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BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH CAPUTO FRACTIONAL DERIVATIVES AND RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS IN BOUNDARY CONDITIONS

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Abstract. In this paper, we study a new class of boundary value problems consisting of a fractional differential inclusion with two fractional derivatives of Caputo type supplemented with boundary conditions which consist by Riemann-Liouville type fractional integral or one fractional derivative of Caputo type and one fractional Riemann-Liouville type integral. Some new existence results for convex as well as non-convex multivalued maps are obtained based on standard fixed point theorems. Some illustrative examples are also provided.

Keywords. Riemann-Liouville fractional integral; Caputo fractional derivative; Fractional differential inclusion; Existence; Fixed point theorem.

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1. Introduction

In recent years, as an extended concept of integral differential equations, fractional differential equations are widely concerned in various fields of science such as physics, chemistry, biology, economics, control theory, signal and image processing, and blood flow phenomenon.

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The boundary value problem of fractional equations has emerged as a new branch in the fields of differential equations for their deep backgrounds. For a detailed account of applications and recent results on initial and boundary value problems of fractional differential equations, we refer the reader to a series of books and papers ([1]-[17]) and references cited therein.

In this paper, we study a new class of boundary value problems consisting of a fractional differential inclusion with two fractional derivatives of Caputo type supplemented with boundary conditions which consist by Riemann-Liouville type fractional integral or one fractional derivative of Caputo type and one fractional Riemann-Liouville type integral. More precisely, we consider the following boundary value problems which consist from the differential inclusion

$$\left(\lambda D^{\alpha}+(1-\lambda)D^{\beta}\right)x(t)\in F(t,x(t)),\quad t\in(0,T), \tag{1}$$

which includes two Caputo type fractional derivatives, supplemented by boundary conditions consisting of two Riemann-Liouville fractional integrals

$$x(0)=0,\quad \mu I^{\delta_1}x(T)+(1-\mu)I^{\delta_2}x(T)=\delta_3, \tag{2}$$

or one fractional derivative of Caputo type and one Riemann-Liouville fractional integral

$$x(0)=0,\quad \mu D^{\gamma}x(T)+(1-\mu)I^{\delta}x(T)=\eta, \tag{3}$$

where D^{ϕ} is the Caputo fractional derivative of order $\phi\in\{\alpha,\beta,\gamma\}$ such that $1<\alpha,\beta\leq 2$, $\delta_3,\eta\in\mathbb{R}$, I^{χ} is the Riemann-Liouville fractional integral of order $\chi\in\{\delta_1,\delta_2,\delta\}$, $0<\lambda\leq 1$, $0\leq\mu\leq 1$ are given constants and $F:[0,T]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Recently, Niyom *et al.* [18] studied the following boundary value problem which contains four Riemann-Liouville fractional derivatives, two in differential equations and two in integral conditions, of the form

$$\begin{aligned}&\left(\lambda D^{\alpha}+(1-\lambda)D^{\beta}\right)x(t)=f(t,x(t)),\quad t\in(0,T),\\ &x(0)=0,\quad \mu D^{\gamma_1}x(T)+(1-\mu)D^{\gamma_2}x(T)=\gamma_3.\end{aligned} \tag{4}$$

The existence and uniqueness results were proved via the Banach contraction principle, the Krasnoselskii fixed point theorem and the Leray-Schauder nonlinear alternative. Problem (4) was studied in [19] with two Caputo fractional derivatives in differential equations and two derivatives or two integrals or one derivative and one integral in boundary conditions via the Sadovski's fixed point theorem. In [19], we also studied a multi-valued case of problem (4) via nonlinear alternative for contractive maps.

In this paper, we establish some existence results for problems (1)-(2) and (1)-(3). In Section 3, we study problem (1)-(2) when the right hand side has convex as well as non-convex values. In the case of convex values (upper semicontinuous case) we use the nonlinear alternative of Leray-Schauder type. When the right hand side is not necessarily convex valued (lower semicontinuous case) we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. Finally, in the last result (Lipschitz case) we prove the existence of solutions for problem (1)-(2) with not necessary non-convex valued right hand side, by applying a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. In Section 4, we consider problem (1)-(3), and prove existence results by applying Krasnoselskii's multi-valued fixed point theorem and nonlinear alternative for contractive maps. The methods used are routine, however their exposition in the framework of problem (1)-(2) and (1)-(3) is new.

2. Preliminaries

2.1. Basic materials for fractional calculus

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to \mathbb{R}$ is defined by

$$J^{\alpha} g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds,$$

provided that the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the Gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0,\infty) \to \mathbb{R}$ is defined by

$$D^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds, \quad n-1 < \alpha < n,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of real number α, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. The Caputo derivative of order q for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$$^cD^q f(t) = D^q\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right),\quad t>0,\quad n-1<q<n.$$

Remark 2.4. If $f(t)\in C^n[0,\infty)$, then

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}ds = I^{n-q}f^{(n)}(t),\ t>0,\ n-1<q<n.$$

Lemma 2.5. *For $q>0$, the general solution of the fractional differential equation $^cD^q x(t)=0$ is given by*

$$x(t) = c_0 + c_1 t + \ldots + c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$ ($n=[q]+1$).

In view of Lemma 2.5, one sees that

$$I^q\, {}^cD^q x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1}t^{n-1}, \tag{5}$$

for some $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$ ($n=[q]+1$).

2.2. Useful lemmas

Lemma 2.6. *The boundary value problem*

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right)x(t) = \omega(t), \quad t\in(0,T), \\ x(0)=0, \quad \mu I^{\delta_1}x(T) + (1-\mu)I^{\delta_2}x(T) = \delta_3, \end{cases} \tag{6}$$

is equivalent to the following integral equation

$$\begin{aligned} x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds \\ &+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}\omega(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}\omega(s)ds\Bigg),\ t\in[0,T], \end{aligned} \tag{7}$$

where the non zero constant Λ *is defined by*

$$\Lambda = \frac{\mu T^{1+\delta_1}}{\Gamma(2+\delta_1)} + \frac{(1-\mu)T^{1+\delta_2}}{\Gamma(2+\delta_2)}. \tag{8}$$

Proof. From the first equation of (6), we have

$$D^{\alpha}x(t) = \frac{\lambda-1}{\lambda}D^{\beta}x(t) + \frac{1}{\lambda}\omega(t), \quad t \in J. \tag{9}$$

Taking the Riemann-Liouville fractional integral of order α to both sides of (9), we get

$$x(t) = \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds + C_1 + C_2 t,$$

for $C_1, C_2 \in \mathbb{R}$. The first boundary condition of (6) implies that $C_1 = 0$. Hence

$$x(t) = \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds + C_2 t. \tag{10}$$

Applying the Riemann-Liouville fractional integral of order $\psi \in \{\delta_1, \delta_2\}$ such that $0 < \psi < \alpha - \beta$ to (10), we have

$$\begin{aligned} I^{\psi}x(t) &= \frac{\lambda-1}{\lambda\Gamma(\psi+\alpha-\beta)}\int_0^t (t-s)^{\psi+\alpha-\beta-1}x(s)ds \\ &\quad + \frac{1}{\lambda\Gamma(\psi+\alpha)}\int_0^t (t-s)^{\psi+\alpha-1}\omega(s)ds + C_2\frac{t^{1+\psi}}{\Gamma(2+\psi)}. \end{aligned}$$

Substituting the values $\psi = \delta_1$ and $\psi = \delta_2$ to the above relation and using the second condition of (6), we obtain

$$\begin{aligned} \delta_3 &= \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &+ \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}\omega(s)ds + \frac{\mu T^{1+\delta_1}}{\Gamma(2+\delta_1)}C_2 \\ &+ \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &+ \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}\omega(s)ds + \frac{(1-\mu)T^{1+\delta_2}}{\Gamma(2+\delta_2)}C_2, \end{aligned}$$

which leads to

$$\begin{aligned} C_2 &= \frac{1}{\Lambda}\Bigg[\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}\omega(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}\omega(s)ds\Bigg]. \end{aligned}$$

Substituting the value of constant C_2 into (10), we deduce the integral equation (7). The converse follows by direct computation. This completes the proof.

