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INTEGRAL INEQUALITIES VIA GENERALIZED (α, m)-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce the concept of generalized (α, m)-convex functions. Some new integral inequalities related to left hand side of the Hermite-Hadamard type for the class of functions whose second derivative at certain powers are generalized (α, m)-functions are proved. Some special cases are also discussed. The idea and technique of this paper may stimulate further research in this field.

Keywords. Generalized convex function; (α, m)-convex function; Hermite-Hadamard's type inequality; Holder's inequality.

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1. Introduction

Convex analysis plays a significant role in pure and applied mathematics especially in optimization theory and nonlinear programming due to its symmetry in shape and properties of convex sets and functions. Innovative techniques and calculation yielded different directions for study convex analysis. Several new classes of convex functions and convex sets have been introduced and investigated, which make this area of research very attractive and useful. Consequently various new inequalities related to these new classes of convex functions have been derived by many researchers.

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In recent years, many generalizations related to convex analysis has been given by various researchers. These generalizations include the idea of strongly convex, log convex, quasi convex, s-convex, h-convex, ϕ-convex, mid convex and many more; see [1]-[7] and the references therein.

Let I be an interval and $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \quad \forall a,b \in I, \quad t \in [0,1]. \tag{1.1}$$

This double inequality is known as the Hermite-Hadamard inequality for convex functions; see [6, 7] and the references therein. In recent years, much attention has been given to derive the Hermite-Hadamard type inequalities for various types of convex functions; see [8, 9, 10] and the references therein.

Convex functions have been generalized in various directions using interesting and innovative techniques to study a wide class of unrelated problems in a unified and general framework. Gordji *et al.* [11] introduced a new class of convex functions involving the bifunction, which is called ϕ-convex functions. These ϕ-convex functions are not convex functions. For the recent developments, see [11, 12, 13, 14] and the references therein.

Motivated and inspired by the ongoing research activities, we introduce and study a new class of φ-convex functions, which is called generalized (α,m)-convex functions. We derive several new integral inequalities via generalized (α,m)-convex functions. Several special cases, which can be obtained from our min results, are also discussed. The technique of this paper may motivate further research.

2. Preliminaries

In this section, we consider the basic concepts and results, which are needed to obtain our main results.

Definition 2.1. [13] Let I be an interval in real line $\mathbb{R}$. A function $f : I = [a,b] \to \mathbb{R}$ is said to be generalized convex (ϕ-convex) function, if there exists a bifunction $\eta(.,.) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, such that

$$f(ta+(1-t)b) \leq f(b)+t\eta(f(a),f(b)) \quad \forall a,b \in I, t \in [0,1].$$

If $\eta(x,y) = x - y$ in the above definition, we obtain the classical definition of a convex functions. Every convex function is a generalized convex function, but the converse may not be true.

Example 2.2. [14] *For a convex function f, we may find another function η other than the function $\eta(x,y) = x - y$ such that f is generalized convex. Consider $f(x) = x^2$ and $\eta(x,y) = 2x + y$. Then we have*

$$\begin{aligned} f(tx+(1-t)y) &\leq tx^2 + y^2 + t(1-t)2xy \\ &\leq tx^2 + y^2 + t(1-t)(x^2+y^2) \\ &\leq y^2 + t(x^2 + x^2 + y^2) \\ &= y^2 + t(2x^2 + y^2) \\ &= f(y) + t\eta(f(x), f(y)), \end{aligned}$$

for all $x, y \in R$ and $t \in (0,1)$. Also the facts $x^2 \leq y^2 + (2x^2 + y^2)$ and $y^2 \leq y^2$, $\forall x, y \in R$ show the correctness of inequality for $t = 1$ and $t = 0$ respectively which means that f is a generalized convex function. Note that function $f(x) = x^2$ is generalized convex w.r.t all $\eta(x,y) = ax + by$ with $a \geq 1, b \geq -1$ and $x, y \in R$.

