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A MULTIPLICITY RESULT FOR A SINGULAR PROBLEM WITH SUBCRITICAL NONLINEARITIES

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Abstract. In this paper, the multiplicity of positive solutions for the Laplacian singular problems is obtained based on the Nehari manifold approach and some variational techniques.

Keywords. Nehari manifold; Fibering map; Singular equation; Weak solution.

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1. Introduction and main results

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $2^* = \frac{2N}{N-2}$. The purpose of this paper is to study the following singular and sub-critical problem:

$$(\mathrm{P}_\lambda)\begin{cases} -\operatorname{div}(M(x)\nabla u) = u^{-\gamma} + \lambda u^p, & \text{in } \Omega, \\ \\ u|_{\partial\Omega} = 0, \quad u > 0, & \text{in } \Omega, \end{cases}$$

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where λ is a positive parameter and $0<\gamma<1<p<2^*-1$. $M(\alpha,\beta)$ is a symmetric matrix and the set of bounded and measurable matrices $M:\Omega\to\mathbb{R}^{N^2}$, i.e.

$$\exists 0<\alpha\leq\beta \text{ such that } \alpha|\xi|^2\leq M(x)\xi\xi, \quad |M(x)|\leq\beta. \tag{1.1}$$

For all $u\in H_0^1(\Omega)$, we define

$$E_\lambda(u)=\frac{1}{2}\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x-\frac{1}{1-\gamma}\int_\Omega |u|^{1-\gamma}\mathrm{d}x-\frac{\lambda}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x.$$

We may obtain the multiplicity of solutions for problem (P_λ) by investigating suitable minimization problems for the functional E_λ. Notice that from (3.1) and the fact that the matrix are symmetric means that norm $\int_\Omega M(x)\nabla u\cdot\nabla u\,dx$ and norm $\int_\Omega \nabla u\cdot\nabla u\,dx$ are equivalent, that is, a function $u\in H_0^1(\Omega)$ is called a weak solution of problem (P_λ) if $u(x)>0$ in Ω and satisfies the equation

$$\int_\Omega M(x)\nabla u\cdot\nabla\phi\,\mathrm{d}x=\int_\Omega |u|^{-\gamma}\phi\,\mathrm{d}x+\lambda\int_\Omega |u|^p\phi\,\mathrm{d}x, \quad \forall\,\phi\in H_0^1(\Omega), \tag{1.2}$$

where $H_0^1(\Omega)$ is the Sobolev space equipped with norm

$$\|u\|=\left[\int_\Omega |\nabla u|^2\,\mathrm{d}x\right]^{1/2}.$$

Let $D(p,\gamma)$ be a constant and Λ be the manifolds, respectively

$$D(p,\gamma)=\frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}}\alpha^{\frac{p+\gamma}{1+\gamma}}|\Omega|^{\frac{p+\gamma}{1+\gamma}},$$
$$\Lambda:=\{t(u)u:\ \ u\in H_0^1\setminus\{0\}\},$$

where $t(u)$ are the zeros of the map $\Phi_u:(0,\infty)\to\mathbb{R}$ defined as

$$\begin{aligned}\Phi_u(t) &= \frac{1}{t^p}\frac{d}{dt}E_\lambda(tu),\\ &= t^{1-p}\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x-\lambda\int_\Omega |u|^p\,dx-t^{-\gamma-p}\int_\Omega a(x)|u(x)|^{1-\gamma}\,dx.\end{aligned}$$

Then,

$$\Phi_u'(t) = (1-p)t^{-p}\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x+(\gamma+p)t^{-\gamma-p-1}\int_\Omega a(x)|u(x)|^{1-\gamma}dx$$

and it is easy to see that $tu\in\Lambda$ if and only if $\Phi_u'(t)=0$. In particular, $u\in\Lambda$ if and only if $\Phi_u'(1)=0$.

To obtain the multiplicity of solutions, we decompose $\Lambda = \Lambda^+ \cup \Lambda^0 \cup \Lambda^-$ with $\Lambda^+, \Lambda^0, \Lambda^-$ measurable sets defined as follows:

$$\Lambda^+ = \{u \in \Lambda : (1+\gamma)\int_\Omega M(x)\nabla u \cdot \nabla u \,\mathrm{d}x - \lambda(p+\gamma)\int_\Omega |u|^{p+1}\,\mathrm{d}x > 0\},$$
$$\Lambda^0 = \{u \in \Lambda : (1+\gamma)\int_\Omega M(x)\nabla u \cdot \nabla u \,\mathrm{d}x - \lambda(p+\gamma)\int_\Omega |u|^{p+1}\,\mathrm{d}x = 0\},$$
$$\Lambda^- = \{u \in \Lambda : (1+\gamma)\int_\Omega M(x)\nabla u \cdot \nabla u \,\mathrm{d}x - \lambda(p+\gamma)\int_\Omega |u|^{p+1}\,\mathrm{d}x < 0\}.$$

The main goal of this paper is to investigate the question of the existence and multiplicity of solutions in respect to the parameters λ and γ. To obtain multiple (at least two distinct, positive) solutions of problem (P_λ), we combine some well-known fibering maps (i.e., maps of the form $t \to E_\lambda(tu)$, see (ALVES-EL HAMIDI [1], BROWN-ZHANG [2]) and minimization on the suitable subset of Nehari manifold.

Specifically, the main result can be described as follows:

Theorem 1.1. *Suppose $1 < p < 2^* - 1$ and $0 < \gamma < 1$. Then problem (P_λ) has at least two positive solutions for all $\lambda \in (0, D(p,\gamma))$.*

The problems of this type are important in many fields of sciences, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behaviour of electric, gravitational, and fluid potentials.