Lemma 2.7. *The boundary value problem*

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right)x(t) = \omega(t), \quad t \in (0,T), \\ x(0) = 0, \quad \mu D^{\gamma}x(T) + (1-\mu)I^{\delta}x(T) = \eta, \end{cases} \tag{11}$$

is equivalent to the following integral equation

$$\begin{aligned} x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds \\ &+ \frac{t}{\Lambda_1}\Bigg(\eta - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma)}\int_0^T (T-s)^{\alpha-\beta-\gamma-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1}\omega(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta+\alpha-\beta)}\int_0^T (T-s)^{\delta+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1}\omega(s)ds\Bigg), \ t \in J := [0,T], \end{aligned} \tag{12}$$

where the non zero constant Λ_1 *is defined by*

$$\Lambda_1 = \frac{\mu T^{1-\gamma}}{\Gamma(2-\gamma)} + \frac{(1-\mu)T^{1+\delta}}{\Gamma(2+\delta)}. \tag{13}$$

Proof. The proof is similar to that of Lemma 2.6 and omitted.

2.3. Basic material for multivalued maps

Here we outline some basic concepts of multivalued analysis; see [21, 22] and the references therein.

Let $\mathscr{U} := C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with norm $\|x\| = \sup\{|x(t)|,\ t \in [0,T]\}$. Also by $L^1([0,T],\mathbb{R})$ we denote the space of functions $x : [0,T] \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_0^T |x(t)|dt$. For a normed space $(X, \|\cdot\|)$, let

$$\begin{aligned}
\mathscr{P}_{cl}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is closed}\},\\
\mathscr{P}_{b}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\},\\
\mathscr{P}_{cl,b}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is closed and bounded}\},\\
\mathscr{P}_{cp}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is compact}\}, \text{and}\\
\mathscr{P}_{cp,c}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is compact and convex}\}.
\end{aligned}$$

A multi-valued map $G : X \to \mathscr{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$.

(ii) is *bounded* on bounded sets if $G(Y) = \cup_{x\in Y} G(x)$ is bounded in X for all $Y \in \mathscr{P}_b(X)$ (i.e. $\sup_{x\in Y}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0) \subseteq N$.

(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap Y \neq \emptyset\}$ is open for any open set Y in X.

(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathscr{P}_b(X)$; If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

(vi) is said to be *measurable* if for every $y \in X$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by *FixG*.

3. Existence results for problem (1)-(2)

3.1. The Carathéodory case

In this subsection, we consider the case when F has convex values and prove an existence result based on nonlinear alternative of the Leray-Schauder type, assuming that F is Carathéodory.

Definition 3.1. A multivalued map $F : [0,T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [0,T]$.

For each $y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ on } [0,T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 3.2. ([21, Proposition 1.2]) *If $G : X \to \mathscr{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 3.3. [24] *Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathscr{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathscr{P}_{cp,c}(C(J,X)),\ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 3.4. (Nonlinear alternative for Kakutani maps) [20]. *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathscr{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Definition 3.5. A function $x \in C^2(J,\mathbb{R})$ is a solution of problem (1)-(2) if $x(0)=0,\ \mu I^{\gamma_1}x(T)+(1-\mu)I^{\gamma_2}x(T)=\delta_3$, and there exists function $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,x(t))$ a.e. on J and

$$\begin{aligned} x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\ &+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v(s)ds\Bigg),\ t \in J, \end{aligned} \tag{14}$$

where $\Lambda \neq 0$ is defined by (8).

Theorem 3.6. *Assume that:*

(H_1) *$F:[0,T]\times\mathbb{R} \to \mathscr{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*

(H_2) *there exists a continuous nondecreasing function $\Phi:[0,\infty)\to(0,\infty)$ and a function $p \in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\Phi(\|x\|) \text{ for each } (t,x) \in [0,T]\times\mathbb{R};$$

(H_3) *there exists a constant $M>0$ such that*

$$\frac{(1-\Omega_1)M}{\Phi(M)\Omega_2 + T|\delta_3|/\Lambda} > 1, \quad \Omega_1 < 1,$$

where

$$\Omega_1 = \frac{|\lambda-1|T^{\alpha-\beta}}{\lambda}\left\{\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{T^{1-\gamma_1}}{\Lambda\Gamma(\delta_1+\alpha-\beta+1)} + \frac{(1-\mu)T^{1-\gamma_2}}{\Lambda\Gamma(\delta_2+\alpha-\beta+1)}\right\},$$

$$\begin{aligned} \Omega_2 &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}p(s)ds + \frac{T}{\Lambda}\Bigg[\frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}p(s)ds \\ &+ \frac{(1-\mu)}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}p(s)ds\Bigg]. \end{aligned}$$

Then boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. To transform problem (1)-(2) into a fixed point problem, we define an operator $\mathscr{F}: C([0,T],\mathbb{R}) \longrightarrow \mathscr{P}(C([0,T],\mathbb{R}))$ by

$$\mathscr{F}(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}): \\ h(t) = \left\{ \begin{array}{l} \dfrac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\displaystyle\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \dfrac{1}{\lambda\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}v(s)ds \\ +\dfrac{t}{\Lambda}\Bigg(\delta_3 - \dfrac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\displaystyle\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ -\dfrac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\displaystyle\int_0^T (T-s)^{\delta_1+\alpha-1}v(s)ds \\ -\dfrac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\displaystyle\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ -\dfrac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\displaystyle\int_0^T (T-s)^{\delta_2+\alpha-1}v(s)ds\Bigg), \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$. It is obvious that the fixed points of $\mathscr{F}$ are solutions of the boundary value problem (1)-(2).

Next, we are in a position to show that $\mathscr{F}$ satisfies the assumptions of the Leray-Schauder nonlinear alternative (Lemma 3.4). The proof consists of several steps.

Step 1. $\mathscr{F}(x)$ *is convex for each* $x \in C([0,T],\mathbb{R})$.

This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Step 2. $\mathscr{F}$ *maps bounded sets (balls) into bounded sets in* $C([0,T],\mathbb{R})$.

For a positive number r, let $B_r = \{x \in C([0,T],\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathscr{F}(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\ &+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v(s)ds\Bigg),\ t \in J. \end{aligned}$$

Then, for $t \in J$, we have

$$\begin{aligned}
|h(t)| &\leq \frac{\Phi(r)}{\lambda\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}p(s)ds + \frac{T\Phi(r)}{\Lambda}\Bigg[\frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}p(s)ds \\
&+ \frac{(1-\mu)}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}p(s)ds\Bigg] + \frac{T|\delta_3|}{\Lambda} \\
&+ \frac{|\lambda-1|T^{\alpha-\beta}}{\lambda}\left\{\frac{1}{\gamma(\alpha-\beta+1)} + \frac{T^{1-\gamma_1}}{\Lambda\Gamma(\delta_1+\alpha-\beta+1)} + \frac{(1-\mu)T^{1-\gamma_2}}{\Lambda\Gamma(\delta_2+\alpha-\beta+1)}\right\}r.
\end{aligned}$$

Consequently, one has $\|h\| \leq \Phi(r)\Omega_2 + r\Omega_1 + T|\delta_3|/\Lambda$.

