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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SYSTEM OF IMPULSIVE DIFFERENTIAL EQUATIONS ON THE HALF-LINE

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Abstract. In this paper, we study the existence and uniqueness, continuous dependence on initial conditions and the boundedness of solutions for a system of impulsive differential equations using the fixed point approach in vector Banach spaces. The compactness and $u.s.c.$ of operator solutions are also investigated.

Keywords. Impulsive differential equation; Fixed point; Generalized metric space; Vector metric space.

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1. Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [7] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [6]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and biology. A comprehensive introduction to the basic

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theory is well developed in the monographs by Benchohra *et al.* [2], Graef *et al.* [5], Lasksh-mikantham *et al.* [1], and Samoilenko and Perestyuk [15]. and the references therein.

The classical Banach contraction principle was extended for contractive maps on spaces endowed with a vector-valued metric by Perov in 1964 [11] and Perov and Kibenko [12]. Up to now, there have been a number of attempts to generalize the Perov fixed point theorem in several directions and also there have been a number of applications in various fields of nonlinear analysis, for systems of ordinary differential and semilinear differential equations. Recently Precup [13] established the role of vector-valued metric convergence in the study of semilinear operator systems. In recent years, many authors studied the existence of solutions for systems of differential equations by using the vector version fixed point theorem; see [3, 10, 14, 8, 9] and the references therein.

In this paper, we consider the following system of impulsive differential equations

$$\begin{cases} x'(t) = f(t,x,y), t \in J := [0,\infty), t \neq t_k,\ k = 1,\ldots, \\ y'(t) = g(t,x,y), t \in J, \quad t \neq t_k,\ k = 1,\ldots, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k), y(t_k)), \quad k = 1,\ldots, \\ y(t_k^+) - y(t_k^-) = \bar{I}_k(x(t_k), y(t_k)), \quad k = 1,\ldots, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $x_0, y_0 \in \mathbb{R}$, $f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are a given functions, $I_k, \bar{I}_k \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The notations $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$ stand for the right and the left limits of the functions y at $t = t_k$, respectively.

This paper is organized as follows. In Section 2, we introduce all the background material used in this paper such as some properties of generalized Banach spaces and fixed point theory. In Sections 3 and 4, we state and prove our main results by using Perov's and Krasnoselskii fixed point type theorems in generalized Banach spaces.

2. Preliminaries

In this section, we introduce notations and definitions which are used throughout this paper.

Definition 2.1. Let X be a nonempty set and consider space $\mathbb{R}^m_+$ endowed with the usual component-wise partial order. The mapping $d : X \times X \to \mathbb{R}^m_+$ which satisfies all the usual axioms of the metric is called a generalized metric in Perov's sense and (X,d) is called a generalized metric space.

Let (X,d) be a generalized metric space in Perov's sense. For $r := (r_1,\dots,r_m) \in \mathbb{R}^m_+$, we denote by

$$B(x_0,r) = \{x \in X : d(x_0,x) < r\},$$

the open ball centered at x_0 with radius r, and

$$\overline{B(x_0,r)} = \{x \in X : d(x_0,x) \le r\},$$

the closed ball centered at x_0 with radius r.

In the case of generalized metric spaces in the sense of Perov, the notions of convergent sequence, Cauchy sequence, completeness, and open and closed subsets are similar to those for usual metric spaces.

If, $v,r \in \mathbb{R}^m, v := (v_1,\dots,v_m)$, and $r := (r_1,\dots,r_m)$, then by $v \le r$ we mean $v_i \le r_i$ for each $i \in \{1,\dots,m\}$ and by $v < r$ we mean $v_i < r_i$ for each $i \in \{1,\dots,m\}$. Also $|v| := (|v_1|,\dots,|v_m|)$ and $\max(u,v) := (\max(u_1,v_1),\dots,\max(u_m,v_m))$. If $c \in \mathbb{R}$, then $v \le c$ means $v_i \le c$ for each $i \in \{1,\dots,m\}$.

Definition 2.2. A square matrix A of real numbers is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.

Lemma 2.3. *Let $A \in \mathscr{M}_{m,m}(\mathbb{R}^+)$. Then the following statements are equivalent:*

- *A is a matrix convergent to zero;*
- *The eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A-\lambda I) = 0$;*
- *The matrix $I-A$ is non-singular and $(I-A)^{-1} = I+A+\cdots+A^n+\cdots$;*
- *The matrix $I-A$ is non-singular and $(I-A)^{-1}$ has nonnegative elements;*
- *$A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for any $q \in \mathbb{R}^m$.*

Example 2.4. *Some examples of matrices convergent to zero are:*

(1) $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, *where $a,b \in \mathbb{R}_+$ and $a+b < 1$;*

(2) $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, *where* $a, b \in \mathbb{R}_+$ *and* $a + b < 1$*;*

(3) $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, *where* $a, b, c \in \mathbb{R}_+$ *and* $\max\{a, c\} < 1$.