The study of semilinear and quasilinear elliptic problems involving singular nonlinearity has received considerable attention in recent years. Historically, problem (P_λ) with $M(x) = I$ has quite an extensive literature (see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]) with their references therein. It seems that problem (P_λ), has attracted less attention. Firstly, BOCCARDO in [17] has shown by using sub-super solutions arguments that problem (P_λ) have a strictly positive solution. Moreover

(1) If $0 < \gamma \leq 1$, then $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;
(2) If $\gamma > 1$, then $u \in W_{loc}^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Furthermore, $u^{\frac{\gamma+1}{2}}$ belongs to $W_0^{1,2}(\Omega)$, which is the meaning of $u = 0$ on the boundary of Ω. Then, in [18] it is proved, using variational methods, the existence of at least two different solutions of (P_λ) belonging to $H_0^1(\Omega)$ when $\gamma < 1$ and $2 < p+1 < 2^*$. Recently, in [19] the previous result is extended by studying the multiplicity of solutions of the problem (P_λ) for every $\gamma > 0$.

The organization of the paper is as follows: The section 2 is devoted to prove some lemmas in preparation for the proof of our main result. An existence result of two solutions is presented (Theorem 1.1) in Section 3.

2. Preliminaries

In this section, we give some results which will be used for the proof of our main result.

Lemma 2.1. *Let $\lambda \in (0, D(p,\gamma))$. Then, there exist t_0^+ and t_0^- satisfying $0 < t_0^+ < t_{\max} < t_0^-$ such that*

$$\Phi(t_0^+) = \lambda \int_\Omega |u|^{p+1}\,\mathrm{d}x = \Phi(t_0^-).$$

Proof. First, for $\Phi'(t) = 0$, it is simple to verify that Φ attains it's maximum at

$$t_{\max} = \Big[\frac{(p+\gamma)\int_\Omega |u|^{1-\gamma}dx}{(p-1)\int_\Omega M(x)\nabla u \nabla u\,\mathrm{d}x}\Big]^{1/(1+\gamma)}.$$

Moreover, $\Phi'(t) > 0$ for all $0 < t < t_{\max}$ and $\Phi'(t) < 0$ for all $t > t_{\max}$. We find from (1.1) that

$$\alpha\|u\|_2^2 \le \int_\Omega M(x)\nabla u \cdot \nabla u\,\mathrm{d}x \le \beta \int_\Omega \nabla u \cdot \nabla u\,\mathrm{d}x \le \beta\|u\|^2. \tag{2.1}$$

By Hölder's inequality, one has

$$\int_\Omega |u|^{p+1}dx \le |u|_2^{p+1}|\Omega|^{\frac{-(p+1)}{2}}, \tag{2.2}$$

$$\int_\Omega |u|^{1-\gamma}dx \le |u|_2^{1-\gamma}|\Omega|^{\frac{\gamma-1}{2}}. \tag{2.3}$$

In view of (2.1), (2.2) and (2.3), one sees that

$$
\begin{aligned}
&\Phi(t_{\max}) - \lambda \int_\Omega |u|^{p+1} dx \\
&= \frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}} \frac{\big(\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x\big)^{\frac{p+\gamma}{1+\gamma}}}{\big(\int_\Omega |u|^{1-\gamma}dx|\Omega|^{\frac{\gamma-1}{2}}\big)^{\frac{p-1}{1+\gamma}}} - \int_\Omega |u|^{p+1}dx \\
&\geq \frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}} \frac{(\alpha\|u\|^2)^{\frac{p+\gamma}{1+\gamma}}}{\big(|u|_2^{1-\gamma}|\Omega|^{\frac{\gamma-1}{2}}\big)^{\frac{p-1}{1+\gamma}}} - \lambda |u|_2^{p+1}|\Omega|^{\frac{-(p+1)}{2}} \\
&= \frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}} \frac{\alpha^{\frac{p+\gamma}{1+\gamma}}|u|^{p+1}}{\big(|\Omega|^{\frac{\gamma-1}{2}}\big)^{\frac{p-1}{1+\gamma}}} - \lambda |\Omega|^{\frac{-(p+1)}{2}}|u|_2^{p+1} \\
&= \Big[\frac{1+\gamma}{p-1}\Big(\frac{p-1}{p+\gamma}\Big)^{\frac{p+\gamma}{1+\gamma}} \frac{\alpha^{\frac{p+\gamma}{1+\gamma}}}{\big(|\Omega|^{\frac{\gamma-1}{2}}\big)^{\frac{p-1}{1+\gamma}}} - \lambda |\Omega|^{\frac{-(p+1)}{2}}\Big]|u|_2^{p+1} \\
&= |\Omega|^{\frac{-(p+1)}{2}}(D(p,\gamma)-\lambda)|u|_2^{p+1} > 0,
\end{aligned}
\tag{2.4}
$$

for all $\lambda \in (0, D(p,\gamma))$. Therefore, there exist t_0^+ and t_0^- such that

$$\Phi(t_0^+) = \lambda \int_\Omega |u|^{p+1}dx = \Phi(t_0^-)$$

and

$$\Phi'(t_0^+) < 0 < \Phi'(t_0^-),$$

that is, $t_0^+ u \in \Lambda^+$ and $t_0^- u \in \Lambda^-$. This completes the proof of Lemma 2.1.

Lemma 2.2. *Let $u \in \Lambda^+$ (respectively Λ^-) and let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined by:*

$$
\begin{aligned}
f(t,s) = t^{\gamma+1}\int_\Omega M(x)\nabla(u+s\varphi)\cdot\nabla(u+s\varphi)dx - \int_\Omega (u+s\varphi)^{1-\gamma}dx \\
- \lambda t^{p+\gamma}\int_\Omega (u+s\varphi)^{p+1}dx.
\end{aligned}
\tag{2.5}
$$

Then, applying the implicit function theorem to the function f at the point $(1,0)$, we obtain a $\overline{\varepsilon} > 0$ and a continuous function $t = t(s) > 0$ such that

$$t(0) = 1, \quad t(s)(u+s\varphi) \in \Lambda^+ \ \textit{(respectively } \Lambda^-), \quad \forall\, s \in \mathbb{R},\ |s| < \varepsilon$$

for all $\varphi \in H_0^1(\Omega)$ with $u > 0$.