Example 2.3. [14] *There are functions, which are generalized convex functions, but not convex. Consider $f : \mathbb{R} \to \mathbb{R}$ as*

$$f(x) = \begin{cases} -x, & \text{if } x \geq 0, \\ x, & \text{if } x < 0, \end{cases}$$

and $\eta : [-\infty, 0] \times [-\infty, 0] \to \mathbb{R}$ as

$$\eta(x,y) = \begin{cases} x & \text{if } y = 0, \\ -y & \text{if } x = 0, \\ -x-y & \text{if } x < 0, y < 0. \end{cases}$$

One can easily check that$f$ is a generalized convex function. Also, it is obvious that f is not a convex function.

We now introduce a new concept of generalized (α,m)-convex functions with respect to an arbitrary bifunction, which is the main motivation.

Definition 2.4. A function $f : I = [a,b] \to \mathbb{R}$, $a < b$ is said to be generalized (α,m)-convex function with respect to a bifunction $\eta(.,.) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, where $(\alpha,m) \in [0,1]^2$ if

$$f(ta+m(1-t)b) \leq m(1-t^{\alpha})f(b)+t^{\alpha}[f(b)+\eta(f(a),f(b))], \quad \forall a,b \in I, t \in [0,1].$$

Now we are in a position to discuss some special cases.

(1) If (α,m)=$(1,m)$, then we obtain generaized-m convex functions.
(2) If (α,m)=$(1,1)$, then we have ordinary generalized convex functions.
(3) If (α,m)=$(1,0)$, then we obtain starshaped generalized convex functions.

If $\eta(f(a),f(b)) = f(a)-f(b)$, then Definition 2.4 reduces to the following.

Definition 2.5. [15] A function $f : I = [a,b] \to \mathbb{R}$ is said to be (α,m)-convex, where (α,m) $\in[0,1]^2$ if

$$f(ta+m(1-t)b) \leq t^{\alpha}f(a)+m(1-t^{\alpha})[f(b)], \quad \forall a,b \in I, t \in [0,1].$$

We denote by $K_m^a(b)$, the class of all (α,m)-cnvex function on $[a,b]$ for which $f(0) \leq 0$.

Definition 2.6. The beta function, also called the Euler integral of the first kind, is a special function defined by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad x,y > 0,$$

where $\Gamma(.)$ is the Gamma function.

We also need the following result to derive our main results.

Lemma 2.7. [16] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0, where I^0 is the interior of I and $a < b$ such that $f'' \in L[a,b]$, where a, $b \in I$. Then*

$$\begin{aligned} & f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ = & \frac{(b-a)^2}{16}\left[\int_0^1 t^2 f''(t(\frac{a+b}{2})+(1-t)a\mathrm{d}t+\int_0^1 (t-1)^2 f''(tb+(1-t)(\frac{a+b}{2})\mathrm{d}t\right], \end{aligned}$$

where $L[a,b]$ is the set of integrable functions on [a,b].

3. Main results

We are now in a position to prove our main results.

Theorem 3.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable functions on I^0, where I^0 is the interior of I and $a < b$ such that $f'' \in L[a,b]$, where $a,\ b \in I$. If $|f''|$ is a generalized (α,m)-convex function on $[a,b]$ for $\alpha \in [0,1]$ and $m \in (0,1]$, then*

$$\begin{aligned}
&|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
&\leq \frac{(b-a)^2}{48((\alpha+3)}\Big[m\alpha|[f''(\frac{a}{m})]| + |f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))|\Big] \\
&\quad + \frac{(b-a)^2}{48(\alpha+1)(\alpha+2)(\alpha+3)}\Big[6|f''(\frac{a+b}{2m}) + \eta(f''(b), f''(\frac{a+b}{2m}))| \\
&\quad - m(\alpha+1)(\alpha+2)(\alpha+3)|[f''(\frac{a+b}{2m})]|\Big].
\end{aligned} \tag{3.1}$$

