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SUCCESSIVE APPROXIMATIONS FOR FUNCTIONAL EVOLUTION EQUATIONS AND INCLUSIONS

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Abstract. This paper deals with the global convergence of successive approximations as well as the uniqueness of solutions for some classes of partial functional evolution equations and inclusions. We prove a theorem on the global convergence of successive approximations to the unique solution of the problems.

Keywords. Functional evolution equation; Functional evolution inclusion; Generalized solution; Global convergence; Successive approximation.

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1. Introduction

Recently, several authors have considered extensively the problem

$$x'(t) = A(t)x(t) + f(t, x_t), \tag{2.1}$$

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when $A(t) = A$. The Existence of mild solutions is developed by Heikkila and Lakshmikantham [1], Kamenski *et al.* [2] and Hino and Murakami [3] for some semi-linear functional differential equations with the finite delay. Recently, there has been a significant development in functional evolution equations and inclusions; see Abbas and Benchohra [4], Baghli and Benchohra [5, 6], Baliki and Benchohra [7, 8], Benchohra, Henderson and Medjadj [9], Benchohra and Medjadj [10] and the references therein. The convergence of successive approximations for ordinary functional differential equations as well as for integral functional equations is a well established property. It has been studied by De Blasi and Myjak [11], Chen [12], Faina [13], Shin [14] and the references therein.

In this article, we discuss the global convergence of successive approximations for the evolution equation

$$u'(t) = A(t)u(t) + f(t, u(t)); \text{ if } t \in I := [0, T], \tag{2.2}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{2.3}$$

where $T > 0$, $f : I \times E \to E$ is a given function, $(E, \|\cdot\|_E)$ is a (real or complex) Banach space, and $\{A(t)\}_{t>0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of bounded linear operators $\{U(t,s)\}_{(t,s)\in I\times I}$ for $0 \le s \le t \le T$ from E into E.

Next, we discuss the global convergence of successive approximations for the partial evolution inclusion

$$u'(t) \in A(t)u(t) + F(t, u(t)), \text{ if } t \in I, \tag{2.4}$$

with the initial condition (2.3), where $F : I \times E \to \mathscr{P}(E)$ is a compact valued multi-valued map, and $\mathscr{P}(E)$ is the family of all nonempty subsets of the Banach space E.

This paper initiates the convergence of successive approximations for functional evolution equations and inclusions. The organization is as follows. In Section 2, some preliminary results are introduced. The main results and an illustrative example are presented in Section 3. In Section 4, a conclusion is provided.

2. Preliminaries

Let $B(E)$ be the Banach space of all bounded linear operators from E into E with the norm

$$\|N\|_{B(E)} = \sup_{\|u\|=1} \|N(u)\|_E.$$

A measurable function $u : I \to E$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. For properties of the Bochner integral, see, for instance, Yosida [15]. As usual, by $L^1(I,E)$ we denote the Banach space of measurable functions $u : I \to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^T \|u(t)\|_E dt,$$

and denote by $L^\infty(I)$ the Banach space of measurable functions $u : I \to E$ which are essentially bounded with the norm defined by

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(t)\|_E \leq c,\ a.e.\ t \in I\}.$$

As usual, by $\mathscr{C} := C(I)$ we denote the Banach space of all continuous functions from I into E with norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{t\in I} \|u(t)\|_E.$$

Definition 2.1. [16] The function $f : I \times E \to E$ is said to be Carathéodory if

(i) $t \longmapsto f(t,u)$ is measurable for each $u \in E$;

(ii) $u \longmapsto f(t,u)$ is continuous for almost all $t \in I$.

The function f is said to be L^∞-Carathéodory if $(i), (ii)$ and the following condition holds;

(iii) for every positive integer k, there exists a function $h_k \in L^\infty(I,\mathbb{R}_+)$ such that

$$\|f(t,u)\|_E \leq h_k(t) \quad for\ all\ \|u\|_E \leq k \quad and\ almost\ each\ t \in I.$$

Let (X,d) be a metric space. We use the following notations:

$$\mathscr{P}_{bd}(X) = \{Y \in \mathscr{P}(X) : Y\ bounded\},\ \mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : Y\ closed\},$$

$$\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y\ compact\},\ \ and\ \ \mathscr{P}_{cv}(X) = \{Y \in \mathscr{P}(X) : Y\ convex\}.$$

A multivalued map $G : X \to \mathscr{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded* on bounded sets if $G(B)$ is bounded in X for each bounded set B of X, i.e.,

$$\sup_{x\in B}\{\sup\{\ \|u\|_{L^\infty} :\ u \in G(x)\}\} < \infty.$$

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. For each $u \in \mathscr{C}$, define the set of selectors $S_{F\circ u}$ by

$$S_{F\circ u} = \{v \in L^\infty(J) : v(t) \in F(t,u(t)) \ , \ a.e. \ t \in J\}.$$

For more details on multivalued maps, we refer to the books of Deimling [17] and Górniewicz [18].