Step 3. $\mathscr{F}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.

Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then, for each $h \in \mathscr{B}(x)$, we obtain

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leq \frac{\Phi(r)}{\lambda\Gamma(\alpha)}\left[\int_0^{t_1}[t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]p(s)ds + \int_{t_1}^{t_2}(t_1-s)^{\alpha-1}p(s)ds\right] \\
&+ \frac{|t_2-t_1|}{\Lambda}\Bigg[|\delta_3| + \frac{\Phi(r)\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}p(s)ds \\
&+ \frac{\Phi(r)(1-\mu)}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}p(s)ds\Bigg] \\
&+ \frac{r|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}\left\{t_2^{\alpha-\beta} - t_1^{\alpha-\beta} - (t_2-t_1)^{\alpha-\beta}\right\} \\
&+ \frac{r|\lambda-1|T^{\alpha-\beta}}{\lambda}\left\{\frac{\mu T^{-\gamma_1}}{\Gamma(\delta_1+\alpha-\beta+1)} + \frac{(1-\mu)T^{-\gamma_1}}{\Gamma(\delta_2+\alpha-\beta+1)}\right\}|t_2-t_1|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathscr{B} : C([0,T],\mathbb{R}) \to \mathscr{P}(C([0,T],\mathbb{R}))$ is completely continuous. Since $\mathscr{F}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph. Thus, in our next step, we show that

Step 4. $\mathscr{F}$ has a closed graph.

Let $x_n \to x_*, h_n \in \mathscr{F}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathscr{F}(x_*)$. Associated with $h_n \in \mathscr{F}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_n(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_n(s)ds \\ &+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v_n(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v_n(s)ds\Bigg),\ t \in J. \end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_*(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_*(s)ds \\ &+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v_*(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v_*(s)ds\Bigg),\ t \in J. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(v)(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\ &+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v(s)ds\Bigg),\ t \in J. \end{aligned}$$

Observe that $\|h_n(t)-h_*(t)\|\to 0$, as $n\to\infty$, and thus, it follows by Lemma 3.3 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$\begin{aligned}
h_*(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_*(s)ds\\
&+ \frac{t}{\Lambda}\Bigg(\delta_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds\\
&- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v_*(s)ds\\
&- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds\\
&- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v_*(s)ds\Bigg),\ t\in J,
\end{aligned}$$

for some $v_*\in S_{F,x_*}$.

Step 5. We show there exists an open set $U\subseteq C([0,T],\mathbb{R})$ with $x\notin\theta\mathscr{F}(x)$ for any $\theta\in(0,1)$ and all $x\in\partial U$.

Let $\theta\in(0,1)$ and $x\in\theta\mathscr{F}(x)$. Then there exists $v\in L^1([0,T],\mathbb{R})$ with $v\in S_{F,x}$ such that, for $t\in[0,T]$, we have

$$\begin{aligned}
x(t) &= \theta\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds+\theta\frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds\\
&+ \frac{\theta t}{\Lambda}\Bigg(\delta_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds\\
&- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v(s)ds\\
&- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds\\
&- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v(s)ds\Bigg),\ t\in J.
\end{aligned}$$

Using the computations of the second step above, we have

$$\begin{aligned}
|x(t)| &\leq \frac{\Phi(\|x\|)}{\lambda\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}p(s)ds \\
&+ \frac{T\Phi(\|x\|)}{\Lambda}\left[\frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}p(s)ds\right. \\
&+ \left.\frac{(1-\mu)}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}p(s)ds\right] + \frac{T|\delta_3|}{\Lambda} \\
&+ \frac{|\lambda-1|T^{\alpha-\beta}}{\lambda}\left\{\frac{1}{\gamma(\alpha-\beta+1)}\right. \\
&+ \left.\frac{T^{1-\gamma_1}}{\Lambda\Gamma(\delta_1+\alpha-\beta+1)} + \frac{(1-\mu)T^{1-\gamma_2}}{\Lambda\Gamma(\delta_2+\alpha-\beta+1)}\right\}\|x\| \\
&= \Phi(\|x\|)\Omega_2 + \Omega_1\|x\| + T|\delta_3|/\Lambda,
\end{aligned}$$

which implies that

$$\frac{(1-\Omega_1)\|x\|}{\Phi(\|x\|)\Omega_2 + T|\delta_3|/\Lambda} \leq 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(I,\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\mathscr{F} : \overline{U} \to \mathscr{P}(C(I,\mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $x \in \partial U$ such that $x \in \theta\mathscr{F}(x)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that $\mathscr{F}$ has a fixed point $x \in \overline{U}$ which is a solution of problem (1)-(2). This completes the proof.

3.2. The lower semicontinuous case

In the next result, F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [25] for lower semi-continuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and let $G : X \to \mathscr{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathscr{L} \otimes \mathscr{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathscr{J} \times \mathscr{D}$, where $\mathscr{J}$ is Lebesgue measurable in $[0,T]$ and $\mathscr{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathscr{A}$ of

$L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v \in \mathscr{A}$ and measurable $\mathscr{J} \subset [0,T] = J$, the function $u\chi_{\mathscr{J}} + v\chi_{J-\mathscr{J}} \in \mathscr{A}$, where $\chi_{\mathscr{J}}$ stands for the characteristic function of $\mathscr{J}$.

Definition 3.7. Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,T]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathscr{F} : C([0,T]\times\mathbb{R}) \to \mathscr{P}(L^1([0,T],\mathbb{R}))$ associated with F as

$$\mathscr{F}(x) = \{w \in L^1([0,T],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\},$$

which is called the Nemytskii operator associated with F.

Definition 3.8. Let $F : [0,T]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathscr{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 3.9. [26] *Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,T],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.10. *Assume that (H_2), (H_3) and the following condition holds:*

(H_4) *$F : [0,T]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*
- (a) *$(t,x) \longmapsto F(t,x)$ is $\mathscr{L}\otimes\mathscr{B}$ measurable,*
- (b) *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,T]$.*

Then boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. From Lemma 3.9, one sees that there exists a continuous function $f : C^2([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(x) \in \mathscr{F}(x)$ for all $x \in C([0,T],\mathbb{R})$. Consider the problem

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right)x(t) = f(x(t)), \quad t \in (0,T), \\ x(0) = 0, \quad \mu D^{\gamma_1}x(T) + (1-\mu)D^{\gamma_2}x(T) = \delta_3. \end{cases} \tag{15}$$

Observe that if $x \in C^2([0,T],\mathbb{R})$ is a solution of (15), then x is a solution to problem (1)-(2). In order to transform problem (15) into a fixed point problem, we define operator $\overline{\mathscr{F}}$ as

$$\begin{aligned}
\overline{\mathscr{F}}x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(x(s))ds \\
&+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\
&- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}f(x(s))ds \\
&- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\
&- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}f(x(s))ds\Bigg),\ t \in J.
\end{aligned}$$

It can easily be shown that $\overline{\mathscr{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.6. So we omit it. This completes the proof.

