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PERIODIC SOLUTIONS FOR FOURTH-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH LINEAR AUTONOMOUS DIFFERENCE OPERATORS

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Abstract. In this article, we discuss the properties of linear autonomous difference operator $(Ax)(t) = x(t) - c(t)x(t-\delta(t))$. By applying the Green's function of fourth-order differential equations and a fixed point theorem in cones, we obtain some sufficient conditions for the existence, nonexistence, multiplicity of positive periodic solutions for a generalized fourth-order neutral differential equation with variable parameters.

Keywords. Neutral operator; Linear autonomous difference operator; Positive periodic solution; Green function.

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1. Introduction

Neutral functional differential equations manifest themselves in many fields including biology, mechanics and economics [1-4]. For example, in population dynamics, since a growing population consume more (or less) food than a matured one, depending on individual species, this leads to neutral functional equations [1]. These equations also arise in classical "cobweb" models in economics where current demand depends on price but supply depends on the previous periodic [2]. The study on neutral functional differential equations is more intricate than

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ordinary delay differential equations. In recent years, there are a good amount of work on periodic solutions for neutral differential equations; see [5-19] and the references cited therein. For example, in [13], Wu and Wang discussed the second-order neutral delay differential equation

$$(x(t)-cx(t-\delta))''+a(t)x(t)=\lambda b(t)f(x(t-\tau(t))). \tag{1.1}$$

Based on a fixed point theorem, they obtained some existence results of positive periodic solutions for (1.1). Recently, Du, Guo, Ge and Lu [15] studied a kind of generalized Liénard neutral differential equation

$$(x(t)-c(t)x(t-\delta))''+f(x(t))x'(t)+g(x(t-\tau(t))=e(t). \tag{1.2}$$

By means of the Mawhin's continuation theorem, they obtained sufficient conditions for the existence of periodic solutions of (1.2). Recently, Ren, Cheng, Siegmud [17] considered second-order neutral functional differential equation

$$(x(t)-cx(t-\delta(t)))''=-a(t)x(t)+\lambda b(t)f(x(t-\tau(t))). \tag{1.3}$$

By an application of the fixed-point index theorem, they obtained the sufficient conditions for the existence, multiplicity and nonexistence of positive periodic solutions of (1.3).

In general, most of the existing results are concentrated on second-order neutral functional differential equations, while studies on fourth-order neutral functional differential equations are rather infrequent, especially on the positive periodic solutions for fourth-order neutral functional differential equations with variable parameters. Motivated by [13, 15, 17], we consider the following fourth-order neutral functional differential equation

$$(x(t)-c(t)x(t-\delta(t)))^{(4)}=a(t)x(t)-\lambda b(t)f(x(t-\tau(t))), \tag{1.4}$$

here λ is a positive parameter; $f\in C(\mathbb{R},[0,\infty))$, and $f(x)>0$ for $x>0$; $a\in C(\mathbb{R},(0,\infty))$ with $\max\{a(t):t\in[0,\omega]\}<(\frac{\pi}{\omega})^4$, $b\in C(\mathbb{R},(0,\infty))$, $c,\ \delta,\ \tau\in C(\mathbb{R},\mathbb{R})$, $a(t)$, $b(t)$, $c(t)$, $\delta(t)$ and $\tau(t)$ are ω-periodic functions.

Notice here that neutral operator $(Ax)(t)=x(t)-cx(t-\delta(t))$ is a natural generalization of the familiar operator $(A_1x)(t)=x(t)-cx(t-\delta)$, $(A_2x)(t)=x(t)-c(t)x(t-\delta)$, $(A_3x)(t)=x(t)-cx(t-\delta(t))$. But A possesses a more complicated nonlinearity than A_1, A_2, A_3. For example, neutral operators A_1 is homogeneous in the following sense $(A_1x)'(t)=(A_1x')(t)$, whereas neutral operator A in general is inhomogeneous. As a consequence, many of the new results for differential equations with the neutral operator A will not be a direct extension of known theorems for neutral differential equations.

The paper is organized as follows. In Section 2, we first analyze qualitative properties of generalized neutral operator A which will be used to further study differential equations with this neutral operator. In Section 3, we consider a fourth-order constant coefficient linear differential equations and present their Green's functions and properties for those equations. In Section 4, by an application of the fixed-point index theorem we obtain sufficient conditions for the existence, multiplicity and nonexistence of positive periodic solutions of fourth-order neutral differential equations. We also give an example to illustrate our main results.

2. Analysis of the generalized neutral operator with linear autonomous difference operators

Let $c_\infty = \max_{t\in[0,\omega]} |c(t)|$, $c_0 = \min_{t\in[0,\omega]} |c(t)|$. Let $X = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t), t \in \mathbb{R}\}$ with norm $\|x\| = \max_{t\in[0,\omega]} |x(t)|$, and let $C^+_\omega = \{x \in C(\mathbb{R},(0,\infty)) : x(t+\omega) = x(t)\}$, $C^-_\omega = \{x \in C(\mathbb{R},(-\infty,0)) : x(t+\omega) = x(t)\}$. Then $(X,\|\cdot\|)$ is a Banach space. A cone K in X is defined by $K = \{x \in X : x(t) \geq \alpha\|x\|, \forall\, t \in \mathbb{R}\}$, where α is a fixed positive number with $\alpha < 1$. Moreover, define operators A, $B : C_\omega \to C_\omega$ by

$$(Ax)(t) = x(t) - c(t)x(t-\delta(t)),\ (Bx)(t) = c(t)x(t-\delta(t)).$$

Lemma 2.1. (see [20]) *If $|c(t)| \neq 1$, then operator A has a continuous inverse A^{-1} on C_ω, satisfying*

(1) $$\left(A^{-1}f\right)(t) = \begin{cases} f(t) + \sum\limits_{j=1}^{\infty}\prod\limits_{i=1}^{j} c(D_i)x\left(t - \sum\limits_{i=1}^{j}\delta(D_i)\right), \ for\ |c(t)| < 1,\ \forall f \in C_\omega, \\ -\frac{f(t+\delta(t))}{c(t+\delta(t))} - \sum\limits_{j=1}^{\infty} \frac{f\left(t+\delta(t)+\sum\limits_{i=1}^{j}\delta(D'_i)\right)}{c(t+\delta(t))\prod\limits_{i=1}^{j} c(D'_i)}, \ for\ |c(t)| > 1,\ \forall f \in C_\omega. \end{cases}$$

(2) $$\left|\left(A^{-1}f\right)(t)\right| \leq \begin{cases} \frac{\|f\|}{1-c_\infty}, \ for\ c_\infty < 1\ \forall\ f \in C_\omega, \\ \frac{\|f\|}{c_0-1}, \ for\ c_0 > 1\ \forall\ f \in C_\omega. \end{cases}$$