Proof. Using Lemma 2.7 and the generalized $(\alpha,m)-convexity$, we have

$$\begin{aligned}
&|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
\leq\ & \frac{(b-a)^2}{16}\Big[\int_0^1 t^2|f''(t(\frac{a+b}{2}) + (1-t)a|\mathrm{d}t + \int_0^1 (t-1)^2|f''(tb + (1-t)(\frac{a+b}{2})|\mathrm{d}t\Big] \\
\leq\ & \frac{(b-a)^2}{16}\Big[\int_0^1 t^2\Big[(m(1-t^\alpha))|[f''(\frac{a}{m})]| + t^\alpha|f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))|\mathrm{d}t \\
& + \int_0^1 (t-1)^2(m(1-t^\alpha))|[f''(\frac{a+b}{2m})] + t^\alpha[f''(\frac{a+b}{2m}) + \eta(f''(b), f''(\frac{a+b}{2m}))|\mathrm{d}t\Big] \\
=\ & \frac{(b-a)^2}{16}\Big[\frac{m\alpha}{3(\alpha+3)}|[f''(\frac{a}{m})]| + \frac{1}{(\alpha+3)}|f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))| \\
& - m[\frac{1}{3} + \frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}|[f''(\frac{a+b}{2m})]| \\
& + \frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}|[f''(\frac{a+b}{2m}) + \eta(f''(b), f''(\frac{a+b}{2m}))|\Big] \\
=\ & \frac{(b-a)^2}{48((\alpha+3)}\Big[m\alpha|[f''(\frac{a}{m})]| + |f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))|\Big] \\
& + \frac{(b-a)^2}{48(\alpha+1)(\alpha+2)(\alpha+3)}\Big[m(\alpha+1)(\alpha+2)(\alpha+3)|[f''(\frac{a+b}{2m})]| \\
& + 6|f''(\frac{a+b}{2m}) + \eta(f''(b), f''(\frac{a+b}{2m}))|\Big],
\end{aligned}$$

which is the required conclusion.

Remark 3.2. If $\eta(f(b), f(a)) = f(b) - f(a)$, then Theorem 3.1 reduces to a result given by Sun and Liu [16].

Corollary 3.3. *If $\alpha = 1$ in Theorem 3.1, then*

$$\begin{aligned}
&|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
\leq\ & \frac{(b-a)^2}{48((1+3)}\left[m|[f''(\frac{a}{m})]| + |f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))|\right] \\
&+\frac{(b-a)^2}{48(1+1)(1+2)(1+3)}\left[m(1+1)(1+2)(1+3)|[f''(\frac{a+b}{2m})]|\right. \\
&\left.+6|f''(\frac{a+b}{2m}) + \eta(f''(b), f''(\frac{a+b}{2m}))|\right] \\
=\ & \frac{(b-a)^2}{192}\left[m|[f''(\frac{a}{m})]| + 3|f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))|\right. \\
&\left.-5m|[f''(\frac{a+b}{2m})]| + |f''(\frac{a+b}{2m}) + \eta(f''(b), f''(\frac{a+b}{2m}))|\right].
\end{aligned}$$

Corollary 3.4. *If $\alpha = m = 1$ in Theorem 3.1, then*

$$\begin{aligned}
|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \ \leq\ & \frac{(b-a)^2}{192}\left[|[f''(a)]| + 3|f''(a) + \eta(f''(\frac{a+b}{2}), f''(a)|\right. \\
&\left.-3|[f''(\frac{a+b}{2})]| + |f''(\frac{a+b}{2}) + \eta(f''(b), f''(\frac{a+b}{2}))|\right].
\end{aligned}$$