Proof. From (2.5), we see that

$$f_t(t,s) = (\gamma+1)t^{\gamma}\int_{\Omega} M(x)\nabla(u+s\varphi)\cdot\nabla(u+s\varphi)dx \\ -\lambda(p+\gamma)t^{p+\gamma-1}\int_{\Omega}(u+s\varphi)^{p+1}dx.$$

It is clear that the function $f_t(t,s)$ is continuous in $\mathbb{R}\times\mathbb{R}$. Moreover, since $u\in\Lambda^+\subset\Lambda$, we obtain $f(1,0)=0$, that is

$$f_t(1,0) = (1+\gamma)\int_{\Omega} M(x)\nabla u\cdot\nabla u dx - \lambda(p+\gamma)\int_{\Omega} u^{p+1}dx < 0.$$

Thus, using the implicit function theorem to f at the point $(1,0)$, we obtain $\overline{\varepsilon}>0$ and a continuous function $t=t(s)>0$ such that

$$t(0)=1, \quad t(s)(u+s\varphi)\in\Lambda, \quad \forall\, s\in\mathbb{R},\ |s|<\overline{\varepsilon}.$$

Hence, putting $\varepsilon>0$ smaller enough, we get

$$t(s)(u+s\varphi)\in\Lambda^+, \quad \forall\, s\in\mathbb{R},\ |s|<\varepsilon.$$

In the same way, we prove that $u\in\Lambda^-$. The proof of the Lemma 2.2 is completed.

Lemma 2.3. *Suppose $\lambda\in(0,D(p,\gamma))$, then $\Lambda^{\pm}\neq\emptyset$ and $\Lambda^0=\{0\}$. Moreover, Λ^- is a closed set in $H_0^1(\Omega)-$topology.*

Proof. First, according to the Lemma 2.1, $\Lambda^{\pm}$ are non-empty for $\lambda\in(0,D(p,\gamma))$. Now, we proceed by contradiction to prove that $\Lambda^0=\{0\}$ for all $\lambda\in(0,D(p,\gamma))$. For this, let us suppose that there exists $0\neq u_0\in\Lambda^0$. Then we get

$$(1+\gamma)\int_{\Omega} M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x - \lambda(p+\gamma)\int_{\Omega}|u_0|^{p+1}dx = 0,$$

which implies that

$$0 = \int_{\Omega} M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x - \lambda\int_{\Omega}|u_0|^{p+1}dx - \int_{\Omega}|u_0|^{1-\gamma}dx \\ = \frac{p-1}{p+\gamma}\int_{\Omega} M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x - \int_{\Omega}|u_0|^{1-\gamma}dx.$$

From (2.4), we obtain

$$
\begin{aligned}
0 &< \Big[\frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{\beta\|u_0\|^{\frac{2(p+\gamma)}{1+\gamma}}}{\big(|\Omega|^{\frac{2^*+\gamma-1}{2^*}}\big)^{\frac{p-1}{1+\gamma}}}-\lambda|\Omega|^{\frac{2^*-p-1}{2^*}}\Big]|u_0|_{2^*}^{p+1}\\
&< \frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{(\beta\|u_0\|)^{\frac{2(p+\gamma)}{1+\gamma}}}{\big(\int_\Omega|u_0|^{1-\gamma}dx\big)^{\frac{p-1}{1+\gamma}}}-\lambda\int_\Omega|u_0|^{p+1}dx\\
&= \frac{1+\gamma}{p-1}\big(\frac{p-1}{p+\gamma}\big)^{\frac{p+\gamma}{1+\gamma}}\frac{(\beta\|u_0\|)^{\frac{2(p+\gamma)}{1+\gamma}}}{\big(\frac{p-1}{p+\gamma}\beta\|u_0\|\big)^{\frac{2(p-1)}{1+\gamma}}}-\frac{1+\gamma}{p+\gamma}(\beta\|u_0\|)^2=0,
\end{aligned}
\tag{2.6}
$$

for all $\lambda\in(0,D(p,\gamma))$, which is impossible. Thus $\Lambda^0=\{0\}$ for all $\lambda\in(0,D(p,\gamma))$.

Now, to prove that Λ^- is closed for all $0<\lambda<D(p,\gamma)$, we consider a sequence $\{u_n\}\subset\Lambda^-$ such that $u_n\to u$ in $H_0^1(\Omega)$ as $n\to\infty$. Then $u\in\Lambda^-$. Using the definition of Λ^-, we get

$$
\begin{aligned}
&\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x-\int_\Omega|u_n|^{1-\gamma}\mathrm{d}x-\lambda\int_\Omega|u_n|^{p+1}\,\mathrm{d}x=0,\\
&(1+\gamma)\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\mathrm{d}x-\lambda(p+\gamma)\int_\Omega|u_n|^{p+1}\,\mathrm{d}x<0.
\end{aligned}
\tag{2.7}
$$

Hence, using Fubini's Theorem, we get that

$$
\begin{aligned}
&\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x-\int_\Omega|u|^{1-\gamma}dx-\lambda\int_\Omega|u|^{p+1}\,\mathrm{d}x=0,\\
&(1+\gamma)\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x-\lambda(p+\gamma)\int_\Omega|u|^{p+1}\,\mathrm{d}x\le 0.
\end{aligned}
$$

Therefore, $u\in\Lambda^0\cup\Lambda^-$. Moreover, from (2.1), (2.2) and (2.7), we obtain

$$
|u_n|_2\le\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\mathrm{d}x\le\Big[\frac{\lambda(p+\gamma)}{\alpha(1+\gamma)}|\Omega|^{\frac{-(p+1)}{2}}\Big]^{1/(1-p)},\quad\forall\,u_n\in\Lambda^-,
\tag{2.8}
$$

which contradicts that $u=0$. Thus $u\in\Lambda^-$ for all $\lambda\in(0,D(p,\gamma))$. Hence the proof of the Lemma 2.3 is now completed.

Now, we prove the coercivity and the boundedness of the functional E_λ.