Consider $H_d : \mathscr{P}(X) \times \mathscr{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(\mathscr{A},\mathscr{B}) = \max\{\sup_{a\in\mathscr{A}} d(a,\mathscr{B}), \sup_{b\in\mathscr{B}} d(\mathscr{A},b)\},$$

where $d(\mathscr{A},b) = \inf_{a\in\mathscr{A}} d(a,b)$, $d(a,\mathscr{B}) = \inf_{b\in\mathscr{B}} d(a,b)$. Then $(\mathscr{P}_{bd,cl}(X),H_d)$ is a metric space and $(\mathscr{P}_{cl}(X),H_d)$ is a generalized (complete) metric space; see [19] and the references therein.

Definition 2.2. [18] A multivalued map $F : I \times E \to \mathscr{P}(E)$ is said to be Carathéodory if

(i) $t \longmapsto F(t,u)$ is measurable for each $u \in E$,
(ii) $u \longmapsto F(t,u)$ is upper semicontinuous for almost all $t \in I$.

The multi-function F is said to be L^∞-Carathéodory if $(i),(ii)$ and the following condition holds;

(iii) for each $c > 0$, there exists $\sigma_c \in L^\infty(I,\mathbb{R}_+)$ such that

$$\begin{aligned} \|F(t,u)\|_{\mathscr{P}} &= \sup\{\|f\|_{L^\infty} : f \in F(t,u)\} \\ &\leq \sigma_c(t) \ \text{ for all } \|u\|_E \leq c \text{ and for } a.e. \ t \in I. \end{aligned}$$

In what follows, for the family $\{A(t),\ t \geq 0\}$ of closed densely defined linear unbounded operators on the Banach space E, we assume that it satisfies the following assumptions (see [20], p. 158).

(P_1) The domain $D(A(t))$ is independent of t and is dense in E.
(P_2) For $t \geq 0$, the resolvent $R(\lambda,A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant K independent of λ and t such that

$$\|R(t,A(t))\| \leq K(1+|\lambda|)^{-1}, \text{ for } Re\lambda \leq 0.$$

(P_3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\|(A(t) - A(\theta))A^{-1}(\tau)\| \leq L|t - \tau|^{\alpha}, \text{ for } t, \theta, \tau \in I.$$

Lemma 2.3. ([20], p. 159) *Under assumptions $(P_1) - (P_3)$, the Cauchy problem*

$$u'(t) - A(t)u(t) = 0, \ t \in I, \ u(0) = y_0,$$

has a unique evolution system $U(t,s)$, $(t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t \leq T\}$ satisfying the following properties:

(1) $U(t,t) = I$ where I is the identity operator in E,

(2) $U(t,s)\, U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t \leq T$,

(3) $U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(t,s) \in \Delta$ and for each $u \in E$, the mapping $(t,s) \to U(t,s)u$ is continuous.

More details on evolution systems and their properties can be found in the books of Ahmed [20] and Pazy [21].

3. Successive approximations and uniqueness results

In this section, we present the main results for the global convergence of successive approximations to a unique solution of our problems. First, let us introduce the definition of the mild solution of the problem (2.2)-(2.3).

Definition 3.1. We say that a continuous function $u(\cdot) : I \to E$ is a mild solution of problem (2.2)-(2.3), if u satisfies the following integral equation

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)\, f(s,u(s))ds, \quad \text{for each } t \in I.$$

Define the successive approximations of problem (2.2)-(2.3) as follows:

$$u^{(0)}(t) = U(t,0)u_0; \ t \in I,$$

$$u^{(n+1)}(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s,u^{(n)}(s))ds, \quad \text{for each } t \in I.$$

Set $I_\sigma := [0, \sigma T]$ for any $\sigma \in [0,1]$. Let us introduce the following hypotheses.