(3) $$\int_0^\omega \left|\left(A^{-1}f\right)(t)\right| dt \leq \begin{cases} \frac{1}{1-c_\infty}\int_0^\omega |f(t)|dt, \ for\ c_\infty < 1\ \forall\ f \in C_\omega, \\ \frac{1}{c_0-1}\int_0^\omega |f(t)|dt, \ for\ c_0 > 1\ \forall\ f \in C_\omega. \end{cases}$$

Lemma 2.2. *If $c(t) < 0$ and $\sigma c_\infty < \alpha$, where $\sigma = \frac{1-c_0^2}{1-c_\infty^2} > 1$, we have for $y \in K$ that*

$$\left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right)\|y\| \le \left(A^{-1}y\right)(t) \le \frac{1}{1-c_\infty}\|y\|.$$

Proof. Since $c(t) < 0$, and $|c(t)| \le c_\infty < \sigma c_\infty < \alpha < 1$, by Lemma 2.1, we have for $y \in K$ that

$$\begin{aligned}
(A^{-1}y)(t) &= y(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(D_i)y(s - \sum_{i=1}^{j}\delta(D_i)) \\
&= y(t) + \sum_{j\ge 1 \text{ even}}\prod_{i=1}^{j} c(D_i)y(t - \sum_{i=1}^{j}\delta(D_i)) - \sum_{j\ge 1 \text{ odd}}\prod_{i=1}^{j}|c(D_i)|y(t - \sum_{i=1}^{j}\delta(D_i)) \\
&\ge \alpha||y|| + \alpha\sum_{j\ge 1 \text{ even}} c_0^j||y|| - ||y||\sum_{j\ge 1 \text{ odd}} c_\infty^j \\
&= \frac{\alpha}{1-c_0^2}||y|| - \frac{c_\infty}{1-c_\infty^2}||y|| \\
&= \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right)||y||.
\end{aligned}$$

Lemma 2.3. *If $c(t) > 0$ and $c(t) < 1$, then for $y \in K$ we have*

$$\frac{\alpha}{1-c_0}\|y\| \le \left(A^{-1}y\right)(t) \le \frac{1}{1-c_\infty}\|y\|.$$

Proof. Since $c(t) > 0$, and $c(t) < 1$, $\alpha < 1$, by Lemma 2.2, we have for $y \in K$ that

$$\begin{aligned}
(A^{-1}y)(t) &= y(t) + \sum_{j\ge 1}\prod_{i=1}^{j} c(D_i)y(t - \sum_{i=1}^{j}\delta(D_i)) \\
&\ge \alpha||y|| + \alpha||y||\sum_{j\ge 1} c_0^j \\
&= \frac{\alpha}{1-c_0}||y||.
\end{aligned}$$

3. Green's function of fourth-order differential equations

Lemma 3.1. *For $\rho > 0$ and $h \in X$, equation*

$$\begin{cases} u^{(4)} - \rho^4 u = h(t), \\ u^{(i)}(0) = u^{(i)}(\omega),\ i = 0,1,2,3, \end{cases} \tag{3.1}$$

has a unique solution which is of the form

$$u(t)=\int_0^\omega G(t,s)(-h(s))\mathrm{d}s, \tag{3.2}$$

where

$$G(t,s)=\begin{cases}\frac{\exp(\rho(t-s))+\exp(\rho(s+\omega-t))}{4\rho^3(\exp(\rho\omega)-1)}+\frac{\sin\rho(t-s)-\sin\rho(t-s-\omega)}{4\rho^3(1-\cos\rho\omega)},\ 0\le s\le t\le\omega,\\ \frac{\exp(\rho(t+\omega-s))+\exp(\rho(s-t))}{4\rho^3(\exp(\rho\omega)-1)}+\frac{\sin\rho(s-t)-\sin\rho(s-\omega-t)}{4\rho^3(1-cos\rho\omega)},\ 0\le t\le s\le\omega.\end{cases} \tag{3.3}$$

Proof. It is easy to check that the associated homogeneous equation of (3.1) has the solution

$$v(t)=c_1\exp(\rho t)+c_2\exp(-\rho t)+c_3\cos\rho t+c_4\sin\rho t.$$

Applying the method of variation of parameters, we get

$$c_1'(t)=\frac{\exp(-\rho t)h(t)}{4\rho^3},\ c_2'(t)=\frac{-\exp(\rho t)h(t)}{4\rho^3},$$
$$c_3'(t)=\frac{\sin(\rho t)h(t)}{2\rho^3},\ c_4'(t)=\frac{-\cos(\rho t)h(t)}{2\rho^3}.$$

It follows that

$$c_1(t)=c_1(0)+\int_0^t\frac{\exp(-\rho s)h(s)}{4\rho^3}ds,\ c_2(t)=c_2(0)+\int_0^t-\frac{\exp(\rho s)h(s)}{4\rho^3}ds,$$
$$c_3(t)=c_3(0)+\int_0^t\frac{\sin(\rho s)h(s)}{2\rho^3}ds,\ c_4(t)=c_4(0)+\int_0^t-\frac{cos(\rho s)h(s)}{2\rho^3}ds.$$

Noting that $u(0)=u(\omega),\ u'(0)=u'(\omega),\ u''(0)=u''(\omega),\ u'''(0)=u'''(\omega)$, we obtain

$$c_1(0)=\int_0^\omega\frac{\exp(\rho(\omega-s))}{4\rho^3(1-\exp(\rho\omega))}h(s)ds,$$

$$c_2(0)=\int_0^\omega\frac{\exp(\rho s)}{4\rho^3(1-\exp(\rho\omega))}h(s)ds,$$

$$c_3(0)=-\int_0^\omega\frac{\sin(\rho s)-\sin(\rho(s-\omega))}{4\rho^3(1-\cos\rho\omega)}h(s)ds$$

and

$$c_4(0)=-\int_0^\omega\frac{\cos(\rho(s-\omega))-\cos(\rho s)}{4\rho^3(1-\cos\rho\omega)}h(s)ds.$$

This implies that

$$\begin{aligned}u(t) =& c_1(t)\exp(\rho t)+c_2(t)\exp(-\rho t)+c_3(t)\cos\rho t+c_4(t)\sin\rho t\\ =&\int_0^t\left\{\frac{\exp(\rho(t-s))+\exp(\rho(s+\omega-t))}{4\rho^3(\exp(\rho\omega)-1)}+\frac{\sin\rho(t-s)-\sin\rho(t-s-\omega)}{4\rho^3(1-\cos\rho\omega)}\right\}\\ &\times(-h(s))ds\\ &+\int_t^\omega\left\{\frac{\exp(\rho(t+\omega-s))+\exp(\rho(s-t))}{4\rho^3(\exp(\rho\omega)-1)}+\frac{\sin\rho(s-t)-\sin\rho(s-t-\omega)}{4\rho^3(1-\cos\rho\omega)}\right\}\\ &\times(-h(s))ds\\ =&\int_0^\omega G(t,s)(-h(s))ds,\end{aligned}$$

where $G_1(t,s)$ is defined as in (3.3).