Lemma 2.4. *E_λ is coercive and bounded below on Λ.*

Proof. For all $u \in \Lambda$, we have

$$\begin{aligned}
E_\lambda(u) &= \frac{1}{2}\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega |u|^{1-\gamma}\mathrm{d}x - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x \\
&= \big(\frac{1}{2}-\frac{1}{p+1}\big)\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x - \big(\frac{1}{1-\gamma}-\frac{1}{p+1}\big)\int_\Omega |u|^{1-\gamma}\mathrm{d}x. \\
&\geq \big(\frac{1}{2}-\frac{1}{p+1}\big)\alpha\int_\Omega |\nabla u|^2\,\mathrm{d}x - \big(\frac{1}{1-\gamma}-\frac{1}{p+1}\big)\int_\Omega |u|^{1-\gamma}\mathrm{d}x \\
&\geq \alpha\big(\frac{1}{2}-\frac{1}{p+1}\big)\|u\|^2 - C||u||^{1-\gamma},\ \ \forall\, u\in\Lambda.
\end{aligned}$$

Therefore, E_λ is coercive and bounded below on Λ. This completes the proof of the Lemma 2.4.

3. Proofs of Theorem 1.1

Since $E_\lambda(u) = E_\lambda(|u|)$, we can assume that all the price elements in Λ are nonnegative. On the other hand, from Lemma 2.1, for all $\lambda \in (0, D_{p,\gamma})$, we see that there exist m^+ and m^- such that

$$m^+ = \inf_{u\in\Lambda^+} E_\lambda(u), \quad m^- = \inf_{u\in\Lambda^-} E_\lambda(u).$$

Moreover, for all $u \in \Lambda^+$, we have

$$(1+\gamma)\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x - \lambda(p+\gamma)\int_\Omega |u|^{p+1}dx > 0,$$

and consequently, since $2 < p+1 < 2^*$, $0 < \gamma < 1$ and $u \not\equiv 0$, we have

$$\begin{aligned}
E_\lambda(u) &= \frac{1}{2}\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega |u|^{1-\gamma}\mathrm{d}x - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}dx \\
&= \big(\frac{1}{2}-\frac{1}{1-\gamma}\big)\int_\Omega M(x)\nabla u\cdot\nabla u\,\mathrm{d}x + \lambda\big(\frac{1}{1-\gamma}-\frac{1}{p+1}\big)\int_\Omega |u|^{p+1}\,\mathrm{d}x \\
&< -\beta\frac{1+\gamma}{2(1-\gamma)}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1+\gamma}{(1-\gamma)(p+1)}\int_\Omega |u|^{p+1}\,\mathrm{d}x \\
&< -\beta\frac{1+\gamma}{2(1-\gamma)}\|u\|^2 + \frac{1+\gamma}{(1-\gamma)(p+1)}\|u\|^2 \\
&= -\beta\frac{1+\gamma}{1-\gamma}\big(\frac{1}{2}-\frac{1}{p+1}\big)\|u\|^2 < 0.
\end{aligned}$$

Thus

$$m^+ = \inf_{u\in\Lambda^+} E_\lambda(u) < 0 \quad \text{for all } \lambda \in (0, D(p,\gamma)). \tag{3.1}$$

Proof. The proof is split into two steps.

Step 1. In view of Lemma 2.3, Λ^+, Λ^- are two closed sets in $H_0^1(\Omega)$. First, we consider sequence $\{u_n\} \subset \Lambda^+$ and apply Ekeland's variational principle (see [20]). It follows that

(i) $E_\lambda(u_n) < m^+ + \frac{1}{n}$,

(ii) $E_\lambda(u) \geq E_\lambda(u_n) - \frac{1}{n}\|u - u_n\|$ for all $u \in \Lambda^+$.

Since $E_\lambda(u) = E_\lambda(|u|)$, we can assume that $u_n(x) \geq 0$. Clearly, $\{u_n\}$ is bounded in $H_0^1(\Omega)$, so there exists a sub-sequence $\{u_n\}$ and $u_0 \geq 0$ such that $u_n \rightharpoonup u_0$, weakly in $H_0^1(\Omega)$, and $L^p(\Omega)$, $u_n \to u_0$, strongly in $L^{1-\gamma}(\Omega)$, and $u_n(x) \to u_0(x)$, a.e. in Ω, as $n \to \infty$. Now, from (3.1) and using the weak lower semi-continuity of norm $E_\lambda(u_0) \leq \liminf E_\lambda(u_n) = \inf\limits_{\Lambda^+} E_\lambda$, we see that $u_0 \not\equiv 0$ in Ω.

Claim 1. $u_0(x) > 0$ a.e. in Ω.

Firstly, we start by observing that, since $u_n \in \Lambda^+$, one has

$$(1+\gamma)\|u_n\|^2 - \lambda(p+\gamma)\int_\Omega |u_n|^{p+1}dx > 0. \tag{3.2}$$

It follows that

$$(1+\gamma)\int_\Omega |u_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega |u_n|^{p+1}dx > 0. \tag{3.3}$$

Now, using Hölder's inequality, we get, as $n \to \infty$, that

$$\begin{aligned}\int_\Omega u_n^{1-\gamma}\mathrm{d}x &\leq \int_\Omega u_0^{1-\gamma}\mathrm{d}x + \int_\Omega |\, u_n - u_0 \,|^{1-\gamma}\,\mathrm{d}x \\ &\leq \int_\Omega u_0^{1-\gamma}\mathrm{d}x + C \,\|\, u_n - u_0 \,\|_{L^2(\Omega)}^{1-\gamma} \\ &= \int_\Omega u_0^{1-\gamma}\mathrm{d}x + o(1).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_\Omega u_0^{1-\gamma}\mathrm{d}x &\leq \int_\Omega u_n^{1-\gamma}\mathrm{d}x + \int_\Omega |\, u_n - u_0 \,|^{1-\gamma}\,\mathrm{d}x \\ &\leq \int_\Omega u_0^{1-\gamma}\mathrm{d}x + C \,\|\, u_n - u_0 \,\|_{L^2(\Omega)}^{1-\gamma} \\ &= \int_\Omega u_n^{1-\gamma}\mathrm{d}x + o(1).\end{aligned}$$