(H_1) The evolution system is uniformly continuous and there exists a constant $M \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq M \text{ for every } (t,s) \in \Delta.$$

(H_2) The function $f : I \times E \to E$ is L^∞-Carathéodory,

(H_3) There exist a constant $\rho > 0$ and a Carathéodory function $w : I \times [0,2\rho] \to [0,\infty)$ such that $w(t,\cdot)$ is nondecreasing for almost all $t \in I$, and the inequality

$$\|f(t,u) - f(t,\overline{u})\|_E \leq w(t, \|u - \overline{u}\|_E) \tag{3.1}$$

holds for all $t \in I$ and $u, \overline{u} \in E$ such that $\|u - \overline{u}\|_E \leq 2\rho$.

(H_4) $v \equiv 0$ is the only function in $\mathscr{C}(I_\lambda, [0,2\rho])$ satisfying the integral inequality

$$v(t) \leq M \int_0^t w(s, v(s))ds, \tag{3.2}$$

with $\sigma \leq \lambda \leq 1$.

Theorem 3.2. *Assume that hypotheses $(H_1) - (H_4)$ are satisfied. Then the successive approximations $u^{(n)}$; $n \in \mathbb{N}$ are well defined and converge to the unique mild solution of problem (2.2)-(2.3) uniformly on I.*

Proof. From (H_2), the successive approximations are well defined. Furthermore, the sequences $\{u^{(n)}(t);\ n \in \mathbb{N}\}$ is equi-continuous on I. Indeed, for every positive integer k, there exists a function $h_k \in L^\infty(I, \mathbb{R}_+)$ such that for each $t_1,\ t_2 \in J$ with $t_1 < t_2$, and for all $t \in I$,

$$\begin{aligned}
\|u^{(n)}(t_2) - u^{(n)}(t_1)\|_E &\leq \|U(t_2,0) - U(t_1,0)\|_{B(E)}\|u_0\|_E \\
&\quad + \left\| \int_0^{t_1} (U(t_2,s) - U(t_1,s)) f(s, u^{(n-1)}(s))ds \right\|_E \\
&\quad + \left\| \int_{t_1}^{t_2} U(t_2,s) f(s, u^{(n-1)}(s))ds \right\|_E \\
&\leq \|U(t_2,0) - U(t_1,0)\|_{B(E)}\|u_0\|_E \\
&\quad + \|h_k\|_{L^\infty} \int_0^{t_1} \|U(t_2,s) - U(t_1,s)\|_{B(E)} ds + M\|h_k\|_{L^\infty}(t_2 - t_1) \\
&\longrightarrow 0, \ \text{as } t_1 \to t_2.
\end{aligned}$$

Let

$$\tau := \sup\{\sigma \in [0,1] : \{u^n(t)\} \text{ converges uniformly on } I_\sigma\}.$$

If $\tau = 1$, then we have the global convergence of successive approximations. Suppose $\tau < 1$. Then $\{u^{(n)}(t)\}$ converges uniformly on I_τ. Since this sequence is equi-continuous, one sees that it converges uniformly to a continuous function $\tilde{u}(t)$. If we prove that there exists $\lambda \in (\tau, 1]$ such that $\{u^{(n)}(t)\}$ converges uniformly on I_λ, then it yields a contradiction. Put $u(t) = \tilde{u}(t)$ for $t \in I_\tau$. From (H_3), one sees that there exist a constant $\rho > 0$ and a Carathéodory function $w : I \times [0, 2\rho] \to [0, \infty)$ satisfying inequality (3.1). Also, there exist $\lambda \in [\tau, 1]$ and $n_0 \in \mathbb{N}$ such that, for all $t \in I_\lambda$ and $n, m > n_0$,

$$\|u^{(n)}(t) - u^{(m)}(t)\|_E \le 2\rho.$$

For any $t \in I_\lambda$, we put

$$v^{(n,m)}(t) = \|u^{(n)}(t) - u^{(m)}(t)\|_E,$$

$$v^{(k)}(t) = \sup_{n,m \ge k} v^{(n,m)}(t).$$

Since $v^{(k)}(t)$ is non-increasing, one sees that it is convergent to a function $v(t)$ for each $t \in I_\lambda$. From the equi-continuity of $\{v^{(k)}(t)\}$, it follows that $\lim_{k\to\infty} v^{(k)}(t) = v(t)$ uniformly on I_λ. Furthermore, for $t \in I_\lambda$ and $n, m \ge k$, we have

$$\begin{aligned}
v^{(n,m)}(t) &= \|u^{(n)}(t) - u^{(m)}(t)\|_E \\
&\le \sup_{s \in [0,t]} \|u^{(n)}(s) - u^{(m)}(s)\|_E \\
&\le \int_0^t \|U(t,s)\|_{B(E)} \|f(s, u^{(n-1)}(s)) - f(s, u^{(m-1)}(s))\|_E ds \\
&\le M \int_0^t \|f(s, u^{(n-1)}(s)) - f(s, u^{(m-1)}(s))\|_E ds.
\end{aligned}$$