By direct calculation, we get the solution u satisfies the periodic boundary value condition of the problem (3.1).

Now we present the properties of the Green's functions for (3.1).

Lemma 3.2. *$\int_0^\omega G(t,s)\mathrm{d}s=\frac{1}{\rho^4}$ and if $\rho<\frac{\pi}{\omega}$ holds, then $0<l<G(t,s)\leq L$ for all $t\in[0,\omega]$ and $s\in[0,\omega]$.*

Proof. From (3.3), we find $\int_0^\omega G(t,s)\mathrm{d}s=\frac{1}{\rho^4}$. If $\rho<\frac{\pi}{\omega}$, we get $G(t,s)>0$ for all $t\in[0,\omega]$ and $s\in[0,\omega]$.

Next we compute a lower and an upper bound for $G(t,s)$ for $s\in[0,\omega]$. We have

$$l:=\frac{\exp\left(\frac{\rho\omega}{2}\right)}{2\rho^3(\exp(\rho\omega)-1)}\leq G(t,s)<\frac{1+\exp(\rho\omega)}{4\rho^3(\exp(\rho\omega)-1)}+\frac{1}{2\rho^3(1-\cos\rho\omega)}:=L$$

and the proof is complete.

4. Positive periodic solutions for neutral equations

Define the Banach space X as in Section 2. Denote

$$M=\max\{a(t):t\in[0,\omega]\},\ m=\min\{a(t):t\in[0,\omega]\},\ \rho^4=M,$$

$$k=l(M+m)+\sigma LM,\ k_1=\frac{k-\sqrt{k^2-4\sigma LlMm}}{2\sigma LM},\ \alpha=\frac{l[m-(M+m)c_\infty]}{LM(1-c_\infty)}.$$

It is easy to see that $M,m,\beta,k,k_1>0$.

Now we consider (1.4). Let

$$\overline{f}_0=\overline{\lim_{x\to 0}}\frac{f(x)}{x},\ \overline{f}_\infty=\overline{\lim_{x\to\infty}}\frac{f(x)}{x},\ \underline{f}_0=\underline{\lim_{x\to 0}}\frac{f(x)}{x},\ \underline{f}_\infty=\underline{\lim_{x\to\infty}}\frac{f(x)}{x},$$

and denote

$$\bar{i}_0=\text{ number of 0's in }(\overline{f}_0,\overline{f}_\infty),\quad \underline{i}_0=\text{ number of 0's in }(\underline{f}_0,\underline{f}_\infty);$$

$$\bar{i}_\infty=\text{ number of }\infty\text{'s in }(\overline{f}_0,\overline{f}_\infty),\quad ,\underline{i}_\infty=\text{ number of }\infty\text{'s in }(\underline{f}_0,\underline{f}_\infty).$$

It is clear that $\bar{i}_0,\ \underline{i}_0,\ \bar{i}_\infty,\ \underline{i}_\infty\in\{0,1,2\}$. We will show that (1.4) has $\bar{i}_0$ or $\underline{i}_\infty$ positive w-periodic solutions for sufficiently large or small λ, respectively.

In the following we discuss (1.4) in two cases, namely, the case where $c(t)<0$, and $-c_\infty>-\min\{k_1,\frac{m}{M+m}\}$.

From $-c_\infty>-\frac{m}{M+m}$, we have $\alpha=\frac{l[m-(M+m)c_\infty]}{LM(1-c_\infty)}>\frac{l\left(m-(M+m)\cdot\frac{m}{M+m}\right)}{LM(1-c_\infty)}=0$. So, we get $\alpha>0$. Moreover, we consider equation

$$\sigma LMx^2-kx+lm=0.$$

The equation has a solution $x=k_1=\frac{k-\sqrt{k^2-4\sigma LlMm}}{2\sigma LM}$. From $c_\infty<k_1$, we find

$$\sigma LMc_\infty^2-kc_\infty+lm<0.$$

So, we have

$$\sigma LMc_\infty^2-(l(M+m)+\sigma LM)c_\infty+lm<0,$$

which implies that

$$\sigma c_\infty>\frac{l[m-(M+m)c_\infty]}{LM(1-c_\infty)}=\alpha.$$

On the other hand, the case where $c>0$ and $c_\infty<\min\{\frac{m}{M+m},\frac{LM-lm}{(L-l)M-lm}\}$, (note that $c_\infty<\frac{m}{M+m}$ implies $\alpha>0$; $c_\infty<\frac{LM-lm}{(L-l)M-lm}$ implies $\alpha<1$). Obviously, we have $c_\infty<1$ which makes Lemma 2.1 applicable for both cases, and also Lemma 2.2 or 2.3, respectively.

Let $K=\{x\in X: x(t)\geq\alpha\|x\|\}$ denote the cone in X as defined in Section 2, where α is just as defined above. We also use $K_r=\{x\in K:\|x\|<r\}$ and $\partial K_r=\{x\in K:\|x\|=r\}$. Letting $y(t)=(Ax)(t)$, we find from Lemma 2.1 that $x(t)=(A^{-1}y)(t)$. Hence (1.4) can be transformed into

$$y^{(4)}(t)-a(t)(A^{-1}y)(t)=-\lambda b(t)f((A^{-1}y)(t-\tau(t))),\tag{4.1}$$

which can be further rewritten as

$$y^{(4)}(t)-a(t)y(t)+a(t)H(y(t))=-\lambda b(t)f((A^{-1}y)(t-\tau(t))), \tag{4.2}$$

where $H(y(t))=y(t)-\left(A^{-1}y\right)(t)=-c(t)\left(A^{-1}y\right)(t-\delta(t))$.

Now we discuss the two cases separately.

Case I: $c(t)<0$ and $-c_\infty>-\min\{k_1,\frac{m}{M+m}\}$

Now we consider

$$y^{(4)}(t)-a(t)y(t)+a(t)H(y(t))=h(t),\ h\in C_\omega^-, \tag{4.3}$$

and define operators $T,\hat{H}:X\to X$ by

$$(Th)(t)=\int_t^{t+\omega}G(t,s)(-h(s))ds,\ (\hat{H}y)(t)=-M+a(t)y(t)-a(t)H(y(t)).$$

Clearly $T,\hat{H}$ are completely continuous, $(Th)(t)>0$ for $h(t)<0$ and $\|\hat{H}\|\le(M-m+M\frac{c_\infty}{1-c_\infty})$. By (3.3), the solution of (4.3) can be written in the form

$$y(t)=(Th))(t)+(T\hat{H}y)(t). \tag{4.4}$$

In view of $c(t)<0$, and $-c_\infty>-\min\{k_1,\frac{m}{M+m}\}$, we find

$$\|T\hat{H}\|\le\|T\|\|\hat{H}\|\le\frac{M-m+mc_\infty}{M(1-c_\infty)}<1, \tag{4.5}$$

where we used $\int_t^{t+\omega}G(t,s)ds=\frac{1}{M}$. Hence

$$y(t)=(I-T\hat{H})^{-1}(Th)(t).$$

Define an operator $P:X\to X$ by

$$(Ph)(t)=(I-T\hat{H})^{-1}(Th)(t).$$

Obviously, for any $h\in C_\omega^-$, if $\max\{a(t):t\in[0,\omega]\}<(\frac{\pi}{\omega})^4$, $y(t)=(Ph)(t)$ is the unique positive ω-periodic solution of (4.3).