Definition 2.5. Let (X,d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix A such that

$$d(N(x),N(y)) \leq Ad(x,y), \quad \forall x,y \in X.$$

Theorem 2.6. *Let (X,d) be a complete generalized metric space and $N : X \to X$ be a contractive operator with Lipschitz matrix A. Then N has a unique fixed point x^* and for each $x_0 \in X$ we have*

$$d(N^k(x_0),x^*) \leq A^k(I-A)^{-1}d(x_0,N(x_0)) \textit{ for all } k \in \mathbb{N}.$$

In [16], the following version of the Krasnoselskii fixed point theorem was obtained.

Theorem 2.7. (Krasnoselskii type) [16] *Let $(X, \|\cdot\|)$ be a generalized Banach space. Suppose that A and B map X into X such that*

(i) *A is a completely continuous operator,*
(ii) *B is a contraction with constant $\alpha < 1$,*
(iii) *the set $\mathscr{M} = \{x \in X : \ x = \lambda B(\frac{x}{\lambda}) + \lambda A(x), \lambda \in (0,1)\}$ is bounded.*

Then there exists $x \in \mathscr{M}$ with $A(x) + B(x) = x$.

Denote by $\mathscr{P}(X) = \{Y \subset X : \ Y \neq \emptyset\}$, $\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : \ Y \text{ closed}\}$, $\mathscr{P}_b(X) = \{Y \in \mathscr{P}(X) : \ Y \text{ bounded}\}$. Let (X,d) and (Y,ρ) be two metric spaces and $F : X \to \mathscr{P}(Y)$ be a multi-valued mapping. The map F is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Equivalently, F is *u.s.c.* if the set $F^{+1}(V)$ is open for any open set V in Y.

The mapping F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup\{F(x) : x \in A\} \subset K.$$

Also, F is *compact* if $F(X)$ is relatively compact, and it is called *locally compact* if for each $x \in X$, there exists an open set U containing x such that $F(U)$ is relatively compact.

Theorem 2.8. [4] *Let $F : X \to \mathscr{P}_{cp}(Y)$ be a closed locally compact multifunction. Then F is u.s.c.*

3. Uniqueness and continuous dependence on initial data

In order to define a solution for problem (1.1), we consider the space of piecewise continuous functions

$$PC_b = \{y \in PC([0,\infty),\mathbb{R}) : \ y \text{ is bounded}\},$$

where $PC([0,\infty),\mathbb{R}) = \{y : [0,\infty) \to \mathbb{R}, y_k \in C((t_k,t_{k+1}],\mathbb{R}),\ k = 0,\dots, y(t_k^-)$ and $y(t_k^+)$ exists and satisfies $y(t_k) = y(t_k^-)$ for $k = 1,\dots\}$. PC_b is a Banach space with norm $\|y\|_b = \sup\{|y(t)| : t \in [0,\infty)\} < \infty$.

Definition 3.1. A function $(x,y) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ is said to be a solution of (1.1) if and only if

$$\begin{cases} x(t) = x_0 + \int_0^t f(s,x(s),y(s))ds + \sum\limits_{0<t_k<t} I_k(x(t_k),y(t_k)), t \in J, \\ y(t) = y_0 + \int_0^t g(s,x(s),y(s))ds + \sum\limits_{0<t_k<t} \bar{I}_k(x(t_k),y(t_k)), t \in J. \end{cases}$$

In this section we assume the following conditions:

(H_1)**:** There exist functions $l_i \in L^1(J,\mathbb{R}^+), i = 1,\dots,4$, such that

$$|f(t,x,y) - f(s,\bar{x},\bar{y})| \leq l_1(t)|x-\bar{x}| + l_2(t)|y-\bar{y}|, \text{ for all } x,\bar{x},y,\bar{y} \in \mathbb{R}$$

and

$$|g(t,x,y) - g(s,\bar{x},\bar{y})| \leq l_3(t)|x-\bar{x}| + l_4(t)|y-\bar{y}|, \text{ for all } x,\bar{x},y,\bar{y} \in \mathbb{R}.$$

(H_2)**:** There exist constants $a_{1k}, a_{2k} \geq 0,\ k = 1,\dots,$ such that

$$|I_k(x,y) - I_k(\bar{x},\bar{y})| \leq a_{1k}|x-\bar{x}| + a_{2k}|y-\bar{y}|, \text{ for all } x,\bar{x},y,\bar{y} \in \mathbb{R}$$

and

$$\sum_{k=1}^{\infty} |I_k(0,0)| < \infty.$$

(H_3)**:** There exist constants $b_{1k}, b_{2k} \geq 0,\ k = 1, \dots,$ such that

$$|\bar{I}_k(x,y) - \bar{I}_k(\bar{x},\bar{y})| \leq b_{1k}|x-\bar{x}| + b_{2k}|y-\bar{y}|, \text{ for all } x,\bar{x},y,\bar{y} \in \mathbb{R}$$

and

$$\sum_{k=1}^{\infty} |\bar{I}_k(0,0)| < \infty.$$

We will use the Perov fixed point theorem to prove that a solution of problem (1.1) is bounded and tends to zero as $t \to \infty$.