Thus,

$$\int_\Omega u_n^{1-\gamma}\mathrm{d}x = \int_\Omega u_0^{1-\gamma}\mathrm{d}x + o(1). \tag{3.4}$$

It follows from the Brezis-Lieb lemma ([21]) that

$$\|\, u_n \,\|_{L^{p+1}(\Omega)}^{p+1} = \|\, u_0 \,\|_{L^{p+1}(\Omega)}^{p+1} + \|\, u_n - u_0 \,\|_{L^{p+1}(\Omega)}^{p+1} + o(1), \tag{3.5}$$

$$\|\, u_n \,\|_{H_0^1(\Omega)}^{2} = \|\, u_0 \,\|_{H_0^1(\Omega)}^{2} + \|\, u_n - u_0 \,\|_{H_0^1(\Omega)}^{2} + o(1). \tag{3.6}$$

Therefore, it follows from (3.4) and (3.5) that

$$\begin{aligned}&\lim_{n\to\infty}\left[(1+\gamma)\int_\Omega |u_n|^{1-\gamma}dx-\lambda(p-1)\int_\Omega |u_n|^{p+1}dx\right]\\&=(1+\gamma)\int_\Omega |u_0|^{1-\gamma}dx-\lambda(p-1)\int_\Omega |u_0|^{p+1}dx\geq 0.\end{aligned}$$

Now, we assume that

$$(1+\gamma)\int_\Omega |u_0|^{1-\gamma}dx-\lambda(p-1)\int_\Omega |u_0|^{p+1}dx=0. \tag{3.7}$$

Hence, Using (3.4), (3.5) and the weakly lower semi-continuity of the norm, we obtain

$$\begin{aligned}&\int_\Omega M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x-\int_\Omega |u_0|^{1-\gamma}\mathrm{d}x-\lambda\int_\Omega |u_0|^{p+1}\,\mathrm{d}x\\&\leq\beta\|u_0\|^2-\frac{p+\gamma}{p-1}\int_\Omega |u_0|^{1-\gamma}\mathrm{d}x\\&=\beta\|u_0\|^2-\frac{\lambda(p+\gamma)}{1+\gamma}\int_\Omega |u_0|^{p+1}dx\leq 0.\end{aligned} \tag{3.8}$$

Consequently, using (2.6), one has a contradiction, that is

$$(1+\gamma)\int_\Omega |u_0|^{1-\gamma}dx-\lambda(p-1)\int_\Omega |u_0|^{p+1}dx>0. \tag{3.9}$$

Now, let $\varphi\in H_0^1(\Omega),\varphi\geq 0$. From Lemma 2.3 with $u=u_n$, there exits a sequence of continuous functions $f_n=f_n(t)$ such that $f_n(t)(u_n+t\varphi)\in\Lambda^+$ and $f_n(0)=1$. Therefore, we have

$$\begin{aligned}0=&[f_n(t)]^2\int_\Omega M(x)\nabla|u_n+t\varphi|\cdot\nabla|u_n+t\varphi|\,\mathrm{d}x\\&-f_n^{1-\gamma}(t)\int_\Omega (u_n+t\varphi)^{1-\gamma}\mathrm{d}x-\lambda f_n^{p+1}(t)\int_\Omega |u_n+t\varphi|^{p+1}dx.\end{aligned}$$

Hence, for t small enough, we arrive at

$$
\begin{aligned}
0 &= [f_n^2(t)-1]\int_\Omega M(x)\nabla|u_n+t\varphi|\cdot\nabla|u_n+t\varphi|\,\mathrm{d}x+\Bigg(\int_\Omega M(x)\nabla|u_n+t\varphi|\cdot\nabla|u_n+t\varphi|\,\mathrm{d}x\\
&-\int_\Omega M(x)\nabla|u_n|\cdot\nabla|u_n|\,\mathrm{d}x\Bigg)-[f_n^{1-\gamma}(t)-1]\int_\Omega(u_n+t\varphi)^{1-\gamma}\mathrm{d}x\\
&-\int_\Omega[(u_n+t\varphi)^{1-\gamma}-|u_n|^{1-\gamma}]\,\mathrm{d}x-\lambda[f_n^{p+1}(t)-1]\int_\Omega|u_n+t\varphi|^{p+1}\,\mathrm{d}x\\
&-\lambda\int_\Omega(|u_n+t\varphi|^{p+1}-|u_n|^{p+1})\,\mathrm{d}x\\
&\le\beta[f_n^2(t)-1]\int_\Omega\nabla|u_n+t\varphi|\cdot\nabla|u_n+t\varphi|\,\mathrm{d}x+\beta\Bigg(\int_\Omega\nabla|u_n+t\varphi|\cdot\nabla|u_n+t\varphi|\,\mathrm{d}x\\
&-\int_\Omega\nabla|u_n|\cdot\nabla|u_n|\,\mathrm{d}x\Bigg)-[f_n^{1-\gamma}(t)-1]\int_\Omega(u_n+t\varphi)^{1-\gamma}\mathrm{d}x\\
&-\lambda[f_n^{p+1}(t)-1]\int_\Omega|u_n+t\varphi|^{p+1}\,\mathrm{d}x-\lambda\int_\Omega(|u_n+t\varphi|^{p+1}-|u_n|^{p+1})\,\mathrm{d}x\\
&=\beta[f_n^2(t)-1]\|u_n+t\varphi\|^2+\beta(\|u_n+t\varphi\|^2-\|u_n\|^2)\\
&-[f_n^{1-\gamma}(t)-1]\int_\Omega(u_n+t\varphi)^{1-\gamma}\mathrm{d}x-\lambda[f_n^{p+1}(t)-1]\int_\Omega|u_n+t\varphi|^{p+1}\,\mathrm{d}x\\
&-\lambda\int_\Omega(|u_n+t\varphi|^{p+1}-|u_n|^{p+1})\,\mathrm{d}x.
\end{aligned}
$$