Theorem 3.5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I^0, where I^0 is the interior of I such that $f'' \in L[a,b]$, where a, $b \in I$ and $a<b$. If $|f''|$ is a generalized (α,m)-convex function on $[a,b]$ for $\alpha \in [0,1]$ and $m \in (0,1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
&\leq \frac{(b-a)^2}{16}(\frac{1}{\alpha+1})^{\frac{1}{q}}\left[|f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))|\right. \\
&\quad\left.+|f''(\frac{a+b}{2m}) + \eta(f''(b), f''(\frac{a+b}{2m}))| + (m\alpha)^{\frac{1}{q}}[|f''(\frac{a+b}{2m})| + |[f''(\frac{a}{m})]|]\right].
\end{aligned}$$

Proof. From Lemma 2.7 and using the power mean inequality, we have

$$
\begin{aligned}
&|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
&\leq \frac{(b-a)^2}{16}\Big[\int_0^1 t^2|f''(t(\frac{a+b}{2})+(1-t)a|\mathrm{d}t \\
&\quad+\int_0^1 (t-1)^2|f''(tb+(1-t)(\frac{a+b}{2})|\mathrm{d}t\Big] \\
&\leq \frac{(b-a)^2}{16}\Big[\int_0^1 (t^{2p}\mathrm{d}t)^{1-\frac{1}{q}}(\int_0^1 t^2|f''(t(\frac{a+b}{2})+(1-t)a|^q\mathrm{d}t)^{\frac{1}{q}} \\
&\quad+(\int_0^1 (t-1)^{2p}\mathrm{d}t))^{\frac{1}{p}}(\int_0^1 |f''(tb+(1-t)(\frac{a+b}{2})|^q\mathrm{d}t)^{\frac{1}{q}}\Big]
\end{aligned}
\tag{3.2}
$$

Since $|f''|$ is a generalized (α,m)-convex function, we have

$$
\begin{aligned}
&\int_0^1 |f''(t(\frac{a+b}{2})+(1-t)a|^q\mathrm{d}t \\
&\leq \int_0^1 [(m(1-t^{\alpha})|[f''(\frac{a}{m})]|^q+t^{\alpha}|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\mathrm{d}t \\
&=\frac{m\alpha}{\alpha+1}|[f''(\frac{a}{m})]|^q+\frac{1}{\alpha+1}|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q.
\end{aligned}
\tag{3.3}
$$

Combining (3.2) with (3.3), we arrive at

$$
\begin{aligned}
&|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
&\leq \frac{(b-a)^2}{16}(\frac{1}{2p+1})^{\frac{1}{q}}(\frac{1}{\alpha+1})^{\frac{1}{q}}\Big[\big\{m\alpha|[f''(\frac{a}{m})]|^q+|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\big\}^{\frac{1}{q}} \\
&\quad+\big\{m\alpha|f''(\frac{a+b}{2m})|^q+|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m}))|^q\big\}^{\frac{1}{q}}\Big] \\
&=\frac{(b-a)^2}{16}(\frac{1}{\alpha+1})^{\frac{1}{q}}\Big[\big\{m\alpha|[f''(\frac{a}{m})]|^q+|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\big\}^{\frac{1}{q}} \\
&\quad+\big\{m\alpha|f''(\frac{a+b}{2m})|^q+|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m}))|^q\big\}^{\frac{1}{q}}\Big],
\end{aligned}
\tag{3.4}
$$

where we have used the fact that $\frac{1}{3}<(\frac{1}{2p+1})^{\frac{1}{q}}<1$. Let $a_1=m\alpha|[f''(\frac{a}{m})]|^q$,

$$b_1=|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q,$$

$a_2=m\alpha|f''(\frac{a+b}{2m})|^q$ and

$$b_2=|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m}))|^q.$$

Since

$$\sum_{i=1}^{n}(a_i+b_i)^r \le \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r, \tag{3.5}$$

for $0<r<1$, $a_1,a_2,...,a_n \ge 0$ and $b_1,b_2,...,b_n \ge 0$, we see that (3.4) becomes