Theorem 3.2. *Assume that $(H_1)-(H_3)$ are satisfied and the matrix*

$$M = \begin{pmatrix} \|l_1\|_{L^1} + \sum_{k=1}^{\infty} a_{1k} & \|l_2\|_{L^1} + \sum_{k=1}^{\infty} a_{2k} \\ \|l_3\|_{L^1} + \sum_{k=1}^{\infty} b_{1k} & \|l_4\|_{L^1} + \sum_{k=1}^{\infty} b_{2k} \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{R}^+), \tag{3.1}$$

where

$$\sum_{k=1}^{\infty} a_{ik} < \infty \text{ and } \sum_{k=1}^{\infty} b_{ik} < \infty, i = 1,2,$$

converges to zero and $f(\cdot,0,0), g(\cdot,0,0) \in L^1(J,\mathbb{R})$. Then problem (1.1) has unique solution. If we add that and $\sum_{k=1}^{\infty} a_{1k} + \sum_{k=1}^{\infty} a_{2k} + \sum_{k=1}^{\infty} b_{1k} + \sum_{k=1}^{\infty} b_{2k} < 1$, the unique solution of (1.1) is bounded.

Proof. Consider operator $N : PC \times PC \to PC \times PC$ defined by

$$N(x,y) = (N_1(x,y), N_2(x,y)),$$

where

$$N_1(x,y)(t) = x_0 + \int_0^t f(s,x(s),y(s))ds + \sum_{0<t_k<t} I_k(x(t_k),y(t_k)), t \in [0,\infty)$$

and

$$N_2(x,y)(t) = y_0 + \int_0^t g(s,x(s),y(s))ds + \sum_{0<t_k<t} \bar{I}_k(x(t_k),y(t_k)), t \in [0,\infty).$$

We show that N was well defined. Given $(x,y) \in PC_b \times PC_b, t \in [0,\infty)$, we have

$$\begin{aligned}\|N_1(x,y)\|_b &\le |x_0| + \int_0^t |f(s,x(s),y(s))|ds + \sum_{0<t_k<t} |I_k(x(t_k),y(t_k))| \\ &\le \|l_1\|_{L^1}\|x\|_b + \|l_2\|_{L^1}\|y\|_b + \sum_{0<t_k<t}(a_{1k}\|x\|_b + a_{2k}\|y\|_b) \\ &\quad + \|f(\cdot,0,0)\|_{L^1} + \sum_{0<t_k<t}(|I_k(0,0)| + |\bar{I}_k(0,0)|).\end{aligned}$$

Similarly we have

$$\begin{aligned}\|N_2(x,y)\|_b &\le \|l_3\|_{L^1}\|x\|_b + \|l_4\|_{L^1}\|y\|_b + \sum_{0<t_k<t}(b_{1k}\|x\|_b + b_{2k}\|y\|_b) \\ &\quad + \|g(\cdot,0,0)\|_{L^1} + \sum_{0<t_k<t}(|I_k(0,0)| + |\bar{I}_k(0,0)|).\end{aligned}$$

Thus

$$\begin{aligned}\begin{pmatrix} \|N_1(x,y)\|_b \\ \|N_1(x,y)\|_b \end{pmatrix} &\le \begin{pmatrix} \|l_1\|_{L^1} + \sum_{k=1}^\infty a_{1k} & \|l_2\|_{L^1} + \sum_{k=1}^\infty a_{2k} \\ \|l_3\|_{L^1} + \sum_{k=1}^\infty b_{1k} & \|l_4\|_{L^1} + \sum_{k=1}^\infty b_{1k}\|x\| + b_{2k} \end{pmatrix} \begin{pmatrix} \|x\|_b \\ \|y\|_b \end{pmatrix} \\ &\quad + \begin{pmatrix} \|f(.,0,0)\|_{L^1} + \sum_{k=1}^\infty(\|I_k(0,0)\|_b + \|\bar{I}_k(0,0)\|_b) \\ \\ \|g(.,0,0)\|_{L^1} + \sum_{k=1}^\infty(|I_k(0,0)| + |\bar{I}_k(0,0)|) \end{pmatrix}.\end{aligned}$$

This implies that N is well defined.

Clearly, fixed points of N are solutions of problem (1.1). We show that N is a contraction. Let $(x,y),(\bar{x},\bar{y}) \in PC_b \times PC_b$. Then (H_1) and (H_2) imply

$$\begin{aligned}|N_1(x,y)(t) - N_1(\bar{x},\bar{y})(t)| &\le \int_0^t |f(s,x(s),y(s)) - f(s,\bar{x}(s),\bar{y}(s))|ds \\ &+ \sum_{0<t_k<t}^{\infty} |I_k(x(t_k),y(t_k)) - I_k(\bar{x}(t_k),\bar{y}(t_k))| \\ &\le \int_0^t (l_1(s)|x(s) - \bar{x}(s)| + l_2(s)|y(s) - \bar{y}(s)|)\,ds \\ &+ \sum_{0<t_k<t}(a_{1k}|x(t_k) - \bar{x}(t_k)| + a_{2k}|y(t_k) - \bar{y}(t_k)|).\end{aligned}$$