3.3. The Lipschitz case

In this subsection, we prove the existence of solutions for problem (1)-(2) with a not necessary nonconvex valued right hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [27].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathscr{P}(X)\times\mathscr{P}(X) \to \mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathscr{P}_{cl,b}(X),H_d)$ is a metric space (see [28]).

Definition 3.11. A multivalued operator $N : X \to \mathscr{P}_{cl}(X)$ is called

(a) $\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x),N(y)) \le \gamma d(x,y) \text{ for each } x,y \in X.$$

(b) a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.

Lemma 3.12. ([27]) *Let (X,d) be a complete metric space. If $N : X \to \mathscr{P}_{cl}(X)$ is a contraction, then $FixN \ne \emptyset$.*

Theorem 3.13. *Assume that:*

(A_1) *$F:[0,T]\times\mathbb{R}\to\mathscr{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x):[0,T]\to\mathscr{P}_{cp}(\mathbb{R})$ is measurable for each $x\in\mathbb{R}$.*

(A_2) *$H_d(F(t,x),F(t,\bar{x}))\le m(t)|x-\bar{x}|$ for almost all $t\in[0,T]$ and $x,\bar{x}\in\mathbb{R}$ with $m\in L^1([0,T],\mathbb{R}^+)$ and $d(0,F(t,0))\le m(t)$ for almost all $t\in[0,T]$.*

Then boundary value problem (1)-(2) has at least one solution on $[0,T]$ if

$$\begin{aligned}\delta: &= \frac{|\lambda-1|T^{\alpha-\beta}}{\lambda\Gamma(\alpha-\beta)}+\frac{\mu|\lambda-1|T^{\delta_1+\alpha-\beta+1}}{\lambda\Lambda\Gamma(\delta_1+\alpha-\beta)}+\frac{T(1-\mu)|\lambda-1|T^{\delta_2+\alpha-\beta+1}}{\lambda\Lambda\Gamma(\delta_2+\alpha-\beta)}\\ &+\frac{1}{\lambda\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}m(s)ds+\frac{T\mu}{\lambda\Lambda\Gamma(\delta_1+\alpha)}\int_0^T(T-s)^{\delta_1+\alpha-1}m(s)ds\\ &+\frac{T(1-\mu)}{\lambda\Lambda\Gamma(\delta_2+\alpha)}\int_0^T(T-s)^{\delta_2+\alpha-1}m(s)ds<1.\end{aligned}\tag{16}$$

Proof. Consider $\mathscr{F}$ defined at the begin of the proof of Theorem 3.6. Observe that set $S_{F,x}$ is nonempty for each $x\in C([0,T],\mathbb{R})$ by the assumption (A_1). So F has a measurable selection (see Theorem III.6 [29]). Now we show that the operator $\mathscr{F}$ satisfies the assumptions of Lemma 3.12. We show that $\mathscr{F}(x)\in\mathscr{P}_{cl}((C[0,T],\mathbb{R}))$ for each $x\in C([0,T],\mathbb{R})$. Let $\{u_n\}_{n\ge0}\in\mathscr{F}(x)$ be such that $u_n\to u$ $(n\to\infty)$ in $C([0,T],\mathbb{R})$. Then $u\in C([0,T],\mathbb{R})$ and there exists $v_n\in S_{F,x_n}$ such that, for each $t\in[0,T]$,

$$\begin{aligned}u_n(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v_n(s)ds\\ &+ \frac{t}{\Lambda}\Bigg(\delta_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T(T-s)^{\delta_1+\alpha-\beta-1}x(s)ds-\frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T(T\\ &- s)^{\delta_1+\alpha-1}v_n(s)ds\\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T(T-s)^{\delta_2+\alpha-\beta-1}x(s)ds\\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T(T-s)^{\delta_2+\alpha-1}v_n(s)ds\Bigg),\ t\in J.\end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$, we have

$$\begin{aligned} u_n(t) \to v(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\ &+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v(s)ds\Bigg),\ t \in J. \end{aligned}$$

Hence, $u \in \mathscr{F}(x)$. Next we show that there exists $\delta < 1$ such that

$$H_d(\mathscr{F}(x),\mathscr{F}(\bar{x})) \le \delta\|x-\bar{x}\| \text{ for each } x,\bar{x} \in C^2([0,T],\mathbb{R}).$$

Let $x,\bar{x} \in C^2([0,T],\mathbb{R})$ and $h_1 \in \mathscr{F}(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_1(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_1(s)ds \\ &+ \frac{t}{\Lambda}\Bigg(\delta_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v_1(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v_1(s)ds\Bigg),\ t \in J. \end{aligned}$$

By (A_2), we have

$$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w| \le m(t)|x(t)-\bar{x}(t)|,\ t \in [0,T].$$

Define $U:[0,T]\to\mathscr{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|\}.$$

Since $U(t)\cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [29]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in[0,T]$, we have $|v_1(t)-v_2(t)|\le m(t)|x(t)-\bar{x}(t)|$. For each $t\in[0,T]$, let us define

$$\begin{aligned} h_2(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\bar{x}(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_2(s)ds \\ &+ \frac{t}{\Lambda}\left(\delta_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}\bar{x}(s)ds\right. \\ &- \frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}v_2(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}\bar{x}(s)ds \\ &- \left.\frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}v_2(s)ds\right),\ t\in J. \end{aligned}$$

Hence, one has

$$\begin{aligned} |h_1(t)-h_2(t)| &\le \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|x(s)-\bar{x}(s)|ds \\ &+ \frac{T\mu|\lambda-1|}{\lambda\Lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T (T-s)^{\delta_1+\alpha-\beta-1}|x(s)-\bar{x}(s)|ds \\ &+ \frac{T(1-\mu)|\lambda-1|}{\lambda\Lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T (T-s)^{\delta_2+\alpha-\beta-1}|x(s)-\bar{x}(s)|ds \\ &+ \frac{1}{\lambda\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|v_1(s)-v_2(s)|ds \\ &+ \frac{T\mu}{\lambda\Lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}|v_1(s)-v_2(s)|ds \\ &+ \frac{T(1-\mu)}{\lambda\Lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}|v_1(s)-v_2(s)|ds \\ &\le \left\{\frac{|\lambda-1|T^{\alpha-\beta}}{\lambda\Gamma(\alpha-\beta)}+\frac{\mu|\lambda-1|T^{\delta_1+\alpha-\beta+1}}{\lambda\Lambda\Gamma(\delta_1+\alpha-\beta)}+\frac{T(1-\mu)|\lambda-1|T^{\delta_2+\alpha-\beta+1}}{\lambda\Lambda\Gamma(\delta_2+\alpha-\beta)}\right. \\ &+ \frac{1}{\lambda\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}m(s)ds+\frac{T\mu}{\lambda\Lambda\Gamma(\delta_1+\alpha)}\int_0^T (T-s)^{\delta_1+\alpha-1}m(s)ds \\ &+ \left.\frac{T(1-\mu)}{\lambda\Lambda\Gamma(\delta_2+\alpha)}\int_0^T (T-s)^{\delta_2+\alpha-1}m(s)ds\right\}\|x-\bar{x}\|. \end{aligned}$$

It follows that

$$\begin{aligned}\|h_1-h_2\| &\le \Big\{\frac{|\lambda-1|T^{\alpha-\beta}}{\lambda\Gamma(\alpha-\beta)}+\frac{\mu|\lambda-1|T^{\delta_1+\alpha-\beta+1}}{\lambda\Lambda\Gamma(\delta_1+\alpha-\beta)}+\frac{T(1-\mu)|\lambda-1|T^{\delta_2+\alpha-\beta+1}}{\lambda\Lambda\Gamma(\delta_2+\alpha-\beta)}\\ &+ \frac{1}{\lambda\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}m(s)ds\\ &+ \frac{T\mu}{\lambda\Lambda\Gamma(\delta_1+\alpha)}\int_0^T(T-s)^{\delta_1+\alpha-1}m(s)ds\\ &+ \frac{T(1-\mu)}{\lambda\Lambda\Gamma(\delta_2+\alpha)}\int_0^T(T-s)^{\delta_2+\alpha-1}m(s)ds\Big\}\|x-\bar{x}\|.\end{aligned}$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain $H_d(\mathscr{F}(x),\mathscr{F}(\bar{x}))\le\delta\|x-\bar{x}\|$, wehere δ is defined by (16). So $\mathscr{F}$ is a contraction. Therefore, it follows by Lemma 3.12 that $\mathscr{F}$ has a fixed point x which is a solution of (1)-(2). This completes the proof.