Lemma 4.1. *P is completely continuous and*

$$(Th)(t)\le(Ph)(t)\le\frac{M(1-c_\infty)}{m-(M+m)c_\infty}\|Th\|,\ \textit{for all}\ h\in C_\omega^-. \tag{4.6}$$

Proof. By the Neumann expansion of P, we have

$$\begin{aligned} P &= (I - T\hat{H})^{-1}T \\ &= (I + T\hat{H} + (T\hat{H})^2 + \cdots + (T\hat{H})^n + \cdots)T \\ &= T + T\hat{H}T + (T\hat{H})^2 T + \cdots + (T\hat{H})^n T + \cdots . \end{aligned} \tag{4.7}$$

Since T and $\hat{H}$ are completely continuous, so is P. Moreover, by (4.7), and recalling that $\|T\hat{H}\| \leq \frac{M-m+mc_\infty}{M(1-c_\infty)} < 1$, we get

$$(Th)(t) \leq (Ph)(t) \leq \frac{M(1-c_\infty)}{m-(M+m)c_\infty}\|Th\|.$$

Define an operator $Q : X \to X$, by

$$Qy(t) = P(\lambda b(t) f((A^{-1}y)(t-\tau(t)))). \tag{4.8}$$

Lemma 4.2. $Q(K) \subset K$.

Proof. From the definition of Q, it is easy to verify that $Qy(t+\omega) = Qy(t)$. For $y \in K$, we have

$$\begin{aligned} Qy(t) &= P(\lambda b(t) f((A^{-1}y)(t-\tau(t)))) \\ &\geq T(\lambda b(t) f((A^{-1}y)(t-\tau(t)))) \\ &= \lambda \int_t^{t+\omega} G(t,s) b(s) f[(A^{-1}y)(s-\tau(s))] ds \\ &\geq \lambda l \int_0^{\omega} b(s) f[(A^{-1}y)(s-\tau(s))] \mathrm{d}s. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} Qy(t) &= P(\lambda b(t) f((A^{-1}y)(t-\tau(t)))) \\ &\leq \tfrac{M(1-c_\infty)}{m-(M+m)c_\infty} \|T(\lambda b(t) f((A^{-1}y)(t-\tau(t))))\| \\ &= \lambda \tfrac{M(1-c_\infty)}{m-(M+m)c_\infty} \max_{t\in[0,\omega]} \textstyle\int_t^{t+\omega} G(t,s) b(s) f((A^{-1}y)(s-\tau(s))) ds \\ &\leq \lambda \tfrac{M(1-c_\infty)}{m-(M+m)c_\infty} L \textstyle\int_0^{\omega} b(s) f((A^{-1}y)(s-\tau(s))) ds. \end{aligned}$$

Therefore

$$Qy(t) \geq \frac{l[m-(M+m)c_\infty]}{LM(1-c_\infty)}\|Qy\| = \alpha \|Qy\|,$$

i.e., $Q(K) \subset K$. This completes the proof.

From the continuity of P, it is easy to verify that Q is completely continuous in X. Comparing (4.2) to (4.3), it is obvious that the existence of periodic solutions for equation (4.2) is equivalent to the existence of fixed-points for the operator Q in X. Recalling Lemma 4.2, the existence of positive periodic solutions for (4.2) is equivalent to the existence of fixed-points of Q in K. Furthermore, if Q has a fixed-point y in K, it means that $(A^{-1}y)(t)$ is a positive $\omega-$periodic solutions of (1.4).

Lemma 4.3. *If there exists $\eta > 0$ such that*

$$f((A^{-1}y)(t-\tau(t))) \geq (A^{-1}y)(t-\tau(t))\eta, \text{ for } t \in [0,\omega] \text{ and } y \in K,$$

then

$$\|Qy\| \geq \lambda l\eta \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right) \int_0^\omega b(s)\mathrm{d}s\|y\|, \ y \in K.$$

Proof. By Lemma 2.2 and Lemma 4.1, we have for $y \in K$ that

$$\begin{aligned}
Qy(t) &= P\left(\lambda b(t) f((A^{-1}y)(t-\tau(t)))\right) \\
&\geq T\left(\lambda b(t) f((A^{-1}y)(t-\tau(t)))\right) \\
&= \lambda \int_t^{t+\omega} G(t,s) b(s) f((A^{-1}y)(s-\tau(s))) ds \\
&\geq \lambda l\eta \int_0^\omega b(s)(A^{-1}y)(s-\tau(s)) ds \\
&\geq \lambda l\eta \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right) \int_0^\omega b(s) ds\|y\|.
\end{aligned}$$

Hence

$$\|Qy\| \geq \lambda l\eta \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right) \int_0^\omega b(s) ds\|y\|, \ y \in K.$$

Lemma 4.4. *If there exists $\varepsilon > 0$ such that*

$$f((A^{-1}y)(t-\tau(t))) \leq (A^{-1}y)(t-\tau(t))\varepsilon, \text{ for } t \in [0,\omega] \text{ and } y \in K,$$

then

$$\|Qy\| \leq \lambda\varepsilon \frac{LM\int_0^\omega b(s)ds}{m-(M+m)c_\infty}\|y\|, \ y \in K.$$

Proof. By Lemma 2.2 and Lemma 4.1, we have

$$\begin{aligned}\|Qy(t)\| &\leq \lambda \frac{M(1-c_\infty)}{m-(M+m)c_\infty} L \int_0^\omega b(s) f((A^{-1}y)(s-\tau(s)))ds \\ &\leq \lambda \frac{M(1-c_\infty)}{m-(M+m)c_\infty} L\varepsilon \int_0^\omega b(s)(A^{-1}y)(s-\tau(s))ds \\ &\leq \lambda \varepsilon \frac{LM\int_0^\omega b(s)ds}{m-(M+m)c_\infty}\|y\|.\end{aligned}$$

Define

$$F(r) = \max\left\{ f(t) : 0 \leq t \leq \frac{r}{1-c_\infty} \right\},$$
$$f_1(r) = \min\left\{ f(t) : \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2} \right) r \leq t \leq \frac{r}{1-c_\infty} \right\}.$$

Lemma 4.5. *If $y \in \partial K_r$, then*

$$\|Qy\| \geq \lambda l f_1(r) \int_0^\omega b(s)ds.$$

Proof. By Lemma 2.2, we obtain $\left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right) r \leq (A^{-1}y)(t-\tau(t)) \leq \frac{r}{1-c_\infty}$ for $y \in \partial K_r$, which yields $f((A^{-1}y)(t-\tau(t))) \geq f_1(r)$. The Lemma now follows analog to the proof of Lemma 4.3.