Thus

$$\begin{aligned}\|N_1(x,y) - N_1(\bar{x},\bar{y})\|_b &\le (\|l_1\|_{L^1} + \sum_{k=1}^\infty a_{1k})\|x - \bar{x}\|_b \\ &+ (\|l_2\|_{L^1} + \sum_{k=1}^\infty a_{2k})\|y - \bar{y}\|_b.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|N_2(x,y)-N_2(\bar{x},\bar{y})\|_b &\leq (\|l_3\|_{L^1}+\sum_{k=1}^{\infty}b_{1k})\|x-\bar{x}\|_b \\ &+ (\|l_4\|_{L^1}+\sum_{k=1}^{\infty}b_{2k})\|y-\bar{y}\|_b.\end{aligned}$$

It follows that

$$\|N(x,y)-N(\bar{x},\bar{y})\|_b \leq M\begin{pmatrix}\|x-\bar{x}\|_b\\ \|y-\bar{y}\|_b\end{pmatrix},\ \text{for all } (x,y),(\bar{x},\bar{y})\in PC_b\times PC_b.$$

Hence, by Theorem 2.6, the operator N has at least one fixed point which is a solution of problem (1.1).

Now we show that the solution (x,y) is bounded. Letting $t\in[0,\infty)$. we get

$$\begin{aligned}|x(t)| &\leq |x_0|+\int_0^t|f(s,x(s),y(s))|ds+\sum_{0<t_k<t}|I_k(x(t_k),y(t_k))| \\ &\leq |x_0|+\int_0^t(l_1(s)|x|+l_2(s)|y|)ds+\sum_{k=1}^{\infty}a_{1k}|x(t_k)|+\sum_{k=1}^{\infty}a_{2k}|y(t_k)| \\ &\quad +\|f(.,0,0)\|_{L^1}+\|g(.,0,0)\|_{L^1}+\sum_{k=1}^{\infty}|I_k(0,0)+\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|\end{aligned}$$

and

$$\begin{aligned}|y(t)| &\leq |y_0|+\int_0^t(l_3(s)|x(s)|+l_4(s)|y(s)|)ds+\sum_{k=1}^{\infty}b_{1k}|x(t_k)|+\sum_{k=1}^{\infty}b_{2k}|y(t_k)| \\ &\quad +\|f(.,0,0)\|_{L^1}+\|g(.,0,0)\|_{L^1}+\sum_{k=1}^{\infty}|I_k(0,0)+\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|.\end{aligned}$$

Thus

$$\begin{aligned}|x(t)|+|y(t)| &\leq |x_0|+|y_0|+\int_0^t((l_1(s)+l_3(s))|x(s)|+(l_2(s)+l_4(s))|y(s)|)ds \\ &+ (\sum_{k=1}^{\infty}a_{1k}+\sum_{k=1}^{\infty}a_{2k}+\sum_{k=1}^{\infty}b_{1k}+\sum_{k=1}^{\infty}b_{2k})(|x(t_k)|+|y(t_k)|) \\ &\quad +2\|f(.,0,0)\|_{L^1}+2\|g(.,0,0)\|_{L^1}+2\sum_{k=1}^{\infty}|I_k(0,0)+2\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{s\in(0,t)}(|x(s)|+|y(s)|) &\leq |x_0|+|y_0|+\int_0^t (l_1(s)+l_3(s)+l_2(s)+l_4(s))\times \\ & \quad \sup_{s\in[0,t]}(|x(s)|+|y(s)|)ds \\ &+ \left(\sum_{k=1}^{\infty} a_{1k}+\sum_{k=1}^{\infty} a_{2k}+\sum_{k=1}^{\infty} b_{1k}+\sum_{k=1}^{\infty} b_{2k}\right)\sup_{s\in[0,t]}(|x(t_k)|+|y(t_k)|) \\ &+ 2\|f(.,0,0)\|_{L^1}+2\|g(.,0,0)\|_{L^1}+2\sum_{k=1}^{\infty}|I_k(0,0)+2\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|.\end{aligned}$$

This implies that

$$\sup_{s\in(0,t)}(|x(s)|+|y(s)|)\leq \alpha+\int_0^t l(s)\sup_{s\in[0,t]}(|x(s)|+|y(s)|)ds,$$

where

$$\alpha=\frac{|x_0|+|y_0|+2\|f(.,0,0)\|_{L^1}+2\|g(.,0,0)\|_{L^1}+2\sum_{k=1}^{\infty}|I_k(0,0)+2\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|}{1-(\sum_{k=1}^{\infty}a_{1k}+\sum_{k=1}^{\infty}a_{2k}+\sum_{k=1}^{\infty}b_{1k}+\sum_{k=1}^{\infty}b_{2k})}$$

and

$$l(s)=\frac{l_1(s)+l_2(s)+l_3(s)+l_4(s)}{1-(\sum_{k=1}^{\infty}a_{1k}+\sum_{k=1}^{\infty}a_{2k}+\sum_{k=1}^{\infty}b_{1k}+\sum_{k=1}^{\infty}b_{2k})}.$$

Using the Gronwall-Bellman lemma, we obtain

$$\sup_{s\in[0,t]}(|x(s)|+|y(s)|)\leq \alpha\exp\left(\int_0^t l(s)ds\right).$$

Then

$$\|x\|_b+\|y\|_b\leq \alpha\exp\left(\int_0^{\infty} l(s)ds\right).$$

This implies that the solution (x,y) is bounded.

