x \in \lambda \mathsf{W}_\gamma\}$. On the one hand, by the first item we have that

$$\gamma\left(\frac{x}{\|x\|}\right) \frac{x}{\|x\|} \in \mathsf{W}_\gamma,$$

so

$$x \in \frac{\|x\|}{\gamma\left(\frac{x}{\|x\|}\right)} \mathsf{W}_\gamma,$$

which means that

$$\mathbf{j}_{\mathsf{W}_\gamma}(x) \leq \frac{\|x\|}{\gamma\left(\frac{x}{\|x\|}\right)}.$$

On the other hand, if $\lambda > 0$ is so that $x \in \lambda \mathsf{W}_\gamma$, then $\frac{1}{\lambda} x \in \mathsf{W}_\gamma$ and again by the first item we deduce that

$$\frac{1}{\lambda}\|x\| = \left\|\frac{1}{\lambda} x\right\| \leq \gamma\left(\frac{x}{\|x\|}\right),$$

that is,

$$\frac{\|x\|}{\gamma\left(\frac{x}{\|x\|}\right)} \leq \lambda,$$

and hence

$$\frac{\|x\|}{\gamma\left(\frac{x}{\|x\|}\right)} \leq \mathbf{j}_{\mathsf{W}_\gamma}(x).$$

(5) We may assume without any loss of generality that W is a balanced and absorbing subset of X contained in B_X. Define

$$\gamma(x) := \frac{1}{\mathbf{j}_W(x)}$$

for all $x \in \mathsf{S}_X$. Observe the following:

- γ is well defined. Indeed, if $\mathbf{j}_W(x) = 0$ for some $x \in \mathsf{S}_X$, then there exist a null sequence $(\lambda_n)_{n\in\mathbb{N}}$ of strictly positive numbers and a sequence $(w_n)_{n\in\mathbb{N}} \subseteq W$ such that $x = \lambda_n w_n$ for every $n \in \mathbb{N}$. Since $\|w_n\| \leq 1$ for all $n \in \mathbb{N}$ and $(\lambda_n)_{n\in\mathbb{N}}$ converges to 0, we deduce that $x = 0$.
- $\mathsf{V}_\gamma = (\mathbf{j}_W)^{-1}([0,1))$. Indeed, for all nonzero x we have that
$$\gamma\left(\frac{x}{\|x\|}\right) = \frac{1}{\mathbf{j}_W\left(\frac{x}{\|x\|}\right)} = \frac{\|x\|}{\mathbf{j}_W(x)},$$
which implies that $\mathbf{j}_W(x) < 1$ if and only if $\|x\| < \gamma\left(\frac{x}{\|x\|}\right)$.
- $W_\gamma = (j_W)^{-1}([0,1])$. It follows a similar proof as the above bullet.

By taking into account the previous proposition, the following example is nearly trivial.

Example 2.1.3. *Let X be a normed space and consider another norm $|\cdot|$ on X such that $\|\cdot\| \leq |\cdot|$. If we denote by $\mathsf{B}_{|\cdot|}$ to the unit ball induced by the norm $|\cdot|$, then it is not difficult to see that $\mathsf{B}_{|\cdot|} = \mathsf{W}_\gamma$ where*

$$\begin{array}{rccl} \gamma: & \mathsf{S}_X & \to & (0,1] \\ & x & \mapsto & \gamma(x) := \frac{1}{|x|}. \end{array}$$

We will finish this subsection with examples of a W_γ-set which is not linearly closed and a V_γ-set which is not linearly open. Recall that a set in a vector space is linearly open if it is composed only of internal points (see [2, TVS II.26]) and linearly closed if its complementary is linearly open.

Example 2.1.4. *Let X be a normed space and fix arbitrary elements $y, z \in \mathsf{S}_X$ such that $(y,z) \notin \mathscr{R}$. Define*

$$\begin{array}{rccl} \gamma: & \mathsf{S}_X & \to & (0,1] \\ & x & \mapsto & \gamma(x) := \begin{cases} 1/5 & x \in \mathbb{K}y, \\ 3/5 & x \notin \mathbb{K}y \cup \mathbb{K}z, \\ 1 & x \in \mathbb{K}z. \end{cases} \end{array}$$

Note that

- $\frac{2}{5}y \in X \setminus \mathsf{W}_\gamma$ *but it is not an internal point of* $X \setminus \mathsf{W}_\gamma$.
- $\frac{4}{5}z \in \mathsf{V}_\gamma$ *but it is not an internal point of* V_γ.