Lemma 4.6. *If $y \in \partial K_r$, then*

$$\|Qy\| \leq \lambda \frac{LM(1-c_\infty)F(r)}{m-(M+m)c_\infty} \int_0^\omega b(s)ds.$$

Proof. By Lemma 2.2, we find that

$$0 \leq (A^{-1}y)(t-\tau(t)) \leq \frac{r}{1-c_\infty}, \quad y \in \partial K_r,$$

which yields $f((A^{-1}y)(t-\tau(t))) \leq F(r)$. Similar to the proof of Lemma 4.4, we get the desired conclusion.

We quote the fixed point theorem which our results will be based on.

Lemma 4.7. [21] *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$.*

(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$;

(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Now we give our main results on positive periodic solutions for (1.4).

Theorem 4.8. *(a) If $\bar{i}_0 = 1$ or 2, then (1.4) has $\bar{i}_0$ positive ω-periodic solutions for*

$$\lambda > \frac{1}{f_1(1) l \int_0^\omega b(s)ds} > 0;$$

(b) If $\underline{i}_\infty = 1$ or 2, then (1.4) has $\underline{i}_\infty$ positive ω-periodic solutions for

$$0 < \lambda < \frac{m - (M+m)c_\infty}{LM(1 - c_\infty)F(1) \int_0^\omega b(s)ds};$$

(c) If $\bar{i}_\infty = 0$ or $\underline{i}_0 = 0$, then (1.4) has no positive ω-periodic solutions for sufficiently small or sufficiently large $\lambda > 0$, respectively.

Proof. (a) Choose $r_1 = 1$ and take $\lambda_0 = \frac{1}{f_1(r_1) l \int_0^\omega b(s)ds} > 0$. For all $\lambda > \lambda_0$, we have from Lemma 4.5 that

$$\|Qy\| > \|y\|, \text{ for } y \in \partial K_{r_1}. \tag{4.9}$$

Case 1. If $\overline{f}_0 = 0$, we can choose $0 < \bar{r}_2 < r_1$ so that $f(u) \leq \varepsilon u$ for $0 \leq u \leq \bar{r}_2$, where constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \frac{LM \int_0^\omega b(s)ds}{m - (M+m)c_\infty} < 1. \tag{4.10}$$

Letting $r_2 = (1 - c_\infty)\bar{r}_2$, we find $f((A^{-1}y)(t - \tau(t))) \leq \varepsilon (A^{-1}y)(t - \tau(t))$ for $y \in K_{r_2}$. By Lemma 2.2, we have $0 \leq (A^{-1}y)(t - \tau(t)) \leq \frac{\|y\|}{1 - c_\infty} \leq \bar{r}_2$ for $y \in \partial K_{r_2}$. In view of Lemma 4.4 and (4.10), we have for $y \in \partial K_{r_2}$ that

$$\|Qy\| \leq \lambda \varepsilon \frac{LM \int_0^\omega b(s)ds}{m - (M+m)c_\infty} \|y\| < \|y\|.$$

It follows from Lemma 4.7 and (4.9) that

$$i(Q, K_{r_2}, K) = 1, \ i(Q, K_{r_1}, K) = 0.$$

Thus $i(Q, K_{r_1} \backslash \bar{K}_{r_2}, K) = -1$ and Q has a fixed point y in $K_{r_1} \backslash \bar{K}_{r_2}$, which means $(A^{-1}y)(t)$ is a positive ω-positive solution of (1.4) for $\lambda > \lambda_0$.

Case 2. If $\overline{f}_\infty = 0$, there exists a constant $\tilde{H} > 0$ such that $f(u) \leq \varepsilon u$ for $u \geq \tilde{H}$, where constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \frac{LM \int_0^\omega b(s)ds}{m - (M+m)c_\infty} < 1. \tag{4.11}$$

Letting

$$r_3 = \max\{2r_1, \frac{\tilde{H}(1-c_0^2)(1-c_\infty^2)}{\alpha(1-c_\infty^2)-c_\infty(1-c_0^2)}\},$$

we find $f((A^{-1}y)(t-\tau(t))) \le \varepsilon (A^{-1}y)(t-\tau(t))$ for $y \in K_{r_3}$. By Lemma 2.2, we have

$$(A^{-1}y)(t-\tau(t)) \ge \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right)\|y\| \ge \tilde{H}, \quad y \in \partial K_{r_3}$$

Using Lemma 4.4 and (4.11), we have for $y \in \partial K_{r_3}$ that

$$\|Qy\| \le \lambda \varepsilon \frac{LM\int_0^\omega b(s)ds}{m-(M+m)c_\infty}\|y\| < \|y\|.$$

Using Lemma 4.7 and (4.9), one sees that

$$i(Q, K_{r_3}, K) = 1,\ i(Q, K_{r_1}, K) = 0.$$

Then $i(Q, K_{r_3}\backslash \bar{K}_{r_1}, K) = 1$ and Q has a fixed point y in $K_{r_3}\backslash \bar{K}_{r_1}$, which means that $(A^{-1}y)(t)$ is a positive ω-positive solution of (1.4) for $\lambda > \lambda_0$.

Case 3. If $\overline{f}_0 = \overline{f}_\infty = 0$, from the above arguments, there exist $0 < r_2 < r_1 < r_3$ such that Q has a fixed point $y_1(t)$ in $K_{r_1}\backslash \bar{K}_{r_2}$ and a fixed point $y_2(t)$ in $K_{r_3}\backslash \bar{K}_{r_1}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω-periodic solutions of (1.4) for $\lambda > \lambda_0$.