For the next result we prove the continuous dependence of solutions on initial conditions.

Theorem 3.3. *Assume that conditions $(H_1)-(H_3)$ hold, $I_k(0,0)=\bar{I}_k(0,0)$, $k=1,\dots$, $f(t,0,0)=g(t,0,0)=0$, $t\in J$ and matrix M defined in (3.1) converges to zero. For every $(x_0,y_0)\in\mathbb{R}\times\mathbb{R}$, we denote by $(x(t,x_0),y(t,y_0))$ the solution of (1.1). Then $(x_0,y_0)\to(x(\cdot,x_0),y(\cdot,y_0))$ is continuous.*

Proof. Let $(x_0,y_0),(\bar{x}_0,\bar{y}_0)\in\mathbb{R}\times\mathbb{R}$. From Theorem 3.2, we see that there exist $(x(\cdot,x_0),y(\cdot,y_0))$, $(\bar{x}(\cdot,\bar{x}_0),\bar{y}(\cdot,\bar{y}_0))\in PC_b\times PC_b$ such that

$$x(t,x_0)=x_0+\int_0^t f(s,x(s,x_0),y(s,y_0))ds+\sum_{0<t_k<t} I_k(x(t_k,x_0),y(t_k,y_0)),t\in[0,\infty),$$

$$y(t,y_0)=y_0+\int_0^t g(s,x(s,x_0),y(s,y_0))ds+\sum_{0<t_k<t}\overline{I}_k(x(t_k,x_0),y(t_k,y_0)),t\in[0,\infty),$$

$$x(t,\bar{x}_0)=\bar{x}_0+\int_0^t f(s,x(s,\bar{x}_0),y(s,\bar{y}_0))ds+\sum_{0<t_k<t}I_k(x(t_k,\bar{x}_0),y(t_k,\bar{y}_0)),t\in[0,\infty),$$

and

$$\bar{y}(t,\bar{y}_0)=\bar{y}_0+\int_0^t g(s,x(s,\bar{x}_0),y(s,\bar{y}_0))ds+\sum_{0<t_k<t}\overline{I}_k(x(t_k,\bar{x}_0),y(t_k,\bar{y}_0)),t\in[0,\infty).$$

In view of Theorem 3.2, we deduce that

$$\begin{aligned}&\|x(\cdot,x_0)-\bar{x}(\cdot,\bar{x}_0)\|_b+\|y(\cdot,y_0)-\bar{y}(\cdot,\bar{y}_0)\|_b\\ &\leq\frac{|x_0-\bar{x}_0|+|y_0-\bar{y}_0|}{1-(\sum_{k=1}^\infty a_{1k}+\sum_{k=1}^\infty a_{2k}+\sum_{k=1}^\infty b_{1k}+\sum_{k=1}^\infty b_{2k})}\exp\left(\int_0^\infty l(s)ds\right).\end{aligned}$$

Then

$$\|x(\cdot,x_0)-\bar{x}(\cdot,\bar{x}_0)\|_b+\|y(\cdot,y_0)-\bar{y}(\cdot,\bar{y}_0)\|_b\to 0, \text{ as } (x_0,y_0)\to(\bar{x}_0,\bar{y}_0).$$

4. Existence and compactness of solution sets

In this section we present an application of the Krasnoselskii type fixed point theorem to problem (1.1).

Lemma 4.1. *Let $M\subset PC_b$. Then M is relatively compact if it satisfies the following conditions:*
(a) M is uniformly bounded in $PC_\ell(\mathbb{R}^+,\mathbb{R}^n)$.
(b) The functions belonging to M are almost equicontinuous on $\mathbb{R}^+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.
(c) The functions from M are equiconvergent, that is, given $\varepsilon>0$, there corresponds $T(\varepsilon)>0$ such that $|x(\tau_1)-x(\tau_2)|<\varepsilon$ for any $\tau_1,\tau_2\geq T(\varepsilon)$ and $x\in M$.