4. Existence results for problem (1)-(3)

4.1. Existence result via Krasnoselskii's multi-valued fixed point theorem

Lemma 4.1. (Krasnoselskii's fixed point theorem [30]) *Let X be a Banach space, $Y\in\mathscr{P}_{b,cl,c}(X)$ and $A,B:Y\to\mathscr{P}_{cp,c}(X)$ be two multivalued operators. Then there exists $y\in Y$ such that $y\in Ay+By$ provided that A and B satisfy the conditions: (i) $Ay+By\subset Y$ for all $y\in Y$; (ii) A is contraction, and (iii) B is u.s.c and compact.*

Definition 4.2. *A function $x\in C^2(J,\mathbb{R})$ is a solution of problem (1)-(3) if $x(0)=0,\ \mu D^{\gamma}x(T)+(1-\mu)D^{\delta}x(T)=\eta$, and there exists function $v\in L^1(J,\mathbb{R})$ such that $v(t)\in F(t,x(t))$ a.e. on J and*

$$\begin{aligned}x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds\\ &+\frac{t}{\Lambda_1}\Bigg(\eta-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma)}\int_0^T(T-s)^{\alpha-\beta-\gamma-1}x(s)ds\\ &-\frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T(T-s)^{\alpha-\gamma-1}v(s)ds-\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta+\alpha-\beta)}\int_0^T(T-s)^{\delta+\alpha-\beta-1}x(s)ds\\ &-\frac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\int_0^T(T-s)^{\delta+\alpha-1}v(s)ds\Bigg),\ t\in J:=[0,T],\end{aligned}\tag{17}$$

where $\Lambda\neq 0$ is defined by (13).

Theorem 4.3. *Assume that (H_1) holds. In addition, we suppose that:*

(B_1) *there exists a continuous function* $p \in C([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t), \text{ for each } (t,x) \in [0,T]\times\mathbb{R};$$

(B_2) $K_1 := \dfrac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \dfrac{T^{\alpha-\beta-\gamma+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma+1)} + \dfrac{T^{\delta+\alpha-\beta+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\delta+\alpha-\beta+1)} < 1.$

Then boundary value problem (1)-(3) has at least one solution on $[0,T]$.

Proof. Based on Lemma 2.7, we introduce an operator $\mathscr{G} : \mathscr{U} \to \mathscr{P}(\mathscr{U})$ as follows:

$$\mathscr{G}(x) = \left\{ \begin{array}{l} h \in C(J,\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\displaystyle\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \dfrac{1}{\lambda\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}v(s)ds \\ +\dfrac{t}{\Lambda_1}\left(\eta - \dfrac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma)}\displaystyle\int_0^T (T-s)^{\alpha-\beta-\gamma-1}x(s)ds\right. \\ -\dfrac{\mu}{\lambda\Gamma(\alpha-\gamma)}\displaystyle\int_0^T (T-s)^{\alpha-\gamma-1}v(s)ds \\ -\dfrac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta+\alpha-\beta)}\displaystyle\int_0^T (T-s)^{\delta+\alpha-\beta-1}x(s)ds \\ \left.-\dfrac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\displaystyle\int_0^T (T-s)^{\delta+\alpha-1}v(s)ds\right), \end{array} \right. \end{array} \right\} \tag{18}$$

for $v \in S_{F,x}$, where $S_{F,x}$ denote the set of selections of F defined by

$$S_{F,x} := \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,x(t)) \text{ for a.e. } t \in J\}.$$

Let us introduce the operator $\mathscr{A} : \mathscr{U} \longrightarrow \mathscr{U}$ by

$$\mathscr{A}(x) = \left\{ \begin{array}{l} \dfrac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\displaystyle\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds \\ +\dfrac{t}{\Lambda_1}\left(\eta - \dfrac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma)}\displaystyle\int_0^T (T-s)^{\alpha-\beta-\gamma-1}x(s)ds\right. \\ \left.-\dfrac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta+\alpha-\beta)}\displaystyle\int_0^T (T-s)^{\delta+\alpha-\beta-1}x(s)ds\right), \end{array} \right. \tag{19}$$

and the multi-valued operator $\mathscr{B} : \mathscr{U} \longrightarrow \mathscr{P}(\mathscr{U})$ by

$$\mathscr{B}(x) = \left\{ \begin{array}{l} h \in C(J,\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\lambda\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}v(s)ds \\ +\dfrac{t}{\Lambda_1}\left(\eta - \dfrac{\mu}{\lambda\Gamma(\alpha-\gamma)}\displaystyle\int_0^T (T-s)^{\alpha-\gamma-1}v(s)ds\right. \\ \left.+\dfrac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\displaystyle\int_0^T (T-s)^{\delta+\alpha-1}v(s)ds\right), \end{array} \right. \end{array} \right\} \tag{20}$$

for $v\in S_{F,x}$. Observe that $\mathscr{G}=\mathscr{A}+\mathscr{B}$, where the operator $\mathscr{G}:\mathscr{U}\longrightarrow\mathscr{P}(\mathscr{U})$ is given by (18). We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the conditions of Lemma 4.1 on $[0,T]$ in several steps. We begin by showing that the operators $\mathscr{A}$ and $\mathscr{B}$ define the multivalued operators $\mathscr{A},\mathscr{B}:B_r\to\mathscr{P}_{cp,c}(\mathscr{U})$, where $B_r=\{x\in\mathscr{U}:\|x\|\le r\}$ is a bounded set in $\mathscr{U}$. We next only prove that operator $\mathscr{B}$ is compact-valued on B_r and convex for all $x\in\mathscr{U}$. Note that operator $\mathscr{B}$ is equivalent to the composition $\mathscr{L}\circ S_F$, where $\mathscr{L}$ is the continuous linear operator on $L^1(J,\mathbb{R})$ into $\mathscr{U}$, defined by

$$\begin{aligned}\mathscr{L}(v)(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds+\frac{t}{\Lambda_1}\Bigg(\eta-\frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T(T-s)^{\alpha-\gamma-1}v(s)ds\\ &\quad+\frac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\int_0^T(T-s)^{\delta+\alpha-1}v(s)ds\Bigg).\end{aligned}$$