(b) Let $r_1 = 1$ and take $\lambda_0 = \frac{m-(M+m)c_\infty}{LM(1-c_\infty)F(r_1)\int_0^\omega b(s)ds} > 0$. If $\lambda < \lambda_0$, we find from Lemma 4.6 that

$$\|Qy\| < \|y\|,\ y \in \partial K_{r_1}. \tag{4.12}$$

Case 1. If $\underline{f}_0 = \infty$, we can choose $0 < \bar{r}_2 < r_1$ so that $f(u) \ge \eta u$ for $0 \le u \le \bar{r}_2$, where the constant $\eta > 0$ satisfies

$$\lambda l \eta \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right)\int_0^\omega b(s)ds > 1. \tag{4.13}$$

Letting $r_2 = (1-c_\infty)\bar{r}_2$, we have $f((A^{-1}y)(t-\tau(t))) \ge \eta (A^{-1}y)(t-\tau(t))$ for $y \in K_{r_2}$. By Lemma 2.2, we have $0 \le (A^{-1}y)(t-\tau(t)) \le \frac{\|y\|}{1-c_\infty} \le \bar{r}_2$ for $y \in \partial K_{r_2}$. Using Lemma 4.3 and (4.13), we find

$$\|Qy\| \ge \lambda l \eta \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right)\int_0^\omega b(s)ds\|y\| > \|y\|.$$

It follows from Lemma 4.7 and (4.12) that

$$i(Q, K_{r_2}, K) = 0,\ i(Q, K_{r_1}, K) = 1,$$

which implies $i(Q,K_{r_1}\backslash\bar{K}_{r_2},K)=1$ and Q has a fixed point y in $K_{r_1}\backslash\bar{K}_{r_2}$. Therefore $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.4) for $0<\lambda<\lambda_0$.

Case 2. If $\underline{f}_\infty=\infty$, there exists a constant $\tilde{H}>0$ such that $f(u)\geq\eta u$ for $u\geq\tilde{H}$, where constant $\eta>0$ satisfies

$$\lambda l\eta\left(\frac{\alpha}{1-c_0^2}-\frac{c_\infty}{1-c_\infty^2}\right)\int_0^\omega b(s)ds>1. \tag{4.14}$$

Letting

$$r_3=\max\{2r_1,\frac{\tilde{H}(1-c_0^2)(1-c_\infty^2)}{\alpha(1-c_\infty^2)-c_\infty(1-c_0^2)}\},$$

we have $f((A^{-1}y)(t-\tau(t)))\geq\eta(A^{-1}y)(t-\tau(t))$ for $y\in K_{r_3}$. By Lemma 2.2, we have

$$(A^{-1}y)(t-\tau(t))\geq\left(\frac{\alpha}{1-c_0^2}-\frac{c_\infty}{1-c_\infty^2}\right)\|y\|\geq\tilde{H}$$

for $y\in\partial K_{r_3}$. Using Lemma 4.3 and (4.14), we have for $y\in\partial K_{r_3}$ that

$$\|Qy\|\geq\lambda l\eta\left(\frac{\alpha}{1-c_0^2}-\frac{c_\infty}{1-c_\infty^2}\right)\int_0^\omega b(s)ds\|y\|>\|y\|.$$

It follows from Lemma 4.7 and (4.12) that

$$i(Q,K_{r_3},K)=0,\ i(Q,K_{r_1},K)=1,$$

i.e., $i(Q,K_{r_3}\backslash\bar{K}_{r_1},K)=-1$ and Q has a fixed point y in $K_{r_3}\backslash\bar{K}_{r_1}$. This implies that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.4) for $0<\lambda<\lambda_0$.

Case 3. If $\underline{f}_0=\underline{f}_\infty=\infty$, we find that Q has a fixed point y_1 in $K_{r_1}\backslash\bar{K}_{r_2}$ and a fixed point y_2 in $K_{r_3}\backslash\bar{K}_{r_1}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω-periodic solution of (1.4) for $0<\lambda<\lambda_0$.

(c) If $y\in K$, we find from Lemma 2.2 that $(A^{-1}y)(t-\tau(t))\geq\left(\frac{\alpha}{1-c_0^2}-\frac{c_\infty}{1-c_\infty^2}\right)\|y\|>0$ for $t\in[0,\omega]$.

Case 1. If $\underline{i}_0=0$, we have $\underline{f}_0>0$ and $\underline{f}_\infty>0$. Letting $b_1=\min\{\frac{f(u)}{u};u>0\}>0$, we obtain

$$f(u)\geq b_1u,\ u\in[0,+\infty).$$

Assume that $y(t)$ is a positive ω-periodic solution of (1.4) for $\lambda>\lambda_0$, where

$$\lambda_0=\frac{(1-c_0^2)(1-c_\infty^2)}{lb_1[\alpha(1-c_\infty^2)-c_\infty(1-c_0)^2]\int_0^\omega b(s)ds}>0.$$

Note that $Qy(t) = y(t)$ for $t \in [0, \omega]$. If $\lambda > \lambda_0$, we find from Lemma 4.3 that

$$\|y\| = \|Qy\| \geq \lambda l b_1 \left(\frac{\alpha}{1 - c_0^2} - \frac{c_\infty}{1 - c_\infty^2} \right) \int_0^\omega b(s) ds \|y\| > \|y\|,$$

which is a contradiction.

Case 2. If $\bar{i}_\infty = 0$, we have $\overline{f}_0 < \infty$ and $\overline{f}_\infty < \infty$. Letting $b_2 = \max\{\frac{f(u)}{u} : u > 0\} > 0$, we obtain

$$f(u) \leq b_2 u, \ u \in [0, \infty).$$

Assume that $y(t)$ is a positive ω-periodic solution of (1.4) for $0 < \lambda < \lambda_0$, where

$$\lambda_0 = \frac{m - (M + m)c_\infty}{b_2 L M \int_0^\omega b(s) ds}.$$

Since $Qy(t) = y(t)$ for $t \in [0, \omega]$, it follows from Lemma 4.4 that

$$\|y\| = \|Qy\| \leq \lambda b_2 \frac{L M \int_0^\omega b(s) ds}{m - (M + m)c_\infty} \|y\| < \|y\|,$$

which is a contradiction. This completes the proof.

Theorem 4.9. *(a) If there exists a constant $b_1 > 0$ such that $f(u) \geq b_1 u$ for $u \in [0, +\infty)$, then (1.4) has no positive ω-periodic solution for*

$$\lambda > \frac{(1 - c_0^2)(1 - c_\infty^2)}{l b_1 [\alpha(1 - c_\infty^2) - c_\infty(1 - c_0^2)] \int_0^\omega b(s) ds}.$$

(b) If there exists a constant $b_2 > 0$ such that $f(u) \leq b_2 u$ for $u \in [0, +\infty)$, then (1.4) has no positive ω-periodic solution for

$$0 < \lambda < \frac{m - (M + m)c_\infty}{b_2 L M \int_0^\omega b(s) ds}.$$

Proof. From the proof of (c) in Theorem 4.1, we obtain this theorem immediately.

Theorem 4.10. *Assume $\underline{i}_0 = \bar{i}_0 = \underline{i}_\infty = \bar{i}_\infty = 0$, and that one of the following conditions holds:*
(1) $\overline{f}_0 \leq \underline{f}_\infty$;
(2) $\underline{f}_0 > \overline{f}_\infty$;
(3) $\underline{f}_0 \leq \underline{f}_\infty \leq \overline{f}_0 \leq \overline{f}_\infty$;

(4) $\underline{f}_\infty \le \underline{f}_0 \le \overline{f}_\infty \le \overline{f}_0$.