Theorem 4.2. *Let (H_1) be satisfied and the following conditions: (H_4) There exist $\alpha_k,\beta_k\geq 0,\ k=1,\dots,$ such that*

$$|I_k(x,y)|\leq\alpha_k|x|+\beta_k|y|+c_k,\quad\forall(x,y)\in\mathbb{R}\times\mathbb{R}.$$

(H_5) There exist $\overline{\alpha_k},\overline{\beta_k}\geq 0,\ k=1,\dots,$ such that

$$|\overline{I_k}(x,y)|\leq\overline{\alpha_k}|x|+\overline{\beta_k}|y|+\overline{c_k},\ \textit{for all}\ (x,y)\in\mathbb{R}\times\mathbb{R}.$$

If

$$M_* = \begin{pmatrix} \|l_1\|_{L^1} & \|l_2\|_{L^1} \\ \|l_3\|_{L^1} & \|l_4\|_{L^1} \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{R}^+) \tag{4.1}$$

converges to zero and $\sum_{k=1}^{\infty}\alpha_k + \sum_{k=1}^{\infty}\overline{\alpha}_k + \sum_{k=1}^{\infty}\beta_k + \sum_{k=1}^{\infty}\overline{\beta}_k < 1, \sum_{k=1}^{\infty}c_k < \infty$ *and* $\sum_{k=1}^{\infty}\overline{c}_k < \infty$, *then problem (1.1) has at least one bounded solution.*

Proof. Let $N : PC_b \times PC_b \to PC_b \times PC_b$ be the operator defined in Theorem 3.1. $N = A + B$, where $A, B : PC_b \times PC_b \to PC_b \times PC_b$ are defined by

$$B(x(t),y(t)) = (B_1(x(t),y(t)),\ B_2(x(t),y(t)),\ \ t \in J,$$

where

$$\begin{cases} B_1(x,y) &= x_0 + \int_0^t f(s,x(s),y(s))ds, \\ B_2(x,y)) &= y_0 + \int_0^t g(s,x(s),y(s))ds, \end{cases}$$

and

$$A(x(t),y(t)) = (A_1(x(t),y(t)),\ A_2(x(t),y(t)),\ \ t \in J,$$

where

$$\begin{cases} A_1(x,y) &= \sum_{0<t_k<t}^{\infty} I_k(x(t_k),y(t_k)), \\ A_2(x,y)) &= \sum_{0<t_k<t}^{\infty} \overline{I}_k(x(t_k),y(t_k)). \end{cases}$$

Step 1. B is a contraction. Let $(x,y),(\overline{x},\overline{y}) \in PC_b \times PC_b$. Then

$$\begin{aligned} |B_1(x(t),y(t)) - B_1(\overline{x}t),\overline{y}(t))| &\leq \int_0^t |f(s,x(s),y(s)) - f(s,\overline{x}(s),\overline{y}(s))| \\ &\leq \int_0^t (l_1(s)|x(s)-\overline{x}(s)| + l_2(s)|y(s)-\overline{y}(s)|)\,ds. \end{aligned}$$

Hence

$$\|B_1(x,y) - B_1(\overline{x},\overline{y})\|_b \leq \|l_1\|_{L^1}\|x-\overline{x}\|_b + \|l_2\|_{L^1}\|y-\overline{y}\|_b.$$

Similarly, we have

$$\|B_2(x,y) - B_2(\overline{x},\overline{y})\|_b \leq \|l_3\|_{L^1}\|x-\overline{x}\|_b + \|l_4\|_{L^1}\|y-\overline{y}\|_b.$$

Therefore

$$\|B(x,y) - B(\overline{x},\overline{y})\|_b \leq \begin{pmatrix} \|l_1\|_{L^1} & \|l_2\|_{L^1} \\ \|l_3\|_{L^1} & \|l_4\|_{L^1} \end{pmatrix} \begin{pmatrix} \|x-\overline{x}\|_b \\ \|y-\overline{y}\|_b \end{pmatrix}.$$

Step 2. A is continuous. Given $(x_n, y_n) \to (x, y)$ in $PC_b \times PC_b$, we see that there exists $M, M^{'} > 0$ such that

$$\|x_n\|_b \leq M \text{ and } \|y_n\|_b \leq M' \text{ for every } n \in \mathbb{N},$$

and

$$|(A_1 x_n)(t) - (A_1 x)(t)| \leq \sum_{0 \leq t_k < t} |I_k(x_n, y_n) - I_k(x, y)|.$$

Since $\sum \alpha_k < \infty$, and $\sum \beta_k < \infty$, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \alpha_k < \frac{\varepsilon}{6M}, \ \sum_{k=k_0}^{\infty} \beta_k < \frac{\varepsilon}{6M'}.$$

Using the fact that $\lim_{k\to\infty} t_k = \infty$, we see that there exists $n_0 \in \mathbb{N}$, such that for each $k \geq n_0 \Rightarrow t_k \geq k_0$. From (H_3), we get

$$\begin{aligned}
& \| A_1(x_n, y_n) - A_1(x, y) \|_{PC_b} \\
& \leq \sum_{0 \leq t_k \leq t_{n_0 - 1}} |I_k(x_n, y_n) - I_k(x, y)| + \sum_{k=k_0}^{\infty} (2M\alpha_k + 2M'\beta_k) \\
& \leq \sum_{k=1}^{k_0 - 1} |I_k(x_n, y_n) - I_k(x, y)| + \frac{2\varepsilon}{3}.
\end{aligned}$$

Using the fact that I_k are continuous functions, we have

$$\sum_{k=0}^{n_0 - 1} |I_k(x_n, y_n) - I_k(x, y)| \to 0 \text{ as } n \to \infty.$$

Hence $\|A_1(x_n, y_n) - A_1(x, y)\|_b \to 0$ as $n \to \infty$. Similarly, we have $\|A_2(x_n, y_n) - A_2(x, y)\|_b \to 0$ as $n \to \infty$. It follows that $\|A(x_n, y_n) - A(x, y)\|_b \to 0$ as $n \to \infty$.