$$\begin{aligned}
&|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
\le\ & \frac{(b-a)^2}{16}(\frac{1}{\alpha+1})^{\frac{1}{q}}\Big[\{m\alpha|[f''(\frac{a}{m})]|^q+|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\}^{\frac{1}{q}} \\
&+\{m\alpha|f''(\frac{a+b}{2m})|^q+|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m}))|^q\}^{\frac{1}{q}}\Big] \\
=\ & \frac{(b-a)^2}{16}(\frac{1}{\alpha+1})^{\frac{1}{q}}\Big[|f''(\frac{a}{m})+\eta(f''\frac{a+b}{2},f''(\frac{a}{m}))| \\
&+|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m}))|+(m\alpha)^{\frac{1}{q}}\big[|f''(\frac{a+b}{2m})|+|[f''(\frac{a}{m})]|\big]\Big],
\end{aligned}$$

where $0<\frac{1}{q}<1$. This completes the proof.

Corollary 3.6. *If $\alpha=m=1$ in Theorem 3.5, then*

$$\begin{aligned}
|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \le \frac{(b-a)^2}{16}(\frac{1}{2})^{\frac{1}{q}}\Big[&|f''(a)+\eta(f''(\frac{a+b}{2}),f''(a)| \\
&+|f''(\frac{a+b}{2})+\eta(f''(b),f''(\frac{a+b}{2}))| \\
&+|f''(\frac{a+b}{2})|+|[f''(a)]|\Big].
\end{aligned}$$

Theorem 3.7. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^0, where I^0 is the interior of I such that $f''\in L[a,b]$, where a, $b\in I$ and $a<b$. If $|f''|$ is a generalized (α,m)-convex function on $[a,b]$ for $\alpha\in[0,1]$ and $m\in$ (0,1] and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}
&|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
&\le \frac{(b-a)^2}{16}\Big[\{\frac{m\alpha}{(q+1)(q+\alpha+1)}|[f''(\frac{a}{m})]|^q \\
&+\frac{1}{(q+\alpha+1)})|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\}^{\frac{1}{q}} \\
&+\{m(\frac{1}{q+1}-(\frac{q}{q+\alpha+1})\frac{\Gamma(\alpha+1)\Gamma(q}{\Gamma(\alpha+q+1))}|f''(\frac{a+b}{2m})|^q \\
&+(\frac{q}{q+\alpha+1})\frac{\Gamma(\alpha+1)\Gamma(q}{\Gamma(\alpha+q+1)}|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m})|^q\}^{\frac{1}{q}}\Big].
\end{aligned} \tag{3.6}$$

Proof. From Lemma 2.7 and using the Holder's inequality, we have

$$\begin{aligned}
&|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x|\\
&\leq \frac{(b-a)^2}{16}\Bigg[\int_0^1[t(t|f''(\frac{a+b}{2})+(1-t)a|)]\mathrm{d}t\\
&\quad+\int_0^1[(t-1)((t-1)|f''(tb+(1-t)(\frac{a+b}{2}))|)]\mathrm{d}t\Bigg] \qquad (3.7)\\
&\leq \frac{(b-a)^2}{16}\Bigg[(\int_0^1 t^p\mathrm{d}t)^{\frac{1}{p}}(\int_0^1 t^q|f''(t(\frac{a+b}{2})+(1-t)a|^q\mathrm{d}t)^{\frac{1}{q}}\\
&\quad+(\int_0^1(t-1)^p\mathrm{d}t)^{\frac{1}{p}}(\int_0^1(t-1)^q|f''(tb+(1-t)(\frac{a+b}{2})|^q\mathrm{d}t)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Since $|f''|$ is a generalized (α,m)-convex function, we have

$$\begin{aligned}
&\int_0^1|f''(t(\frac{a+b}{2})+(1-t)a|^q\mathrm{d}t\\
&\leq\int_0^1[(m(1-t^\alpha)|[f''(\frac{a}{m})]|^q+t^\alpha|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\mathrm{d}t. \qquad (3.8)
\end{aligned}$$