If

$$\frac{(1-c_0^2)(1-c_\infty^2)}{l[\alpha(1-c_\infty^2)-c_\infty(1-c_0^2)]\int_0^\omega b(s)ds\max\{\underline{f}_0,\overline{f}_0,\underline{f}_\infty,\overline{f}_\infty\}}$$
$$<\lambda<\frac{m-(M+m)c_\infty}{LM\int_0^\omega b(s)ds\min\{\underline{f}_0,\overline{f}_0,\underline{f}_\infty,\overline{f}_\infty\}},$$

then (1.4) has one positive ω-periodic solution.

Proof. Case 1. If $\overline{f}_0 \le \underline{f}_\infty$, then

$$\frac{(1-c_0^2)(1-c_\infty^2)}{l[\alpha(1-c_\infty^2)-c_\infty(1-c_0^2)]\int_0^\omega b(s)ds}<\lambda<\frac{m-(M+m)c_\infty}{LM\int_0^\omega b(s)ds}.$$

It is easy to see that there exists an $0<\varepsilon<f_\infty$ such that

$$\frac{(1-c_0^2)(1-c_\infty^2)}{(\bar{f}_\infty-\varepsilon)l[\alpha(1-c_\infty^2)-c_\infty(1-c_0^2)]\int_0^\omega b(s)ds}<\lambda<\frac{m-(M+m)c_\infty}{(\underline{f}_0+\varepsilon)LM\int_0^\omega b(s)ds}.$$

For the above ε, we choose $\bar{r}_1>0$ such that $f(u)\le(\underline{f}_0+\varepsilon)u$ for $0\le u\le\bar{r}_1$. Letting $r_1=(1-c_\infty)\bar{r}_1$, we have

$$f((A^{-1}y)(t-\tau(t)))\le(\underline{f}_0+\varepsilon)(A^{-1}y)(t-\tau(t))$$

for $y\in K_{r_1}$. By Lemma 2.2, we have

$$0\le(A^{-1}y)(t-\tau(t))\le\frac{\|y\|}{1-c_\infty}\le\bar{r}_1$$

for $K\in\partial K_{r_1}$. Using Lemma 4.4 we have for $y\in\partial K_{r_1}$ that

$$\|Qy\|\le\lambda(\underline{f}_0+\varepsilon)\frac{LM\int_0^\omega b(s)ds}{m-(M+m)c_\infty}\|y\|<\|y\|.$$

On the other hand, there exists a constant $\tilde{H}>0$ such that $f(u)\ge(\overline{f}_\infty-\varepsilon)u$ for $u\ge\tilde{H}$. Letting

$$r_2=\max\{2r_1,\frac{\tilde{H}(1-c_0^2)(1-c_\infty^2)}{\alpha(1-c_\infty^2)-c_\infty(1-c_0^2)}\},$$

we have

$$f((A^{-1}y)(t-\tau(t)))\ge(\overline{f}_\infty-\varepsilon)(A^{-1}y)(t-\tau(t))$$

for $y\in K_{r_2}$. By Lemma 2.2, we have

$$(A^{-1}y)(t-\tau(t))\ge\left(\frac{\alpha}{1-c_0^2}-\frac{c_\infty}{1-c_\infty^2}\right)\|y\|\ge\tilde{H}$$

for $y \in \partial K_{r_2}$. Using Lemma 4.3, for $y \in \partial K_{r_2}$, we have

$$\|Qy\| \geq \lambda l(\overline{f}_\infty - \varepsilon)\left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right)\int_0^\omega b(s)ds\|y\| > \|y\|.$$

It follows from Lemma 4.7 that

$$i(Q, K_{r_1}, K) = 1,\ i(Q, K_{r_2}, K) = 0.$$

Thus $i(Q, K_{r_2}\backslash\bar{K}_{r_1}, K) = -1$ and Q has a fixed point y in $K_{r_2}\backslash\bar{K}_{r_1}$. So $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.4).

Case 2. If $\underline{f}_0 > \overline{f}_\infty$, we have

$$\frac{(1-c_0^2)(1-c_\infty^2)}{\bar{f}_0 l[\alpha(1-c_\infty^2) - c_\infty(1-c_0^2)]\int_0^\omega b(s)ds} < \lambda < \frac{m-(M+m)c_\infty}{\underline{f}_\infty LM\int_0^\omega b(s)ds}.$$

It is easy to see that there exists an $0 < \varepsilon < f_0$ such that

$$\frac{(1-c_0^2)(1-c_\infty^2)}{(\bar{f}_0-\varepsilon) l[\alpha(1-c_\infty^2) - c_\infty(1-c_0^2)]\int_0^\omega b(s)ds} < \lambda < \frac{m-(M+m)c_\infty}{(\underline{f}_\infty+\varepsilon) LM\int_0^\omega b(s)ds}.$$

For the above ε, we choose $\bar{r}_1 > 0$ such that $f(u) \geq (\overline{f}_0 - \varepsilon)u$ for $0 \leq u \leq \bar{r}_1$. Letting $r_1 = (1-c_\infty)\bar{r}_1$, we have

$$f((A^{-1}y)(t-\tau(t))) \geq (\overline{f}_0 - \varepsilon)(A^{-1}y)(t-\tau(t))$$

for $y \in K_{r_1}$. By Lemma 2.2, we have

$$0 \leq (A^{-1}y)(t-\tau(t)) \leq \frac{\|y\|}{1-c_\infty} \leq \bar{r}_1$$

for $y \in \partial K_{r_1}$. Using Lemma 4.4, we have for $y \in \partial K_{r_1}$ that

$$\|Qy\| \geq \lambda l(\overline{f}_0 - \varepsilon)\left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right)\int_0^\omega b(s)ds\|y\| > \|y\|.$$

On the other hand, there exists a constant $\tilde{H} > 0$ such that $f(u) \leq (\underline{f}_\infty + \varepsilon)u$ for $u \geq \tilde{H}$. Letting

$$r_2 = \max\{2r_1, \frac{\tilde{H}(1-c_0^2)(1-c_\infty^2)}{\alpha(1-c_\infty) - c_\infty(1-c_0^2)}\},$$

we have

$$f((A^{-1}y)(t-\tau(t))) \leq (\underline{f}_\infty + \varepsilon)(A^{-1}y)(t-\tau(t))$$

for $y \in K_{r_2}$. By Lemma 2.2, we have

$$(A^{-1}y)(t-\tau(t)) \geq \left(\frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2}\right)\|y\| \geq \tilde{H}$$

for $y \in \partial K_{r_2}$. Thus by Lemma 4.4, we have for $y \in \partial K_{r_2}$ that

$$\|Qy\| \leq \lambda(\underline{f}_\infty + \varepsilon)\frac{LM\int_0^\omega b(s)ds}{m-(M+m)c_\infty}\|y\|.$$

It follows from Lemma 4.7 that

$$i(Q, K_{r_1}, K) = 0 \; i(Q, K_{r_2}, K) = 1.$$

Thus $i(Q, K_{r_2}\backslash\bar{K}_{r_1}, K) = -1$ and Q has a fixed point y in $K_{r_2}\backslash\bar{K}_{r_1}$, proving that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.4).