Dividing by $t>0$, and passing to the limit for $t\to 0$, we obtain

$$
\begin{aligned}
0 &\le 2\beta\left(f_n'(0)\|u_n\|^2+\int_\Omega\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x\right)\\
&-f_n'(0)(1-\gamma)\int_\Omega u_n^{1-\gamma}\mathrm{d}x-\lambda(p+1)f_n'(0)\int_\Omega u_n^{p+1}\mathrm{d}x\\
&=f_n'(0)\left((2\beta-p-1)\|u_n\|^2+(p+\gamma)\int_\Omega u_n^{1-\gamma}\mathrm{d}x\right)+2\int_\Omega\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x,
\end{aligned}
\tag{3.10}
$$

where $f_n'(0)\in\overline{\mathbb{R}}$ denotes the right derivative of $f_n(t)$ at zero and since $u_n\in\Lambda^+$, $f_n'(0)\neq-\infty$. For simplicity, we assume that the right derivative of f_n at $t=0$ exists. Moreover, from (3.9) $f_n'(0)$ is uniformly bounded from below. Thus from condition (ii) it follows that for $t>0$ small enough,

$$
E_\lambda(u_n)\le E_\lambda(f_n(t)(u_n+t\varphi))+\frac{1}{n}||f_n(t)(u_n+t\varphi)-u_n||. \tag{3.11}
$$

That is,

$$
\begin{aligned}
&\frac{|f_n(t)-1|}{n}\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\mathrm{d}x + t f_n(t)\frac{\|\varphi\|}{n}\\
&\geq E_\lambda(u_n) - E_\lambda[f_n(t)(u_n+t\varphi)]\\
&= -\frac{1+\gamma}{2(1-\gamma)}\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\mathrm{d}x + \lambda\frac{p+\gamma}{(p+1)(1-\gamma)}\int_\Omega u_n^{p+1}\,\mathrm{d}x\\
&+\frac{1+\gamma}{2(1-\gamma)}f_n^2(t)\int_\Omega M(x)\nabla|u_n+t\varphi|\nabla|u_n+t\varphi|\,\mathrm{d}x\\
&-\lambda\frac{p+\gamma}{(p+1)(1-\gamma)}f_n^{p+1}(t)\int_\Omega|u_n+t\varphi|^{p+1}\,\mathrm{d}x\\
&\geq \frac{\alpha(1+\gamma)}{2(1-\gamma)}(\|u_n+t\varphi\|^2-\|u_n\|^2)+\frac{\alpha(1+\gamma)}{2(1-\gamma)}[f_n(t)-1]\|u_n+t\varphi\|^2\\
&-\lambda\frac{p+\gamma}{(p+1)(1-\gamma)}f_n^{p+1}(t)\int_\Omega(|u_n+t\varphi|^{p+1}-|u_n|^{p+1})\,\mathrm{d}x\\
&-\lambda\frac{p+\gamma}{(p+1)(1-\gamma)}[f_n^{p+1}(t)-1]\int_\Omega u_n^{p+1}\,\mathrm{d}x.
\end{aligned}
$$

Dividing by $t>0$, and passing to the limit $t\to 0$, we obtain

$$
\begin{aligned}
&\frac{1}{n}\left(|f_n'(0)|\|u_n\|+t f_n(t)\|\varphi\|\right)\\
&\geq \frac{f_n'(0)}{1-\gamma}\left[(p+\gamma)\int_\Omega u_n^{1-\gamma}\,\mathrm{d}x-\alpha(p-1)\|u_n\|^2\right]\\
&\quad+\alpha\left(\frac{1+\gamma}{1-\gamma}\right)\int_\Omega\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x-\lambda\left(\frac{p+\gamma}{1-\gamma}\right)\int_\Omega|u_n|^p\varphi\,\mathrm{d}x.
\end{aligned}
\tag{3.12}
$$

In view of (3.9), we see that there exists a positive constant C such that

$$
-\frac{1}{1-\gamma}\left((p+1-2\alpha)\|u_n\|^2-(p+\gamma)\int_\Omega u_n^{1-\gamma}\,\mathrm{d}x\right)-\frac{||u_n||}{n}\geq C>0. \tag{3.13}
$$

Thus, according to (3.12) and (3.13), $f_n'(0)$ is uniformly bounded from above. Consequently, we have

$$
f_n'(0)\quad\text{is uniformly bounded for } n \text{ large enough.} \tag{3.14}
$$

From (3.11), we obtain

$$
\begin{aligned}
&\frac{1}{n}[|f_n(t)-1|\cdot\|u_n\|+f_n(t)\|\varphi\|]\\
&\geq \frac{1}{n}\|f_n(t)(u_n+t\varphi)-u_n\|\\
&\geq E_\lambda(u_n)-E_\lambda[f_n(t)(u_n+t\varphi)]\\
&=\frac{1}{2}\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\mathrm{d}x-\frac{1}{1-\gamma}\int_\Omega u_n^{1-\gamma}\,\mathrm{d}x-\frac{\lambda}{p+1}\int_\Omega u_n^{p+1}\,\mathrm{d}x\\
&-\frac{f_n^2(t)}{2}\int_\Omega M(x)\nabla|u_n+t\varphi|\cdot\nabla|u_n+t\varphi|\,\mathrm{d}x+\frac{f_n^{1-\gamma}(t)}{1-\gamma}\int_\Omega(u_n+t\varphi)^{1-\gamma}\,\mathrm{d}x\\
&-\frac{\lambda f_n^{p+1}(t)}{p+1}\int_\Omega(u_n+t\varphi)^{p+1}\,\mathrm{d}x,\\
&\geq\frac{\alpha}{2}\|u_n\|^2-\frac{1}{1-\gamma}\int_\Omega u_n^{1-\gamma}\,\mathrm{d}x-\frac{\lambda}{p+1}\int_\Omega u_n^{p+1}\,\mathrm{d}x\\
&-\beta\frac{f_n^2(t)}{2}\|u_n+t\varphi\|^2+\frac{f_n^{1-\gamma}(t)}{1-\gamma}\int_\Omega(u_n+t\varphi)^{1-\gamma}\,\mathrm{d}x-\frac{\lambda f_n^{p+1}(t)}{p+1}\int_\Omega(u_n+t\varphi)^{p+1}\,\mathrm{d}x.
\end{aligned}
$$