Using (3.1), we get

$$\begin{aligned}
v^{(n,m)}(t) &\le M \int_0^t w(s, \|u^{(n-1)}(s) - u^{(m-1)}(s)\|_E) ds \\
&= M \int_0^t w(s, v^{(n-1,m-1)}(s)) ds.
\end{aligned}$$

Hence

$$v^{(k)}(t) \le M \int_0^t w(s, v^{(k-1)}(s)) ds.$$

Using the Lebesgue dominated convergence theorem, we get

$$v(t) \le M \int_0^t w(s, v(s)) ds.$$

It follows from (H_2) and (H_4) that $v \equiv 0$ on I_λ. This yields that $\lim_{k\to\infty} v^{(k)}(t) = 0$ uniformly on I_λ. Thus $\{u^{(k)}(t)\}_{k=1}^{\infty}$ is a Cauchy sequence on I_λ. Consequently $\{u^{(k)}(t)\}_{k=1}^{\infty}$ is uniformly convergent on I_λ which yields the contradiction. Thus $\{u^{(k)}(t)\}_{k=1}^{\infty}$ converges uniformly on I to a continuous function $u^*(t)$. Using the Carathéodory condition (iii) and the Lebesgue dominated convergence theorem, we get

$$\lim_{k\to\infty}\int_0^t U(t,s)f(s,u^{(k)})(s)ds = \int_0^t U(t,s)f(s,u^*(s))ds,$$

for each $t \in I$. This yields that u^* is a mild solution of the problem (2.2)-(2.3).

Finally, we show the uniqueness of mild solutions of problem (2.2)-(2.3). Let u_1 and u_2 be two mild solutions of (2.2)-(2.3). As above, we put

$$\tau := \sup\{\sigma \in [0,1] : u_1(t) = u_2(t)\ for\ t \in I_\sigma\},$$

and suppose that $\tau < 1$. There exist a constant $\rho > 0$ and a comparison function $w : I_\tau \times [0,2\rho] \to [0,\infty)$ satisfying inequality (3.1). We choose $\lambda \in (\sigma,1)$ such that

$$\|u_1(t) - u_2(t)\|_E \le 2\rho,\ for\ t \in I_\lambda.$$

Then, for all $t \in I_\lambda$, we obtain

$$\begin{aligned}\|u_1(t) - u_2(t)\|_E &\le \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u_1(s)) - f(s,u_2(s))\|_E dtds\\ &\le M\int_0^t w(s,\|u_1(s) - u_2(s)\|_E)ds.\end{aligned}$$

Again, by (H_2) and (H_4), we get $u_1 - u_2 \equiv 0$ on I_λ. This gives $u_1 = u_2$ on I_λ, which yields a contradiction. Consequently, $\tau = 1$ and the mild solution of problem (2.2)-(2.3) is unique on I.

Now, we present the main result for the global convergence of successive approximations to a unique mild solution of problem (2.4)-(2.3).

Definition 3.3. A continuous function $u \in \mathscr{C}$ is a mild solution of problem (2.4)-(2.3), if there exists $f \in S_{F\circ u}$, such that u satisfies the following integral equation

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds.$$

Define the successive approximations of problem (2.4)-(2.3) as follows:

$$u^{(0)}(t) = U(t,0)u_0;\ t \in I,$$

$$u^{(n+1)}(t) = U(t,0)u_0 + \int_0^T U(t,s)f_n(s)ds;\ t \in I,$$

where $f_n \in S_{F\circ u^{(n)}}$ with $\|f_n\|_{L^\infty} = \|F(t,u^{(n)})\|_{\mathscr{P}}$. Set $I_\sigma := [0,\sigma T]$, for any $\sigma \in [0,1]$. Let us introduce the following hypotheses.

(H_1') The multifunction $F : I \times E \to \mathscr{P}(E)$ is $L^\infty-$Carathéodory.