Case 3. $\underline{f}_0 \leq \underline{f}_\infty \leq \overline{f}_0 \leq \overline{f}_\infty$. The proof is the same as in Case 1.

Case 4.$\underline{f}_\infty \leq \underline{f}_0 \leq \overline{f}_\infty \leq \overline{f}_0$. The proof is the same as in Case 2.

Case II: $c(t) > 0$ and $c_\infty < \min\{\frac{m}{M+m}, \frac{LM-lm}{(L-l)M-lm}\}$

Define

$$f_2(r) = \min\{f(t) : \frac{\alpha}{1-c_0}r \leq t \leq \frac{r}{1-c_\infty}\}.$$

We have the following results.

Theorem 4.11. *(a) If $\bar{i}_0 = 1$ or 2, then (1.4) has i_0 positive ω-periodic solutions for*

$$\lambda > \frac{1}{f_2(1)l\int_0^\omega b(s)ds} > 0.$$

(b) If $\underline{i}_\infty = 1$ or 2, then (1.4) has i_∞ positive ω-periodic solutions for

$$0 < \lambda < \frac{m-(M+m)c_\infty}{LM(1-c_\infty)F(1)\int_0^\omega b(s)ds}.$$

(c) If $\bar{i}_\infty = 0$ or $\underline{i}_0 = 0$, then (1.4) has no positive ω-periodic solution for sufficiently small or large $\lambda > 0$, respectively.

Theorem 4.12. *(a) If there exists a constant $b_1 > 0$ such that $f(u) \geq b_1 u$ for $u \in [0, +\infty)$, then (1.4) has no positive ω-periodic solution for*

$$\lambda > \frac{1-c_0}{l\alpha b_1\int_0^\omega b(s)ds}.$$

(b) If there exists a constant $b_2 > 0$ such that $f(u) \leq b_2 u$ for $u \in [0, +\infty)$, then (1.4) has no positive ω-periodic solution for

$$0 < \lambda < \frac{m-(M+m)c_\infty}{b_2 LM\int_0^\omega b(s)ds}.$$

Theorem 4.13. *Assume $\underline{i}_0 = \bar{i}_0 = \underline{i}_\infty = \bar{i}_\infty = 0$ hold, and that one of the following conditions holds:*
(1) $\overline{f}_0 \le \underline{f}_\infty$;
(2) $\underline{f}_0 > \overline{f}_\infty$;
(3) $\underline{f}_0 \le \underline{f}_\infty \le \overline{f}_0 \le \overline{f}_\infty$;
(4) $\underline{f}_\infty \le \underline{f}_0 \le \overline{f}_\infty \le \overline{f}_0$.
If

$$\frac{1-c_0}{l\alpha\int_0^\omega b(s)ds\max\{\underline{f}_0,\overline{f}_0,\underline{f}_\infty,\overline{f}_\infty\}} < \lambda < \frac{m-(M+m)c_\infty}{LM\int_0^\omega b(s)ds\min\{\underline{f}_0,\overline{f}_0,\underline{f}_\infty,\overline{f}_\infty\}},$$

then (1.4) has one positive ω-periodic solution.

Remark 4.14. In this paper, we calculate the Green's function of fourth-order linear equations. Based on the Green's function and a fixed point theorem cones (Krasnoselskii's fixed point theorem), we obtain the existence, nonexistence, multiplicity of positive periodic solutions for (1.1) with sublinear and superliner cases.

We illustrate our results with an example.

Example 4.15. Consider the following neutral functional differential equation

$$\begin{aligned}&\left(u(t)+\frac{7}{100}\left(1-\frac{1}{2}\sin 2t\right)u(t-\cos^2 t)\right)^{(4)}+\frac{1}{16}\left(1-\frac{1}{2}\sin^2 t\right)u(t)\\&=-\lambda(1-\cos 2t)u^2(t-\tau(t))a^{u(t-\tau(t))},\end{aligned}\tag{4.15}$$

where λ and $0<a<1$ are two positive parameters, $\tau(t+\pi)=\tau(t)$. Comparing (4.15) to (1.4), we see that $\delta(t)=\cos^2 t$, $c(t)=-\frac{7}{100}\left(1-\frac{1}{2}\sin 2t\right)$, $a(t)=\frac{1}{16}(1-\frac{1}{2}\sin^2 t)$, $b(t)=1-\cos 2t$, $\omega=\pi$, $f(u)=u^2a^u$. Clearly, $c_\infty=\frac{7}{100}$, $c_0=\frac{7}{200}$, $M=\frac{1}{16}$, $m=\frac{1}{32}$, $M=\frac{1}{16}<(\frac{\pi}{\pi})^4=1$, $\overline{f}_0=0$, $\overline{f}_\infty=0$, $\bar{i}_0=2$. By Theorem 4.1, we easily get the following conclusion: equation (4.15) has two positive π-periodic solutions for $\lambda>\frac{10}{23\pi r_1}$, where $r_1=\min\{f(0.065),f(\frac{100}{93})\}$.

In fact, by simple computations, we have

$$\rho=\frac{1}{2},\ L=\frac{1+\exp(\rho\omega)}{4\rho^3(\exp(\rho\omega)-1)}+\frac{1}{2\rho^3(1-\cos\rho\omega)}\approx 7.0508,$$

$$l=\frac{\exp\left(\frac{\rho\omega}{2}\right)}{2\rho^3(\exp(\rho\omega)-1)}\approx 2.3038, k\approx 0.6787, k_1\approx 0.1151, \alpha\approx 0.1388,$$

$$c_\infty=\frac{7}{100}<\min\{k_1,\frac{m}{M+m}\}=0.1151, c_\infty=\frac{7}{100}<0.1388=\alpha$$

and

$$f_1(1) = \min\{f(t) : 0.065 \approx \frac{\alpha}{1-c_0^2} - \frac{c_\infty}{1-c_\infty^2} \le t \le \frac{100}{93}\}$$
$$= \min\{f(0.065),\ f(\frac{100}{93})\}$$
$$= r_1,$$

$$\frac{1}{f_1(1) l \int_0^\omega b(s)\mathrm{d}s} = \frac{10}{23\pi r_1}.$$

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