Step 3. From (H_3), we can easily prove that A maps bounded sets into bounded sets in $PC \times PC$. We next show that $A(M)$ is contained in a compact set.

Step 4. A maps bounded sets in $PC_b \times PC_b$ into almost equicontinuous sets of $PC_b \times PC_b$. Letting $r = (r_1, r_2) > 0$, $B_r := \{(x, y) \in PC_b \times PC_b : \|(x, y)\|_\infty \leq r\}$ be a bounded set in $PC \times PC$, $\tau_1, \tau_2 \in [0, \infty)$, $\tau_1 < \tau_2$, and $\phi \in B_r$, we have

$$A\phi(\tau_1) = (A_1\phi(\tau_1), A_2\phi(\tau_1)), \text{ where } \begin{cases} A_1\phi(\tau_1) & = & \sum_{0 \leq t_k \leq \tau_1} I_k(\phi_1(t_k), \phi_2(t_k)) \\ A_2\phi(\tau_1). & = & \sum_{0 \leq t_k \leq \tau_2} I_k(\phi_1(t_k), \phi_2(t_k)) \end{cases}.$$

Then

$$|A_1\phi(\tau_2) - A_1\phi(\tau_1)| \leq \sum_{\tau_1 \leq t_k \leq \tau_2} I_k(\phi_1(t_k), \phi_2(t_k)).$$

Thus $|A_1\phi(\tau_2) - A_1\phi(\tau_1)| \to 0$ as $\tau_1 \to \tau_2$. Similarly, we have $|A_2\phi(\tau_2) - A_2\phi(\tau_1)| \to 0$ as $\tau_1 \to \tau_2$. It follows that $|A\phi(\tau_1) - A\phi(\tau_2)| \to 0$ as $\tau_1 \to \tau_2$.

Step 5. We now show that the set $A(\overline{B}(0,r))$ is equiconvergent, i.e., for every $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that $\|A(\phi(t)) - A(\phi(s))\| \leq \varepsilon$ for every $t,s \geq T(\varepsilon)$ and each $\phi \in \overline{B}(0,r)$. Letting $\phi \in \overline{B}(0,r)$, for every $\varepsilon > 0$, we see that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \alpha_k < \frac{\varepsilon}{2r_1}, \sum_{k=k_0}^{\infty} \beta_k < \frac{\varepsilon}{2r_2},$$

$$\begin{aligned} |A_1\phi(t) - A_1\phi(s)| &\leq \sum_{s \leq t_k \leq t} I_k(\phi_1(t_k), \phi_2(t_k)) \\ &\leq \sum_{s \leq t_k \leq t} (\alpha_k r_1 + \beta_k r_2). \end{aligned}$$

Then, for every $s,t > k_0$, we get

$$|A_1\phi(t) - A_1\phi(s)| \leq r_1 \sum_{k=k_0}^{\infty} \alpha_k + r_2 \sum_{k=k_0}^{\infty} \beta_k.$$

Therefore for all $\phi \in B(0,r)$ and $s,t > k_0$ we have $|A_1\phi(t) - A_1\phi(s)| \leq \varepsilon$. Similarly ,we can prove that there exists $\bar{k}_0 > 0$ such that for all $\phi \in B(0,r)$ and $s,t > k_0$ we have $|A_2\phi(t) - A_2\phi(s)| \leq \varepsilon$. Thus, for every $(\varepsilon, \varepsilon) > 0$ there exists $(k_0, \bar{k}_0) > 0$ such that for all $s,t > k_0$ and $s,t > \bar{k}_0$ we have

$$|A\phi(\tau_1) - A\phi(\tau_2)| \leq (\varepsilon, \varepsilon), \quad \forall \phi \in B(0,r).$$

Step 6. Now, we show that set

$$\mathscr{M} = \{(x,y) \in PC \times PC; (x,y) = \lambda B(\frac{x}{\lambda}, \frac{y}{\lambda}) + \lambda A(x,y), \lambda \in (0,1)\}$$

is bounded. Letting $(x,y) \in \mathscr{M}$, one has

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t \lambda \left| f\left(s, \frac{x(s)}{\lambda}, \frac{y(s)}{\lambda}\right) \right| ds + \sum_{0 < t_k < t} |I_k(x(t_k), y(t_k))| \\ &\leq |x_0| + \int_0^t (l_1(s)|x(s)| + l_2(s)|y(s)|) ds + \sum_{k=1}^{\infty} \alpha_k |x(t_k)| + \sum_{k=1}^{\infty} \beta_k |y(t_k)| + \sum_{k=1}^{\infty} c_k, \end{aligned}$$

$$|y(t)| \leq |y_0| + \int_0^t (l_3(s)|x(s)| + l_4(s)|y(s)|) ds + \sum_{k=1}^{\infty} \overline{\alpha}_k |x(t_k)| + \sum_{k=1}^{\infty} \overline{\beta}_k |y(t_k)| + \sum_{k=1}^{\infty} \overline{c}_k.$$