2.2. Closure of W_γ

Recall that a function $f : A \to \overline{\mathbb{R}}$ between a metric space A and the extended real line is said to be upper semicontinuous at $a \in A$ provided that

$$\limsup_{b \to a} f(b) \leq f(a).$$

Theorem 2.2.1. *Let X be a normed space and $\gamma : \mathsf{S}_X \to (0,1]$ $\mathscr{R}$-invariant. Then W_γ is closed if and only if γ is upper semicontinuous.*

Proof. Assume first that γ is upper semicontinuous. Let $(y_n)_{n\in\mathbb{N}} \subseteq \mathsf{W}_\gamma \setminus \{0\}$ be convergent to some $y \in \mathsf{B}_X \setminus \{0\}$. In accordance with Proposition 2.1.2 (1) and taking into consideration that γ is upper semicontinuous, we have that $\|y_n\| \leq \gamma\left(\frac{y_n}{\|y_n\|}\right)$ for all $n \in \mathbb{N}$, which means that

$$\|y\| \leq \limsup_{n\to\infty} \gamma\left(\frac{y_n}{\|y_n\|}\right) \leq \gamma\left(\frac{y}{\|y\|}\right).$$

By applying again Proposition 2.1.2 (1), we obtain that $y \in \mathsf{W}_\gamma$.

Conversely, suppose that W_γ is closed. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ be convergent to some $x \in \mathsf{S}_X$. In virtue of Proposition 2.1.2 (1), we have that $(\gamma(x_n)x_n)_{n\in\mathbb{N}} \subseteq \mathsf{W}_\gamma \setminus \{0\}$. Since W_γ is closed, we have that

$$\left(\limsup_{n\to\infty} \gamma(x_n)\right) x \in \mathsf{W}_\gamma.$$

Now Proposition 2.1.2 (1) comes again into play to assure that

$$\limsup_{n\to\infty} \gamma(x_n) = \left\| \left(\limsup_{n\to\infty} \gamma(x_n)\right) x \right\| \leq \gamma(x).$$

In order to find the closure of W_γ we will strongly rely on the function

$$\begin{array}{rccl} \overline{\gamma}: & \mathsf{S}_X & \to & (0,1] \\ & a & \mapsto & \overline{\gamma}(x) := \max\left\{\gamma(x), \limsup_{y\to x} \gamma(y)\right\}, \end{array}$$

which trivially verifies the following properties:

- If $x \in \mathsf{S}_X$, then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ converging to x such that $(\gamma(y_n))_{n\in\mathbb{N}}$ converges to $\overline{\gamma}(x)$.

- $\overline{\gamma}$ is upper semi-continuous and $\mathscr{R}$-invariant.

Corollary 2.2.2. *Let X be a normed space and $\gamma\colon \mathsf{S}_X \to (0,1]$ $\mathscr{R}$-invariant. Then* $\operatorname{cl}\left(\mathsf{W}_\gamma\right) = \mathsf{W}_{\overline{\gamma}}$.

Proof.

- $\subseteq$ Since $\gamma \leq \overline{\gamma}$, Proposition 2.1.2 (1) allows us to conclude that $\mathsf{W}_\gamma \subseteq \mathsf{W}_{\overline{\gamma}}$. By keeping in mind that $\overline{\gamma}$ is upper semicontinuous and in accordance with Theorem 2.2.1, we deduce that $\operatorname{cl}\left(\mathsf{W}_\gamma\right) \subseteq \mathsf{W}_{\overline{\gamma}}$.
- $\supseteq$ Let $x \in \mathsf{S}_X$ and $t \in \mathsf{B}_{\mathbb{K}}$. There exists a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ converging to x such that $(\gamma(y_n))_{n\in\mathbb{N}}$ converges to $\overline{\gamma}(x)$. Then $(\gamma(y_n)ty_n)_{n\in\mathbb{N}} \subseteq \mathsf{W}_\gamma$ converges to $\overline{\gamma}(x)tx$.

As an immediate consequence of the previous corollary we obtain a sufficient condition for W_γ to be dense in B_X.

Scholium 2.2.3. Let X be a normed space and $\gamma\colon \mathsf{S}_X \to (0,1]$ $\mathscr{R}$-invariant. If $\gamma^{-1}(\{1\})$ is dense in S_X, then W_γ is dense in B_X.

Proof. It only suffices to bear in mind Corollary 2.2.2 together with the fact that $\mathsf{W}_{\overline{\gamma}} = \mathsf{B}_X$ if and only if $\overline{\gamma}$ is the constant function 1.

2.3. Interior of W_γ

First off, observe that the lack of convexity of W_γ could make possible that W_γ not be a neighborhood of 0 and have non-empty interior.