Suppose that $x\in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t)\in F(t,x(t))$ for almost all $t\in[0,T]$. Since $F(t,x(t))$ is compact for all $t\in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t)\in S_{F,x}$ for almost all $t\in J$. On the other hand, $\mathscr{L}$ is continuous, so $\mathscr{L}(v_n)(t)\to\mathscr{L}(v)(t)$ pointwise on J. In order to show that the convergence is uniform, we have to show that $\{\mathscr{L}(v_n)\}$ is an equi-continuous sequence. Let $t_1,t_2\in J$ with $t_1<t_2$. Then, we have

$$\begin{aligned}&|\mathscr{L}(v_n)(t_2)-\mathscr{L}(v_n)(t_1)|\\ \le\ &\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]p(s)ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}p(s)ds\right|\\ &+\frac{|t_2-t_1|}{\Lambda_1}\Bigg(\frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T(T-s)^{\alpha-\gamma-1}v(s)ds\\ &+\frac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\int_0^T(T-s)^{\delta+\alpha-1}v(s)ds\Bigg)\\ \le\ &\frac{\|p\|}{\Gamma(\alpha+1)}[t_2^\alpha-t_1^\alpha+2(t_2-t_1)^\alpha]+\frac{\|p\||t_2-t_1|}{\Lambda_1}\left(\frac{\mu T^{\alpha-\gamma}}{\lambda\Gamma(\alpha-\gamma+1)}+\frac{(1-\mu)T^{\delta+\alpha}}{\lambda\Gamma(\delta+\alpha+1)}\right).\end{aligned}$$

We see that the right hand of the above inequality tends to zero as $t_2\to t_1$. Thus, sequence $\{\mathscr{L}(v_n)\}$ is equi-continuous and hence it follows by the Arzelá-Ascoli theorem that there exists a uniformly convergent subsequence $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathscr{L}(v_n)\to\mathscr{L}(v)$. Note that $\mathscr{L}(v)\in\mathscr{L}(S_{F,x})$. Hence, $\mathscr{B}(x)=\mathscr{L}(S_{F,x})$ is compact for all $x\in B_r$. So $\mathscr{B}(x)$ is compact. To establish that $\mathscr{B}(x)$ is convex for all $x\in\mathscr{U}$, let $z_1,z_2\in\mathscr{B}(x)$. We select

$v_1, v_2 \in S_{F,x}$ such that

$$\begin{aligned} z_i(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_i(s)ds + \frac{t}{\Lambda_1}\left(\eta - \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1} v_i(s)ds \right. \\ &\quad \left. + \frac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1} v_i(s)ds\right),\ i=1,2, \end{aligned}$$

for almost all $t \in J$. Let $0 \le \omega \le 1$. Then, we have

$$\begin{aligned} & [\omega z_1 + (1-\omega) z_2](t) \\ = & \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\omega v_1(s) + (1-\omega) v_2(s)]ds \\ & + \frac{t}{\Lambda_1}\left(\eta - \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1}[\omega v_1(s) + (1-\omega) v_2(s)]ds \right. \\ & \left. + \frac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1}[\omega v_1(s) + (1-\omega) v_2(s)]ds\right). \end{aligned}$$

Since F has convex values, so $S_{F,x}$ is convex and $\omega v_1(s) + (1-\omega) v_2(s) \in S_{F,x}$. Thus $\omega z_1 + (1-\omega) z_2 \in \mathscr{B}(x)$. Consequently, $\mathscr{B}$ is convex-valued.

The rest of the proof consists of the following steps and claims.

Step 1. We show that $\mathscr{A}$ is a contraction on $\mathscr{U}$. Letting $x, y \in \mathscr{U}$, we have

$$\begin{aligned} \|\mathscr{A}x - \mathscr{A}y\| &\le \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}|x(s)-y(s)|ds \\ &\quad + \frac{T\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma)}\int_0^T (T-s)^{\alpha-\beta-\gamma-1}|x(s)-y(s)|ds \\ &\quad + \frac{T(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\delta+\alpha-\beta)}\int_0^T (T-s)^{\delta+\alpha-\beta-1}|x(s)-y(s)|ds \\ &\le \left\{\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma+1)} \right. \\ &\quad \left. + \frac{T^{\delta+\alpha-\beta+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\delta+\alpha-\beta+1)}\right\}\|x-y\| \\ &= K_1\|x-y\|, \end{aligned}$$

which is contractive since $K_1 < 1$.

Step 2. $\mathscr{B}$ is compact and upper semicontinuous. This will be established in several claims.

CLAIM I. *$\mathscr{B}$ maps bounded sets into bounded sets in $\mathscr{C}$.* Let $B_r = \{x \in \mathscr{C} : \|x\| \le r\}$ be a bounded set in $\mathscr{C}$. Then, for each $h \in \mathscr{B}(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds + \frac{t}{\Lambda_1}\left[\eta - \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1}v(s)ds\right. \\ &\quad \left. - \frac{(1-\mu)}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1}v(s)ds\right]. \end{aligned}$$

Then, for $t \in J$, we have

$$|h(t)| \le \|p\|\frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T}{\Lambda_1}\left[|\eta| + \frac{\|p\|\mu T^{\alpha-\gamma}}{\lambda\Gamma(\alpha-\gamma+1)} + \frac{\|p\|(1-\mu)T^{\delta+\alpha}}{\lambda\Gamma(\delta+\alpha+1)}\right].$$

Thus,

$$\|h\| \le \|p\|\frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T}{\Lambda_1}\left[|\eta| + \frac{\|p\|\mu T^{\alpha-\gamma}}{\lambda\Gamma(\alpha-\gamma+1)} + \frac{\|p\|(1-\mu)T^{\delta+\alpha}}{\lambda\Gamma(\delta+\alpha+1)}\right].$$

CLAIM II: *$\mathscr{B}$ maps bounded sets into equi-continuous sets*. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then, for each $h \in \mathscr{B}(x)$, we obtain

$$\begin{aligned} |h(t_2) - h(t_1)| &\le \frac{\|p\|}{\Gamma(\alpha+1)}[t_2^\alpha - t_1^\alpha + 2(t_2-t_1)^\alpha] \\ &\quad + \frac{|t_2-t_1|}{\Lambda_1}\left(|\eta| + \frac{\|p\|\mu T^{\alpha-\gamma}}{\lambda\Gamma(\alpha-\gamma+1)} + \frac{\|p\|(1-\mu)T^{\delta+\alpha}}{\lambda\Gamma(\delta+\alpha+1)}\right). \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathscr{B} : \mathscr{U} \to \mathscr{P}(\mathscr{U})$ is completely continuous.

Next we show that $\mathscr{B}$ is an upper semi-continuous multi-valued mapping. It is known by Lemma 3.2 that $\mathscr{B}$ will be upper semicontinuous if we establish that it has a closed graph, since already shown to be completely continuous. Thus we will prove that:

CLAIM III: *$\mathscr{B}$ has a closed graph.* Let $x_n \to x_*, h_n \in \mathscr{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathscr{B}(x_*)$. Associated with $h_n \in \mathscr{B}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$,

$$\begin{aligned} h(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds + \frac{t}{\Lambda_1}\left[\eta - \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1}v(s)ds\right. \\ &\quad \left. - \frac{(1-\mu)}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1}v(s)ds\right]. \end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in J$,

$$\begin{aligned} h_*(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_*(s)ds + \frac{t}{\Lambda_1}\Bigg[\eta - \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1} v_*(s)ds \\ &\quad - \frac{(1-\mu)}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1} v_*(s)ds\Bigg]. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1(J,\mathbb{R}) \to \mathscr{U}$ given by

$$\begin{aligned} v \mapsto \Theta(v)(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)ds + \frac{t}{\Lambda_1}\Bigg[\eta \\ &- \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1} v(s)ds \\ &- \frac{(1-\mu)}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1} v(s)ds\Bigg]. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| &= \Bigg\| \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(v_n(s) - v_*(s))ds \\ &+ \frac{t}{\Lambda_1}\Bigg[-\frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1}(v_n(s)-v_*(s))ds \\ &- \frac{(1-\mu)}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1}(v_n(s)-v_*(s))ds\Bigg]\Bigg\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

Thus, it follows by Lemma 3.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have that

$$\begin{aligned} h_*(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_*(s)ds + \frac{t}{\Lambda_1}\Bigg[\eta - \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1} v_*(s)ds \\ &\quad - \frac{(1-\mu)}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1} v_*(s)ds\Bigg], \end{aligned}$$

for some $v_* \in S_{F,x_*}$. Hence $\mathscr{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathscr{B}$ is compact valued and upper semi-continuous.