Dividing by $t>0$, and passing to the limit $t\to 0$, we get

$$
\begin{aligned}
&\frac{1}{n}\left(|f_n'(0)|\|u_n\|+\|\varphi\|\right)\geq-\frac{\beta}{2}f_n'(0)\|u_n\|^2-\beta\int_\Omega\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x\\
&+f_n'(0)\int_\Omega u_n^{1-\gamma}\,\mathrm{d}x+\lambda f_n'(0)\int_\Omega|u_n|^{p+1}\,\mathrm{d}x\\
&+\lambda\int_\Omega|u_n|^p\varphi\,\mathrm{d}x+\liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}x\\
&\geq-\beta\int_\Omega\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x+\lambda\int_\Omega|u_n|^p\varphi\,\mathrm{d}x\\
&+\liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}x.
\end{aligned}
\tag{3.15}
$$

It follows that

$$
\begin{aligned}
&\liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}x\\
&\leq\int_\Omega M(x)\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x-\lambda\int_\Omega|u_n|^p\varphi\,\mathrm{d}x+\frac{1}{n}\left(|f_n'(0)|\|u_n\|+\|\varphi\|\right).
\end{aligned}
\tag{3.16}
$$

Since $[(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}]\geq 0,\quad \forall x\in\Omega,\ \forall t>0$, and using the Fatou's Lemma, we get

$$
\int_\Omega u_n^{-\gamma}\varphi\,\mathrm{d}x\leq\liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}x.
$$

It follows from (3.16) that

$$\int_\Omega u_n^{-\gamma}\varphi\,\mathrm{d}x \le \int_\Omega M(x)\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x - \lambda\int_\Omega u_n^p\varphi\,\mathrm{d}x + f_n'(0)\frac{\|u_n\|+\|\varphi\|}{n}$$

for n large enough. Using (3.14) and applying Fatou's Lemma again, to conclude that $u_0(x)>0$ a.e. in Ω and

$$\int_\Omega M(x)\nabla u_0\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega u_0^{-\gamma}\varphi\,\mathrm{d}x - \lambda\int_\Omega u_0^p\varphi\,\mathrm{d}x \ge 0, \tag{3.17}$$

for all $\varphi\in H_0^1(\Omega)$, $\varphi\ge 0$. Now, we prove that $u_0\in\Lambda^+$ for all $\lambda\in(0.D(p,\gamma))$. Choosing $\varphi=u_0$ in (3.17), we get

$$\int_\Omega M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x \ge \int_\Omega u_0^{1-\gamma}\,\mathrm{d}x + \lambda\int_\Omega u_0^{p+1}\,\mathrm{d}x.$$

Moreover, we find from (3.8) that

$$\int_\Omega M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x \le \int_\Omega u_0^{1-\gamma}\,\mathrm{d}x + \lambda\int_\Omega u_0^{p+1}\,\mathrm{d}x,$$

that is,

$$\int_\Omega M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x = \int_\Omega u_0^{1-\gamma}\,\mathrm{d}x + \lambda\int_\Omega u_0^{p+1}\,\mathrm{d}x. \tag{3.18}$$

This implies $u_0\in\Lambda$. It follows from (3.7) that

$$\lim_{n\to\infty}\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\mathrm{d}x = \int_\Omega u_0^{1-\gamma}\,\mathrm{d}x + \lambda\int_\Omega u_0^{p+1}\,\mathrm{d}x.$$

Therefore, we obtain from (3.18) that $u_n\to u_0$ in $H_0^1(\Omega)$ as $n\to\infty$. In particular, we have

$$(1+\gamma)\int_\Omega M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x - \lambda(p+\gamma)\int_\Omega |u_0|^{p+1}dx > 0,$$

which implies $u_0\in\Lambda^+$.

Claim 2. u_0 is a solution of problem (P_λ).

Our proof is inspired by GHANMI-SAOUDI [22, 23]. Let $\phi\in H_0^1(\Omega)$ and $\varepsilon>0$. We define $\Psi\in H_0^1(\Omega)$ by

$$\Psi := (u_0+\varepsilon\phi)^+,$$

where $(u_0+\varepsilon\phi)^+ = \max\{u_0+\varepsilon\phi, 0\}$. Replacing φ with Ψ in (3.17) and combining with (3.18), we obtain

$$\begin{aligned}
0 &\le \int_\Omega M(x)\nabla u_0\cdot\nabla\Psi\,\mathrm{d}x - \int_\Omega u_0^{-\gamma}\Psi\,\mathrm{d}x - \lambda\int_\Omega u_0^p\Psi\,\mathrm{d}x\\
&= \int_{\{x|u_0+\varepsilon\phi\ge 0\}}\Big(M(x)\nabla u_0\cdot\nabla(u_0+\varepsilon\phi) - u_0^{-\gamma}(u_0+\varepsilon\phi)\,\mathrm{d}x - \lambda u_0^p(u_0+\varepsilon\phi)\Big)\,\mathrm{d}x\\
&= \int_\Omega M(x)\nabla u_0\cdot\nabla u_0\,\mathrm{d}x - \int_\Omega u_0^{1-\gamma}\,\mathrm{d}x - \lambda\int_\Omega u_0^{p+1}\,\mathrm{d}x\\
&+\varepsilon\int_\Omega\Big(M(x)\nabla u_0\cdot\nabla\phi - \lambda u_0^p\phi - u_0^{-\gamma}\phi\Big)\,\mathrm{d}x\\
&-\int_{\{x|u_0+t\phi<0\}}\Big[M(x)\nabla u_0\cdot\nabla(u_0+\varepsilon\phi) - u_0^{-\gamma}(u_0+\varepsilon\phi) - \lambda u_0^p(u_0+\varepsilon\phi)\Big]\,\mathrm{d}x\\
&= \varepsilon\int_\Omega\Big(\nabla u_0\cdot\nabla\phi - u_0^{-\gamma}\phi - \lambda u_0^p\phi\Big)\,\mathrm{d}x\\
&\quad-\int_{\{x|u_0+\varepsilon\phi<0\}}\Big[M(x)\nabla u_0\cdot\nabla(u_0+\varepsilon\phi) - u_0^{-\gamma}(u_0+t\phi) - \lambda u_0^p(u_0+t\phi)\Big]\,\mathrm{d}x\\
&\le \varepsilon\int_\Omega\Big(M(x)\nabla u_0\cdot\nabla\phi - u_0^{-\gamma}\phi - \lambda u_0^p\phi\Big)\,\mathrm{d}x\\
&-\varepsilon\int_{\{x|u_0+\varepsilon\phi<0\}} M(x)\nabla u_0\cdot\nabla\phi\,\mathrm{d}x.
\end{aligned}$$