Combining (3.7) and (3.8), we have

$$\begin{aligned}
&|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x|\\
&\leq\frac{(b-a)^2}{16}(\frac{1}{p+1})^{\frac{1}{p}}\Bigg[\{\frac{m\alpha}{(q+1)(q+\alpha+1)}|[f''(\frac{a}{m})]|^q\\
&\quad+\frac{1}{(q+\alpha+1})|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\}^{\frac{1}{q}}\\
&\quad+\{m(\frac{1}{q+1}-\beta(\alpha+1,q+1)|f''(\frac{a+b}{2m})|^q\\
&\quad+\beta(\alpha+1,q+1)|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m}))|^q\}^{\frac{1}{q}}\Bigg]\\
&=\frac{(b-a)^2}{16}\Bigg[\{\frac{m\alpha}{(q+1)(q+\alpha+1)}|[f''(\frac{a}{m})]|^q\\
&\quad+\frac{1}{(q+\alpha+1})|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\}^{\frac{1}{q}}\\
&\quad+\{m(\frac{1}{q+1}-(\frac{q}{q+\alpha+1})\frac{\Gamma(\alpha+1)\Gamma(q}{\Gamma(\alpha+q+1))}|f''(\frac{a+b}{2m})|^q\\
&\quad+(\frac{q}{q+\alpha+1})\frac{\Gamma(\alpha+1)\Gamma(q}{\Gamma(\alpha+q+1)}|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m})|^q\}^{\frac{1}{q}}\Bigg],
\end{aligned}$$

where we have used the properties of beta and gemma function and the fact that $\frac{1}{2} < (\frac{1}{p+1})^{\frac{1}{p}} < 1$.

Corollary 3.8. *If $\alpha = m = 1$ in Theorem 3.7, then*

$$
\begin{aligned}
& |f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
\leq\ & \frac{(b-a)^2}{16}\Bigg[\big\{\frac{1}{(q+1)(q+2)}|[f''(a)]|^q + (\frac{1}{q+2})|f''(a) + \eta(f''(\frac{a+b}{2}), f''(a))|^q\big\}^{\frac{1}{q}} \\
& + \big\{(\frac{1}{q+1} - (\frac{q}{q+2})\frac{\Gamma(2)\Gamma(q)}{\Gamma(q+2)}|f''(\frac{a+b}{2})|^q \\
& + (\frac{q}{q+2})(\frac{\Gamma(2)\Gamma(q)}{\Gamma(q+2)}|f''(\frac{a+b}{2}) + \eta(f''(b), f''(\frac{a+b}{2}))|^q\big\}^{\frac{1}{q}}\Bigg] \\
=\ & \frac{(b-a)^2}{16}\Bigg[|[f''(a)]|^q + (q+1)|f''(a) + \eta(f''(\frac{a+b}{2}), f''(a))|^q\big\}^{\frac{1}{q}} \\
& + \big\{(q+1)|f''(\frac{a+b}{2})|^q + |f''(\frac{a+b}{2}) + \eta(f''(b), f''(\frac{a+b}{2}))|^q\big\}^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

where we have used the fact that $(\frac{1}{(q+1)(q+2)})^{\frac{1}{q}} \leq 1$.

Theorem 3.9. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 where I^0 is the interior of I such that $f'' \in [a,b]$, where a,b $\in$ I and a<b.If $|f''|$ is a generalized (α,m)-convex function on $[a,b]$ for $\alpha \in [0,1]$ and m $\in$ (0,1] and $q > 1$, then*

$$
\begin{aligned}
& |f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
\leq\ & \frac{(b-a)^2}{16}(\frac{1}{3})^{1-\frac{1}{q}}\Bigg[\big[\frac{m\alpha}{3(\alpha+3)}|[f''(\frac{a}{m})]|^q + \frac{1}{\alpha+3})|f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))|^q\big]^{\frac{1}{q}} \\
& + \big[m(\frac{1}{3} - \frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}|[f''(\frac{a+b}{2m})]|^q \\
& + \frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}|f''(\frac{a}{m}) + \eta(f''(\frac{a+b}{2}), f''(\frac{a}{m}))|^q\big]^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Proof. From Lemma 2.7 and using the power mean inequality, we have