Step 3. Here, we show that $\mathscr{A}(x) + \mathscr{B}(x) \subset B_r$ for all $x \in B_r$. Let us select $r \geq \frac{\|p\|K_2 + |\eta|T/\Lambda}{1-K_1}$, where K_1 defined by (B_2) and

$$K_2 = \frac{T^{\alpha}}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma+1)} + \frac{T^{\delta+\alpha+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\delta+\alpha+1)}. \tag{21}$$

For any $x \in B_r$, we have

$$\begin{aligned}
\|\mathscr{P}x\| &\leq \sup_{t\in J}\Bigg|\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\
&+ \frac{t}{\Lambda_1}\Bigg(\eta-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma)}\int_0^T (T-s)^{\alpha-\beta-\gamma-1}x(s)ds \\
&- \frac{\mu}{\lambda\Gamma(\alpha-\gamma)}\int_0^T (T-s)^{\alpha-\gamma-1}v(s)ds \\
&- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta+\alpha-\beta)}\int_0^T (T-s)^{\delta+\alpha-\beta-1}x(s)ds \\
&- \frac{1-\mu}{\lambda\Gamma(\delta+\alpha)}\int_0^T (T-s)^{\delta+\alpha-1}v(s)ds\Bigg)\Bigg| \\
&\leq \|x\|\left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{T^{\alpha-\beta-\gamma+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma+1)}+\frac{T^{\delta+\alpha-\beta+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\delta+\alpha-\beta+1)}\right] \\
&+ \frac{|\eta|T}{\Lambda} \\
&+ \|p\|\left[\frac{T^{\alpha}}{\lambda\Gamma(\alpha+1)}+\frac{T^{\alpha-\gamma+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma+1)}+\frac{T^{\delta+\alpha+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\delta+\alpha+1)}\right] \\
&\leq rK_1+\|p\|K_2+\frac{|\eta|T}{\Lambda_1}\leq r,
\end{aligned}$$

which implies that $\mathscr{P}B_r \subset B_r$. Hence $\|h\| \leq r$, which means that $\mathscr{A}(x)+\mathscr{B}(x) \subset B_r$ for all $x \in B_r$. Thus, operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Lemma 4.1 and hence we conclude that $x \in \mathscr{A}(x)+\mathscr{B}(x)$ has a solution in B_r. Therefore problem (1)-(2) has a solution in B_r and the proof is completed.

4.2. Existence result via nonlinear alternative for contractive maps

Lemma 4.4. ([31]) *Let X be a Banach space and let D be a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathscr{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathscr{P}_{cp,c}(X)$ be two multi-valued operators satisfying*

(a) *Z_1 is contraction, and*

(b) *Z_2 is upper semicontinuous and compact.*

Then, if $G = Z_1 + Z_2$, either

(i) *G has a fixed point in $\bar{D}$ or*

(ii) *there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

Theorem 4.5. *Assume that (H_1) and (H_2) hold. In addition, we suppose that:*

(B_3) *there exists a constant* $M > 0$ *such that* $\dfrac{(1-K_1)M}{\|p\|\Phi(M)K_2} > 1, \quad K_1 < 1$, *where* K_1, K_2 *are defined by* (B_2) *and (21) respectively.*

Then boundary value problem (1)-(3) has at least one solution on $[0,T]$.

Proof. We consider operator $\mathscr{G} : C([0,T],\mathbb{R}) \longrightarrow \mathscr{P}(C([0,T],\mathbb{R}))$ by (18) and operators $\mathscr{A} : \mathscr{U} \longrightarrow \mathscr{U}$ and $\mathscr{B} : \mathscr{U} \longrightarrow \mathscr{P}(\mathscr{U})$ defined by (19) and (20) respectively. As in Theorem 4.3, one can show that the operators $\mathscr{A}$ and $\mathscr{B}$ are indeed the multivalued operators $\mathscr{A}, \mathscr{B} : B_r \to \mathscr{P}_{cp,c}(\mathscr{U})$ where $B_r = \{x \in \mathscr{U} : \|x\| \leq r\}$ is a bounded set in $\mathscr{U}$, $\mathscr{A}$ is a contraction on $\mathscr{U}$ and $\mathscr{B}$ is u.s.c. and compact. It follows that $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 4.4. Hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \theta\mathscr{A}(x) + \theta\mathscr{B}(x)$ for $\theta \in (0,1)$, then there exists $v \in S_{F,y}$ such that

$$
\begin{aligned}
x(t) &= \theta \frac{\lambda - 1}{\lambda\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s)ds + \theta \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds \\
&+ \theta \frac{t}{\Lambda_1} \Bigg(\eta - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma)} \int_0^T (T-s)^{\alpha-\beta-\gamma-1} x(s)ds \\
&- \frac{\mu}{\lambda\Gamma(\alpha-\gamma)} \int_0^T (T-s)^{\alpha-\gamma-1} v(s)ds - \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta+\alpha-\beta)} \int_0^T (T-s)^{\delta+\alpha-\beta-1} x(s)ds \\
&- \frac{1-\mu}{\lambda\Gamma(\delta+\alpha)} \int_0^T (T-s)^{\delta+\alpha-1} v(s)ds \Bigg), \ t \in J.
\end{aligned}
$$

Using the assumptions, we get

$$
\begin{aligned}
|x(t)| &\leq \|x\| \Bigg[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma+1)} \\
&+ \frac{T^{\delta+\alpha-\beta+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\delta+\alpha-\beta+1)} \Bigg] + \frac{|\eta|T}{\Lambda} \\
&+ \|p\|\Phi(\|x\|) \left[\frac{T^{\alpha}}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma+1)} + \frac{T^{\delta+\alpha+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\delta+\alpha+1)} \right] \\
&\leq \|x\|K_1 + \|p\|\Phi(\|x\|)K_2 + \frac{|\eta|T}{\Lambda_1},
\end{aligned}
$$

which implies that

$$
\frac{(1-K_1)\|x\|}{\|p\|\Phi(\|x\|)K_2} \leq 1. \tag{22}
$$

If condition (ii) of Theorem 4.4 holds, then there exists $\theta \in (0,1)$ and $x \in \partial B_M$ with $x = \theta \mathscr{G}(x)$. Then, x is a solution of (1)-(3) with $\|x\| = M$. Now, by the inequality (22), we get

$$\frac{(1-K_1)M}{\|p\|\Phi(M)K_2} \leq 1,$$

which contradicts (B_3). Hence, $\mathscr{G}$ has a fixed point on J by Theorem 4.4, and consequently problem (1)-(3) has a solution. This completes the proof.

5. Examples

In this section, we present some examples to illustrate our results.