Thus

$$\begin{aligned}|x(t)|+|y(t)| &\leq |x_0|+|y_0|+\int_0^t ((l_1(s)+l_3(s))|x(s)|+(l(s)_2+l(s)_4)|y(s)|)ds \\ &+ (\sum_{k=1}^{\infty}\alpha_k+\sum_{k=1}^{\infty}\overline{\alpha}_k)|x(t_k|+(\sum_{k=1}^{\infty}\beta_k+\sum_{k=1}^{\infty}\overline{\beta}_k)|y(t_k|+\sum_{k=1}^{\infty}c_k+\sum_{k=1}^{\infty}\overline{c}_k.\end{aligned}$$

Hence

$$\begin{aligned}&\sup_{s\in[0,t]}(|x(s)|+|y(s)|) \\ &\leq |x_0|+|y_0|+\int_0^t (l_1(s)+l_3(s)+l_2(s)+l_4(s))\times \sup_{s\in[0,t]}(|x(s)|+|y(s)|)ds \\ &+ (\sum_{k=1}^{\infty}\alpha_k+\sum_{k=1}^{\infty}\overline{\alpha}_k)\sup_{s\in(0,t)}|x(t_k|+(\sum_{k=1}^{\infty}\beta_k+\sum_{k=1}^{\infty}\overline{\beta}_k)\sup_{s\in(0,t)}|y(t_k|+\sum_{k=1}^{\infty}c_k+\sum_{k=1}^{\infty}\overline{c}_k.\end{aligned}$$

This implies that

$$\sup_{s\in(0,t)}(|x(s)|+|y(s)|) \leq \beta+\int_0^t l_*(s)\sup_{s\in[0,t]}(|x(s)|+|y(s)|)ds,$$

where

$$\beta=\frac{|x_0|+|y_0|+\sum_{k=1}^{\infty}c_k+\sum_{k=1}^{\infty}\overline{c}_k}{1-(\sum_{k=1}^{\infty}\alpha_k+\sum_{k=1}^{\infty}\overline{\alpha}_k+\sum_{k=1}^{\infty}\beta_k+\sum_{k=1}^{\infty}\overline{\beta}_k)},$$

and

$$l_*(s)=\frac{l_1(s)+l_3(s)+l_2(s)+l_4(s)}{1-(\sum_{k=1}^{\infty}\alpha_k+\sum_{k=1}^{\infty}\overline{\alpha}_k+\sum_{k=1}^{\infty}\beta_k+\sum_{k=1}^{\infty}\overline{\beta}_k)}.$$

By the Gronwall-Bellman lemma, we have

$$\sup_{s\in(0,t)}(|x(s)|+|y(s)|)\leq \beta e^{\|l_*\|_{L^1}}.$$

Then $\|x\|_b\leq \beta e^{\|l_*\|_{L^1}}$ and $\|y\|_b\leq \beta e^{\|l_*\|_{L^1}}$. In view of Theorem 2.7, we see that problem (1.1) has at least one solution.

By simple modification in the prove we can obtain the following result.

Theorem 4.3. *Let (H_2) be satisfied and the following condition:*
(H_6) There exists $p\in L^1(J,\mathbb{R}_+)$, and let $\psi:\mathbb{R}_+\to(0,\infty)$ be a continuous nondecreasing function such that

$$|f(t,x,y)|\leq p(t)\psi(|x|+|y|),\ \textit{for all}\ (x,y)\in\mathbb{R}\times\mathbb{R}$$

and

$$|g(t,x,y)| \leq p(t)\psi(|x|+|y|), \text{ for all } (x,y) \in \mathbb{R}\times\mathbb{R}.$$

If

$$\bar{M}_* = \begin{pmatrix} \sum_{k=1}^{\infty} a_{1k} & \sum_{k=1}^{\infty} a_{2k} \\ \sum_{k=1}^{\infty} b_{1k} & \sum_{k=1}^{\infty} b_{2k} \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{R}^+) \tag{4.2}$$

converges to zero, then problem (1.1) has unique bounded solution.

Using the nonlinear alternative in generalized Banach spaces, we can also obtain the following result.

Theorem 4.4. *Assume that $(H_4)-(H_6)$ hold. If*

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \overline{\alpha}_k + \sum_{k=1}^{\infty} \beta_k + \sum_{k=1}^{\infty} \overline{\beta}_k < \infty, \sum_{k=1}^{\infty} c_k < \infty \text{ and } \sum_{k=1}^{\infty} \overline{c}_k < \infty,$$

then problem (1.1) has at least one solution. Moreover, the solution set

$$S(x_0,y_0) = \{(x,y) \in PC_b \times PC_b : (x,y) \text{ is solution of } (1.1)\}$$

is compact and $S : (x_0,y_0) \to S(x_0,y_0)$ is u.s.c.

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