Theorem 2.3.1. *Let X be a normed space and $\gamma : \mathsf{S}_X \to (0,1]$ $\mathscr{R}$-invariant. Let $\varepsilon > 0$ and $z \in X \setminus \{0\}$ and $v \in \mathsf{S}_X$.*

(1) $\mathsf{B}_X(0,\varepsilon) \subseteq \mathsf{W}_\gamma$ *if and only if* $\varepsilon \leq \inf\limits_{x\in\mathsf{S}_X} \gamma(x)$.
(2) If $\mathsf{B}_X(z,\varepsilon) \subseteq \mathsf{W}_\gamma$, *then* $\|z\| \leq \inf\left\{\gamma(x) : x \in \mathsf{S}_X \cap \mathsf{B}_X\left(\frac{z}{\|z\|}, \frac{\varepsilon}{\|z\|}\right)\right\}$.
(3) If $\delta := \inf\{\gamma(x) : x \in \mathsf{S}_X \cap \mathsf{B}_X(v,\varepsilon)\} > 0$, *then* $\mathsf{B}_X\left(\frac{(2+\varepsilon)\delta}{2(1+\varepsilon)}v, \frac{\varepsilon\delta}{2(1+\varepsilon)}\right) \subseteq \mathsf{W}_\gamma$.

Proof.

(1) Assume first that $\mathsf{B}_X(0,\varepsilon) \subseteq \mathsf{W}_\gamma$ and fix an arbitrary $x \in \mathsf{S}_X$. Notice that $\varepsilon x \in \mathsf{B}_X(0,\varepsilon) \subseteq \mathsf{W}_\gamma$. From Proposition 2.1.2 (1) we deduce that $\varepsilon = \|\varepsilon x\| \leq \gamma(x)$. Conversely, assume that $\varepsilon \leq \inf_{x\in\mathsf{S}_X} \gamma(x)$ and let $y \in \mathsf{B}_X(0,\varepsilon)$. Since $0 \in \mathsf{W}_\gamma$, we may assume that $y \neq 0$. Then $\|y\| \leq \varepsilon \leq \gamma\left(\frac{y}{\|y\|}\right)$. Again by Proposition 2.1.2 (1), $y \in \mathsf{W}_\gamma$.

(2) Fix an arbitrary element $x \in \mathsf{S}_X \cap \mathsf{B}_X\left(\frac{z}{\|z\|}, \frac{\varepsilon}{\|z\|}\right)$. Notice that $\|z\|x \in \mathsf{B}_X(z,\varepsilon) \subseteq \mathsf{W}_\gamma$. In virtue of Proposition 2.1.2 (1), $\|z\| = \|\|z\|x\| \leq \gamma(x)$.

(3) Let $y \in \mathsf{B}_X\left(\frac{(2+\varepsilon)\delta}{2(1+\varepsilon)}v, \frac{\varepsilon\delta}{2(1+\varepsilon)}\right)$. Notice that

$$\frac{\delta}{1+\varepsilon} = \frac{(2+\varepsilon)\delta}{2(1+\varepsilon)} - \frac{\varepsilon\delta}{2(1+\varepsilon)} \leq \|y\| \leq \frac{(2+\varepsilon)\delta}{2(1+\varepsilon)} + \frac{\varepsilon\delta}{2(1+\varepsilon)} = \delta$$

and

$$\begin{aligned}
\left\|\frac{y}{\|y\|} - v\right\| &= \frac{\|y - \|y\|v\|}{\|y\|} \\
&\leq \frac{\left\|y - \frac{(2+\varepsilon)\delta}{2(1+\varepsilon)}v\right\| + \left\|\frac{(2+\varepsilon)\delta}{2(1+\varepsilon)}v - \|y\|v\right\|}{\|y\|} \\
&= \frac{\left\|y - \frac{(2+\varepsilon)\delta}{2(1+\varepsilon)}v\right\| + \left|\left\|\frac{(2+\varepsilon)\delta}{2(1+\varepsilon)}v\right\| - \|y\|\right|}{\|y\|} \\
&\leq \frac{2\left\|y - \frac{(2+\varepsilon)\delta}{2(1+\varepsilon)}v\right\|}{\|y\|} \\
&\leq 2\frac{\varepsilon\delta}{2(1+\varepsilon)}\frac{1+\varepsilon}{\delta} \\
&= \varepsilon.
\end{aligned}$$

As a consequence, $\frac{y}{\|y\|} \in \mathsf{S}_X \cap \mathsf{B}_X(v,\varepsilon)$, so by hypothesis

$$\|y\| \leq \delta \leq \gamma\left(\frac{y}{\|y\|}\right)$$

which implies in accordance with Proposition 2.1.2 (1) that $y \in \mathsf{W}_\gamma$.