Example 5.1. *Let us consider the following two orders fractional differential inclusion with two orders fractional integral boundary conditions*

$$(23)\qquad \begin{cases} \left(\frac{33}{38}D^{29/15} + \frac{5}{38}D^{16/15}\right)x(t) \in F_1(t,x(t)), \quad t \in [0,3/2], \\ x(0) = 0, \quad \frac{6}{11}I^{5/2}x\left(\frac{3}{2}\right) + \frac{5}{11}I^{7/2}x\left(\frac{3}{2}\right) = \frac{2}{3}, \end{cases}$$

where $F_1(t,x)$ is the multivalue function defined by

$$F_1(t,x) = \left[\frac{\sqrt{t}e^{-t}}{15}\left(\frac{\sin|x|}{7(|x|+1)} + \frac{1}{9}\right), \frac{(\sqrt{t}+1)}{10}\left(\frac{x^2}{5(|x|+1)} + \frac{1}{2}\right)\right].$$

Here $\lambda = 33/38$, $\alpha = 29/15$, $\beta = 16/15$, $\mu = 6/11$, $\delta_1 = 5/2$, $\delta_2 = 7/2$, $\delta_3 = 2/3$, $T = 3/2$. We can find that $\Lambda = 0.2476789227$ and $\Omega_1 = 0.4867106649$. It is easy to see that

$$\|F_1(t,x)\|_{\mathscr{P}} = \sup\{|y| : y \in F_1(t,x)\} \leq \frac{(\sqrt{t}+1)}{10}\left(\frac{|x|}{5} + \frac{1}{2}\right).$$

Set $p(t) = (\sqrt{t}+1)/10$ and $\Phi(x) = (x/5) + (1/2)$. After that, we get $\Omega_2 = 0.3107475249$. From the given data, we can find that there exists a positive constant $M > 9.293923401$ satisfying condition (H_3) of Theorem 3.6. Therefore, by applying Theorem 3.6, we deduce that the boundary value problem (23) has at least one solution on $[0,3/2]$.

Example 5.2. *Let us consider the following two orders fractional differential inclusion with two orders fractional integral boundary conditions*

$$(24)\qquad \begin{cases} \left(\frac{49}{53}D^{17/9} + \frac{4}{53}D^{10/9}\right)x(t) \in F_2(t,x(t)), \quad t \in [0,1/2], \\ x(0) = 0, \quad \frac{2}{5}I^{1/2}x\left(\frac{1}{2}\right) + \frac{3}{5}I^{3/2}x\left(\frac{1}{2}\right) = \frac{2}{7}, \end{cases}$$

where $F_2(t,x)$ is the multivalue function defined by

$$F_2(t,x) = \left[0, \left(\frac{t+3}{t^2+1}\right)\left(\frac{x^2+2|x|}{2(1+|x|)}+\frac{1}{2}\right)\right].$$

Here $\lambda = 49/53$, $\alpha = 17/9$, $\beta = 10/9$, $\mu = 2/5$, $\delta_1 = 1/2$, $\delta_2 = 3/2$, $\delta_3 = 2/7$, $T = 1/2$. By direct computation, we can find that $\Lambda = 0.1382999905$. It is easy to see that

$$\sup\{|x| : x \in F_2(t,x)\} \leq \left(\frac{t+3}{t^2+1}\right)\left(\frac{x^2+2|x|}{2(1+|x|)}+\frac{1}{2}\right).$$

Also we have

$$H_d(F_2(t,x),F_2(t,y)) \leq \left(\frac{t+3}{t^2+1}\right)|x-y|.$$

Setting $m(t) = (t+3)/(t^2+1)$, we can write as $H_d(F_2(t,x),F_2(t,y)) \leq m(t)|x-y|$ such that $d(0,F_2(t,0)) \leq m(t)$. From the above data, we find that $\delta = 0.9640022781 < 1$. Therefore all assumptions of Theorem 3.13 are fulfilled. Thus, by the conclusion of Theorem 3.13, we deduce that the problem (24) with $F_2(t,x)$ given by (24) has at least one solution on $[0,1/2]$.

Example 5.3. *Let us consider the following two orders fractional differential inclusion with mixed fractional derivative and integral boundary conditions*

$$(25) \qquad \begin{cases} \left(\frac{25}{27}D^{19/13}+\frac{2}{27}D^{15/13}\right)x(t) \in F_3(t,x(t)), \quad t \in [0,3/2], \\ x(0) = 0, \quad \frac{9}{16}D^{8/15}x\left(\frac{3}{2}\right)+\frac{7}{16}I^{11/15}x\left(\frac{3}{2}\right) = \frac{1}{3}, \end{cases}$$

where $F_3(t,x)$ is the multivalue function defined by

$$F_3(t,x) = \left[\frac{e^{-x^2}\sin x}{(t^2+2)}, \log\left(\frac{|x|+1}{|x|+3}\right)\frac{\cos x}{(t^2+1)}\right].$$

Here $\lambda = 25/27$, $\alpha = 19/13$, $\beta = 15/13$, $\mu = 9/16$, $\gamma = 8/15$, $\delta = 11/15$, $\eta = 1/3$, $T = 3/2$. We can compute that $\Lambda_1 = 1.324284418$ and $K_1 = 0.1994504661 < 1$. Observe that the following inequality holds

$$\|F_3(t,x)\|_{\mathscr{P}} = \sup\{|x| : x \in F_3(t,x)\} \leq \frac{1}{t^2+1}.$$

Choose $p(t) = 1/(t^2+1)$. Then all conditions of Theorem 4.3 are satisfied. Hence, by applying Theorem 4.3, we obtain a conclusion that boundary value problem (25) has at least one solution on $[0,3/2]$.

Example 5.4. *Let us consider the following two orders fractional differential inclusion with mixed fractional derivative and integral boundary conditions*

$$(26)\qquad \begin{cases} \left(\dfrac{41}{46}D^{13/7}+\dfrac{5}{46}D^{8/7}\right)x(t)\in F_4(t,x(t)), \quad t\in[0,1/2], \\ x(0)=0, \quad \dfrac{13}{31}D^{5/9}x\left(\dfrac{1}{2}\right)+\dfrac{18}{31}I^{4/9}x\left(\dfrac{1}{2}\right)=\dfrac{3}{4}, \end{cases}$$

where $F_4(t,x)$ is the multivalue function defined by

$$F_4(t,x)=\left[\left(\frac{1}{t^2+4}\right)\left(\frac{1}{3}\sin x+\frac{1}{2}e^{-|x|}\right),\left(\frac{1}{t^2+3}\right)\left(\frac{2x^2}{1+|x|}+1\right)\right].$$

Here $\lambda=41/46$, $\alpha=13/7$, $\beta=8/7$, $\mu=13/31$, $\gamma=5/9$, $\delta=4/9$, $\eta=3/4$, $T=1/2$. We can find that $\Lambda_1=0.5146859058$, $K_1=0.1579843375$ and $K_2=0.3824736642$. In addition, we have

$$\|F_4(t,x)\|_{\mathscr{P}}=\sup\{|x|\,:\,y\in F_4(t,x)\}\le\left(\frac{1}{t^2+3}\right)(2|x|+1).$$

Now, we set $p(t)=1/(t^2+3)$ and $\Phi(|x|)=2|x|+1$. So, we get $\|p\|=1/3$. From the obtained detail, we can find that there exists a constant $M>0.2171788872$ satisfying inequality in Theorem 4.5. Thus, from Theorem 4.5, we get that boundary value problem (26) has at least one solution on $[0,1/2]$.

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