$$
\begin{aligned}
&|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
\leq\quad & \frac{(b-a)^2}{16}\Big[\int_0^1 t^2|f''(t(\frac{a+b}{2})+(1-t)a|\mathrm{d}t \\
& +\int_0^1 (t-1)^2|f''(tb+(1-t)(\frac{a+b}{2})|\mathrm{d}t\Big] \\
\leq\quad & \frac{(b-a)^2}{16}\Big[(\int_0^1 t^2\mathrm{d}t)^{1-\frac{1}{q}}(\int_0^1 t^2|f''(t(\frac{a+b}{2})+(1-t)a|^q\mathrm{d}t)^{\frac{1}{q}} \\
& +(\int_0^1 (t-1)^2\mathrm{d}t)^{1-\frac{1}{q}}(\int_0^1 (t-1)^2|f''(tb+(1-t)(\frac{a+b}{2})|^q\mathrm{d}t)^{\frac{1}{q}}\Big].
\end{aligned}
$$

Since $|f''|$ is a generalized (α,m)-convex function, we have

$$
\begin{aligned}
&\int_0^1 t^2|f''(t(\frac{a+b}{2})+(1-t)a|^q\mathrm{d}t \\
&\leq \int_0^1 \big[t^2[(m(1-t^\alpha)|[f''(\frac{a}{m})]|^q+t^\alpha|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\big]\mathrm{d}t \qquad (3.9)\\
&=\frac{m\alpha}{3(\alpha+3)}|[f''(\frac{a}{m})]|^q+\frac{1}{\alpha+3}|f''(\frac{a}{m})+\eta(f''\frac{a+b}{2},f''(\frac{a}{m}))|^q
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 (1-t)^2|f''(tb+(1-t)(\frac{a+b}{2})|^q\mathrm{d}t \\
&\leq \int_0^1 \big[(1-t)^2[(m(1-t^\alpha)|[f''(\frac{a+b}{2m})]|^q+t^\alpha|f''(\frac{a+b}{2m})+\eta(f''(b),f''(\frac{a+b}{2m}))|^q\big]\mathrm{d}t \quad (3.10)\\
&=m(\frac{1}{3}-\beta(\alpha+1,3))|[f''(\frac{a+b}{2m})]|^q+\beta(\alpha+1,3)|f''(\frac{a}{m})+\eta(f''\frac{a+b}{2},f''(\frac{a}{m}))|^q.
\end{aligned}
$$

Combing (3.9) and (3.10), we have

$$
\begin{aligned}
&|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x| \\
\leq\quad & \frac{(b-a)^2}{16}(\frac{1}{3})^{1-\frac{1}{q}}\Big[\big[\frac{m\alpha}{3(\alpha+3)}|[f''(\frac{a}{m})]|^q+\frac{1}{\alpha+3})|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\big]^{\frac{1}{q}} \\
& +\big[m(\frac{1}{3}-\beta(\alpha+1,3)|[f''(\frac{a+b}{2m})]|^q+\beta(\alpha+1,3)|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q\big]^{\frac{1}{q}}
\end{aligned}
$$

$$= \frac{(b-a)^2}{16}(\frac{1}{3})^{1-\frac{1}{q}}\Bigg[[\frac{m\alpha}{3(\alpha+3)}|[f''(\frac{a}{m})]|^q+\frac{1}{\alpha+3}|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q]^{\frac{1}{q}}$$
$$+[m(\frac{1}{3}-\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}|[f''(\frac{a+b}{2m})]|^q$$
$$+\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}|f''(\frac{a}{m})+\eta(f''(\frac{a+b}{2}),f''(\frac{a}{m}))|^q]^{\frac{1}{q}}\Bigg],$$

which is the required conclusion.

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