A function $\gamma : \mathsf{S}_X \to (0,1]$ is said to be rare provided that $\inf\gamma(U) = 0$ for all open subsets U of S_X.

Corollary 2.3.2. *Let X be a normed space and $\gamma : \mathsf{S}_X \to (0,1]$ $\mathscr{R}$-invariant. Then* $\operatorname{int}(\mathsf{W}_\gamma) = \varnothing$ *if and only if γ is rare.*

As a consequence, $\{\mathsf{W}_\gamma : \gamma : \mathsf{S}_X \to (0,1]$ is $\mathscr{R}$-invariant and rare$\}$ is a family of balanced and absorbing sets with empty interior, provided that the set of proper $\mathscr{R}$-invariant functions is not empty. In the next subsection, we show examples of such functions.

2.4. The strict inclusion $\mathscr{U}_0 \cap \mathscr{B}_X \subsetneq \mathscr{A}_X \cap \mathscr{B}_X$

The final subsection begins with a combination of [4, Corollary 2.2 and Theorem 3.2] into a single result with a very simple proof. However first a couple of technical lemmas.

Lemma 2.4.1. *Let X be a separable normed space. If $(x_n)_{n\in\mathbb{N}}$ is a dense sequence in S_X with pairwise different elements, then*

$$\begin{array}{rcl} \gamma\colon \ \mathsf{S}_X & \to & (0,1] \\ x & \mapsto & \gamma(x) := \begin{cases} 1 & x \in \mathsf{S}_X \setminus \{\lambda x_n : \lambda \in \mathsf{S}_{\mathbb{K}}, n \in \mathbb{N}\}, \\ 1/n & x \in \mathbb{K}x_n, \end{cases} \end{array}$$

is an $\mathscr{R}$-invariant rare function.

The proof of the previous lemma is omitted for simplicity.

Lemma 2.4.2. *Let X be an infinite dimensional normed space. Let $\{e_i : i \in I\}$ be a Hamel basis for X contained in S_X except for a countably infinite subset $\{e_{i_n} : n \in \mathbb{N}\}$ which verifies that $\|e_{i_n}\| = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then*

$$\begin{array}{rcl} \gamma\colon \ \mathsf{S}_X & \to & (0,1] \\ x & \mapsto & \gamma(x) := \frac{1}{\sum_{i=1}^k |\lambda_i|} \text{ where } x = \sum_{i=1}^k \lambda_i e_i. \end{array}$$

is an $\mathscr{R}$-invariant rare function.

Proof. Let U be an open subset of S_X and consider any $x \in U$. We can fix an $\varepsilon > 0$ such that

$$\frac{x + \varepsilon n e_{i_n}}{\|x + \varepsilon n e_{i_n}\|} \in U$$

for all $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ so that if $n \geq n_0$, then e_{i_n} does not appear in the decomposition of x. Now observe that

$$\gamma\left(\frac{x + \varepsilon n e_{i_n}}{\|x + \varepsilon n e_{i_n}\|}\right) \leq \frac{\|x + \varepsilon n e_{i_n}\|}{\varepsilon n} \leq \frac{1 + \varepsilon}{\varepsilon n} \to 0 \text{ as } n \to \infty.$$

As a consequence, $\inf \gamma(U) = 0$.

Theorem 2.4.3. *If X is a normed space, then there exists a balanced and absorbing subset of X with empty interior.*

Proof. Simply consider the family $\{\mathsf{W}_\gamma : \gamma : \mathsf{S}_X \to (0,1] \text{ is } \mathscr{R}\text{-invariant and rare}\}$.

Acknowledgments

The work was supported by the MINECO and FEDER under Grant MTM2014-58984-P.

REFERENCES

[1] J. M. Borwein, J. D. Vanderwerff, Convex functions: constructions, characterizations and counterexamples, Encyclopedia of Mathematics and its Applications, 109. Cambridge University Press, Cambridge, 2010.

[2] N. Bourbaki, Topological vector spaces. Chapters 1–5. Translated from the French by H. G. Eggleston and S. Madan. Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987.

[3] F. J. García-Pacheco, Non-continuous linear functionals on topological vector spaces, Banach J. Math. Anal. 2 (2008), 11-15.

[4] F. J. García-Pacheco, E. Naranjo-Guerra, Balanced and absorbing sets with empty interior, Adv. Geom. 16 (2016), 477-480.