(H_2') There exist a constant $\rho > 0$ and a Carathéodory function $w : I \times [0,\rho] \to [0,\infty)$ such that $w(t,\cdot)$ is nondecreasing for almost all $t \in I$, and the inequality

$$H_d(F(t,u),F(t,\overline{u})) \le w(t,\|u-\overline{u}\|_E) \tag{3.3}$$

holds for all $t \in I$ and $u,\overline{u} \in E$ such that $\|u-\overline{u}\|_E \le \rho$.

(H_3') $v \equiv 0$ is the only function in $\mathscr{C}(J_\lambda,[0,\rho])$ satisfying the integral inequality

$$v(t) \le M\int_0^t w(s,v(s))ds, \tag{3.4}$$

with $\sigma \le \lambda \le 1$.

Theorem 3.4. *Assume that hypotheses (H_1), $(H_1')-(H_3')$ are satisfied. Then the successive approximations $u^{(n)}$; $n \in \mathbb{N}$ are well defined and converge to the unique mild solution of problem (2.4)-(2.3) uniformly on I.*

Proof. From (H_1'), the successive approximations are well defined. Furthermore, $\{u^{(n)}(t);\ n \in \mathscr{N}\}$ is equi-continuous on I. Let

$$\tau := \sup\{\sigma \in [0,1] : \{u^{(n)}(t)\} \text{ converges uniformly on } I_\sigma\}.$$

If $\tau = 1$, then we have the global convergence of successive approximations. Suppose $\tau < 1$. Then $\{u^{(n)}(t)\}$ converges uniformly on I_τ. Since this sequence is equi-continuous, one sees that it converges uniformly to a continuous function $\tilde{u}(t)$. If we prove that there exists $\lambda \in (\tau,1]$ such that $\{u^{(n)}(t)\}$ converges uniformly on I_λ. This yields a contradiction. Put $u(t) = \tilde{u}(t)$, for $(t) \in I_\tau$. From (H_2'), there exist a constant $\rho > 0$ and a Carathéodory function $w : I \times [0,\rho] \to [0,\infty)$ satisfying inequality (3.3). Also, there exist $\lambda \in [\tau,1]$ and $n_0 \in \mathbb{N}$ such that, for all $t \in I_\lambda$ and $n,m > n_0$,

$$\|u^{(n)}(t) - u^{(m)}(t)\|_E \le \rho.$$

For any $t \in I_\lambda$, we put

$$v^{(n,m)}(t) = \|u^{(n)}(t) - u^{(m)}(t)\|_E,$$

$$v^{(k)}(t) = \sup_{n,m \geq k} v^{(n,m)}(t).$$

Since $v^{(k)}(t)$ is non-increasing, one finds that it is convergent to a function $v(t)$ for each $t \in I_\lambda$. From the equi-continuity of $\{v^{(k)}(t)\}$, it follows that $\lim_{k\to\infty} v^{(k)}(t) = v(t)$ uniformly on I_λ. Furthermore, for $t \in I_\lambda$ and $n,m \geq k$, there exist $f_{n-1} \in S_{F\circ u^{(n-1)}}$ and $f_{m-1} \in S_{F\circ u^{(m-1)}}$, with $\|f_{n-1}\|_{L^\infty} = \|F(t,u^{(n-1)})\|_{\mathscr{P}}$ and $\|f_{m-1}\|_{L^\infty} = \|F(t,u^{(m-1)})\|_{\mathscr{P}}$, such that

$$\begin{aligned}
v^{(n,m)}(t) &= \|u^{(n)}(t) - u^{(m)}(t)\|_E \\
&\leq \sup_{s\in[0,t]} \|u^{(n)}(s) - u^{(m)}(s)\|_E \\
&\leq \int_0^t \|U(t,s\|_{B(E)} \|f_{n-1}(s) - f_{m-1}(s)\|_E ds \\
&\leq M \int_0^t H_d(F(s,u^{(n-1)}(s)), F(s,u^{(m-1)}(s))ds.
\end{aligned}$$

It follows from (3.3) that

$$\begin{aligned}
v^{(n,m)}(t) &\leq M \int_0^t w(s, \|u^{(n-1)}(s) - u^{(m-1)}(s)\|_E)ds \\
&= M \int_0^t w(s, v^{(n-1,m-1)}(s))ds.
\end{aligned}$$