Since the measure of the domain of integration $\{x : u_0+\varepsilon\phi<0\}$ tends to zero as $\varepsilon\to 0^+$, it follows that $\int_{\{x|u_0+\varepsilon\phi<0\}} M(x)\nabla u_0\cdot\nabla\phi\,\mathrm{d}x\to 0$ as $\varepsilon\to 0^+$. Dividing by ε and letting $\varepsilon\to 0^+$, we get

$$\int_\Omega\Big(M(x)\nabla u_0\cdot\nabla\phi - u_*^{-\gamma}\phi - \lambda u_0^p\phi\Big)\,\mathrm{d}x\ge 0.$$

since the equality holds if we replace φ by $-\varphi$ which implies that u_0 is a positive solution of problem (P_λ).

Step 2. We prove the existence of positive solution to the problem (P_λ) in Λ^-. Similarly to Step 1, consider $\{v_n\}\subset\Lambda^-$ and applying Ekeland's variational principle, we have

(i) $E_\lambda(v_n) < m^- + \frac{1}{n}$,

(ii) $E_\lambda(v)\ge E_\lambda(v_n) - \frac{1}{n}\|v-v_n\|$, for all $v\in\Lambda^-$.

Since $E_\lambda(v) = E_\lambda(|v|)$, we may assume that $v_n(x)\ge 0$. Clearly, $\{v_n\}$ is bounded in $H_0^1(\Omega)$, so there exists a sub-sequence, denoted by $\{v_n\}$, and $v_0\ge 0$ such that $v_n\rightharpoonup v_0$, weakly in $H_0^1(\Omega)$, and $L^p(\Omega)$, $v_n\to v_0$, strongly in $L^{1-\gamma}(\Omega)$, and $v_n(x)\to v_0(x)$, a.e. in Ω, as $n\to\infty$. Moreover, from (3.1) and using the weak lower semi-continuity of norm $E_\lambda(v_0)\le\liminf E_\lambda(v_n) = \inf_{\Lambda^-}E_\lambda$,

we see that $v_0 \not\equiv 0$ in Ω. Now, we prove that $v_0(x) > 0$ a.e. in Ω. Similarly to the arguments in Claim 1, we start by observing that, since $v_n \in \Lambda^-$, one has

$$(1+\gamma)\|v_n\|^2 - \lambda(p+\gamma)\int_\Omega |v_n|^{p+1}dx < 0. \tag{3.19}$$

It follows that

$$(1+\gamma)\int_\Omega |v_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega |v_n|^{p+1}dx < 0. \tag{3.20}$$

In view of (3.4) and (3.5), we find that

$$\begin{aligned}&\lim_{n\to\infty}\left[(1+\gamma)\int_\Omega |v_n|^{1-\gamma}dx - \lambda(p-1)\int_\Omega |v_n|^{p+1}dx\right]\\ &= (1+\gamma)\int_\Omega |v_0|^{1-\gamma}dx - \lambda(p-1)\int_\Omega |v_0|^{p+1}dx \le 0.\end{aligned}$$

Now, repeating the same arguments as in Claim 1, we see that

$$(1+\gamma)\int_\Omega |v_0|^{1-\gamma}dx - \lambda(p-1)\int_\Omega |v_0|^{p+1}dx < 0. \tag{3.21}$$

Now, let $\varphi \in H_0^1(\Omega), \varphi \ge 0$. From Lemma 2.3 with $u = v_n$, there exits a sequence of continuous functions $f_n = f_n(t)$ such that $f_n(t)(v_n + t\varphi) \in \Lambda^-$ and $f_n(0) = 1$. Therefore, using the same arguments as in Claim 1, we find that

$$f_n'(0) \quad \text{is uniformly bounded for } n \text{ large enough.} \tag{3.22}$$

Then, as in Step 1 applying (ii) and (3.22), we conclude that $v_0(x) > 0$ a.e. in Ω and

$$\int_\Omega M(x)\nabla v_0 \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega v_0^{-\gamma}\varphi \,\mathrm{d}x - \lambda\int_\Omega v_0^p\varphi \,\mathrm{d}x \ge 0, \tag{3.23}$$

for all $\varphi \in H_0^1(\Omega)$, $\varphi \ge 0$. Finally, as in the arguments of Claim 2, we obtain that $v_0 \in \Lambda^-$ is a positive solution of problem (P_λ). The proof of the Theorem 2.1 is now completed.

Remark 3.2. Theorem 1.1 shows that for a class of semilinear singular elliptic equations with boundary Dirichlet conditions, the existence of two minimizers of E_λ on Λ^+ and Λ^- respectively. Precisely, we prove that a minimizer of E_λ on Λ^+ (and in Λ^- respectively) is indeed a positive solution of (1.2). This result generalises Theorem 2.1 in [17]. Therefore, this approach could be considered for more general quasilinear operators to get multiplicity results for corresponding singular equations.

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