Hence

$$v^{(k)}(t) \leq M \int_0^t w(s, v^{(k-1)}(s))ds.$$

Using the Lebesgue dominated convergence theorem, we get

$$v(t) \leq M \int_0^t w(s, v(s))ds.$$

It follows from by (H_1') and (H_3') that $v \equiv 0$ on J_λ, which yields $\lim_{k\to\infty} v^{(k)}(t) = 0$ uniformly on I_λ. Thus $\{u^{(k)}(t)\}_{k=1}^\infty$ is a Cauchy sequence on I_λ. Consequently $\{u^{(k)}(t)\}_{k=1}^\infty$ is uniformly convergent on I_λ which yields the contradiction. Thus $\{u^{(k)}(t)\}_{k=1}^\infty$ converges uniformly on I to a continuous function $u^*(t)$. Using (H_1') and the Lebesgue dominated convergence theorem, we get, for each $t \in I$,

$$\lim_{k\to\infty} \int_0^t U(t,s) f_k(s)ds = \int_0^t U(t,s) f_*(s)ds,$$

where $f_k \in S_{F\circ u_k}$ and $f_* \in S_{F\circ u_*}$, with $\|f_k\|_{L^\infty} = \|F(t,u_k)\|_{\mathscr{P}}$ and $\|f_*\|_{L^\infty} = \|F(t,u_*)\|_{\mathscr{P}}$. This yields that u^* is a mild solution of problem (2.4)-(2.3).

Finally, we show the uniqueness of the mild solution of problem (2.4)-(2.3). Let u_1 and u_2 be two solutions of (2.4)-(2.3). As above, put

$$\tau := \sup\{\sigma \in [0,1] : u_1(t) = u_2(t) \; for \; t \in I_\sigma\},$$

and suppose $\tau < 1$. There exist a constant $\rho > 0$ and a comparison function $w : I_\tau \times [0,\rho] \to [0,\infty)$ satisfying inequality (3.2). We choose $\lambda \in (\sigma, 1)$ such that

$$\|u_1(t) - u_2(t)\|_E \le \rho; \; for \; t \in I_\lambda.$$

Then, for all $t \in I_\lambda$, we obtain

$$\begin{aligned}\|u_1(t) - u_2(t)\|_E &\le \|U(t,s)\|_{B(E)} H_d(F(s,u_1(s)), F(s,u_2(s)))ds \\ &\le M \int_0^t w(s, \|u_1(s) - u_2(s)\|_E) ds.\end{aligned}$$

Again, by (H_1') and (H_3'), we get $u_1 - u_2 \equiv 0$ on I_λ. This gives $u_1 = u_2$ on I_λ, which yields a contradiction. Consequently, $\tau = 1$ and the mild solution of the problem (2.4)-(2.3) is unique on I.

As an application of the main results, we consider the following functional evolution problem.

Example 3.5.

$$\begin{cases} \frac{\partial z}{\partial t}(t,x) = a(t,x)\frac{\partial^2 z}{\partial x^2}(t,x) + Q(t,z(t,x)), t \in [0,1], & 7x \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0, & t \in [0,1], \\ z(0,x) = \Phi(x) & x \in [0,\pi], \end{cases} \tag{3.5}$$

where $a(t,x) : [0,1] \times [0,\pi] \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t, $Q : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions. Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,x)w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E, \; w(0) = w(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ (see [22]). For $x \in [0,\pi]$, we have

$$y(t)(x) = z(t,x) \quad t \in [0,1],$$

$$f(t,u(t),x) = Q(t,z(t,x)) \quad t \in [0,1],$$

and

$$u_0(x) = \Phi(x) \quad x \in [0, \pi].$$

Under the above definitions of f, u_0 *and* $A(\cdot)$, *system (3.5) can be represented by functional evolution problem (2.2)-(2.3). Furthermore, more appropriate conditions on* Q *ensure the hypotheses* $(H_1) - (H_4)$. *Consequently, Theorem (3.2) implies that the successive approximations* $u^{(n)}$; $n \in \mathbb{N}$, *defined by*

$$u^{(0)}(t)(x) = \Phi(x); \; x \in [0, \pi],$$

$$u^{(n+1)}(t)(x) = u^{(0)}(t)(x) + \int_0^t U(t,s) f(s, u^{(n)}(s), x) ds; \; x \in [0, \pi],$$

converge to the unique solution of problem (3.5) uniformly on $[0, 1]$.

5. Conclusions

In this paper, we provided some sufficient conditions ensuring the global convergence of successive approximations as well as the uniqueness of solutions for some classes of partial functional evolution equations and inclusions. Such successive approximations are useful, for instance, when it is difficult to find an exact solution like in numerical analyse or optimization.

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