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EXPLICIT AND IMPLICIT ITERATIVE ALGORITHMS FOR STRICT PSEUDO-CONTRACTIONS IN BANACH SPACES

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Abstract. In this paper, we consider explicit and implicit iterative algorithms for finding fixed points of strict pseudo-contractions in a Banach space. Strong convergence of the algorithms is obtained and an example is also provided. The results presented in this article gives a positive answer to the question of Marino Scardamaglia and Karapinar raised in 2016.

Keywords. Banach space; Mann's algorithm; Normalized duality mapping; Strong convergence; Strict pseudo-contraction.

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1. Introduction

Let E be a real Banach space and let C be a nonempty subset of E. Let E^* be the dual space of E and let J denote the normalized duality mapping. Let $T : C \to C$ be a mapping. We denote the fixed point set of T by $Fix(T)$, that is $Fix(T) = \{x \in C : x = Tx\}$. Mapping $T : C \to C$ is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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$T:C\to C$ is said to be κ-strictly pseudo-contractive iff there exists a constant $\kappa\in(0,1)$ such that

$$\langle Tx-Ty, j(x-y)\rangle \le \|x-y\|^2-\kappa\|(I-T)x-(I-T)y\|^2,\quad \forall x,y\in C$$

for some $j(x-y)\in J(x-y)$.

$T:C\to C$ is said to be pseudo-contractive iff

$$\langle Tx-Ty, j(x-y)\rangle \le \|x-y\|^2,\quad \forall x,y\in C$$

for some $j(x-y)\in J(x-y)$;

$T:C\to C$ is said to be strongly pseudo-contractive iff there exists a constant $\kappa\in(0,1)$ such that

$$\langle Tx-Ty, j(x-y)\rangle \le \kappa\|x-y\|^2,\quad \forall x,y\in C$$

for some $j(x-y)\in J(x-y)$.

Remark 1.1. If $T:C\to C$ is a strict pseudo-contraction, then T is Lipschitz continuous and $Fix(T)$ is closed and convex; see [1, 2, 3] and the references therein.

Remark 1.2. The conception of strict pseudo-contraction was introduced by Browder and Petryshyn [4] in a real Hilbert space in 1967. Let H be a real Hilbert space and C be a nonempty subset of H. A mapping $S:C\to C$ is said to be a κ-strict pseudo-contraction iff there exists a $\kappa\in[0,1]$ such that

$$\|Sx-Sy\|^2 \le \|x-y\|+\kappa\|(I-S)x-(I-S)y\|^2,\quad \forall x,y\in C. \tag{1.1}$$

Example 1.3. Let $H=l^2$ and $C=\{(x_1,x_2,\cdots,x_n,\cdots): x_i\ge 0,\ \forall\, i\in\mathbb{N} \text{ and } \sum_{i=1}^{\infty}x_i^2<\infty\}$. Define a mapping $S:C\to C$ by $Sx=(\frac{x_1}{2},-3x_2,-3x_3,\cdots,-3x_n,\cdots)$ for all $x=(x_1,x_2,x_3,\cdots,x_n,\cdots)$ $\in C$. It is easy to see S satisfies (1.1) with $k=\frac{1}{2}$. In fact, for each $x=(x_1,x_2,\cdots,x_n,\cdots), y=(y_1,y_2\cdots,y_n,\cdots)\in C$, we have

$$\begin{aligned}
\|Sx-Sy\|^2 &= \|(\frac{x_1-y_1}{2},-3(x_2-y_2),\cdots,-3(x_n-y_n),\cdots)\|^2\\
&= \frac{(x_1-y_1)^2}{4}+9\sum_{i=2}^{\infty}(x_i-y_i)^2\\
&\le \|x-y\|^2+\frac{1}{2}\|(I-S)x-(I-S)y\|^2\\
&= \frac{9(x_1-y_1)^2}{8}+9\sum_{i=2}^{\infty}(x_i-y_i)^2.
\end{aligned}$$

Hence S is a $\frac{1}{2}$-strict pseudo-contraction. However, it is not a strong pseudo-contraction. Indeed, the class of strongly pseudo-contractive mappings is independent of the class of κ-strict pseudo-contractions; see [3] and the references therein.

Theorem MX. [5] *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a κ-strict pseudo-contraction for some $0 \leq \kappa < 1$. Assume that $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \ \ n \in \mathbb{N}. \tag{1.2}$$

Assume that $\{\alpha_n\} \in (\kappa, 1)$ satisfies $\sum_{n=1}^{\infty}(\alpha_n - \kappa)(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.

In an infinite-dimensional Hilbert space, the normal Mann's iteration algorithm (1.2) has only weak convergence for strict pseudo-contractions (even for nonexpansive mappings). In order to get a strong convergence result, one has to modify the normal Mann's iteration algorithm; see [3, 6, 7, 8, 9, 10, 11].

Recently, Marino, Scardamaglia and Karapinar [12] constructed a new iterative algorithm by modifying the normal Mann's iteration for a strict pseudo-contraction in Hilbert spaces. It needs to mention that the mapping is defined on a nonempty closed cone of Hilbert spaces. More precisely, they gave the following result:

Theorem MSK. [12] *Let H be a Hilbert space and let C be a nonempty closed cone of H. Let $T : C \to C$ be a κ-strict pseudo-contractive mapping such that $Fix(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(\kappa, 1)$ and in $(0,1)$, respectively, satisfying the conditions:*

(i) $\kappa < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \mu_n = 0$;

(iii) $\Sigma_{n=1}^{\infty} \mu_n = \infty$.

Define a sequence $\{x_n\}$ as follows:

$$x_1 \in C, \ x_{n+1} = \alpha_n(1-\mu_n)x_n + (1-\alpha_n)Tx_n, \ n \in \mathbb{N}. \tag{1.3}$$

Then $\{x_n\}$ converges strongly to $p \in Fix(T)$, that is, the unique solution of the variational inequality $\langle -p, y-p \rangle \leq 0$, $\forall y \in Fix(T)$.

Marino, Scardamaglia and Karapinar also posed an open question whether the result in Theorem MSK holds in the framework Banach spaces. Recently, some authors have studied the fixed point problems for strict pseudo-contractions in Banach space; see [3, 13, 14, 15, 16] and the references therein. On the other hand, the implicit iterative algorithms for strict pseudo-contractions are also considered by some authors [1, 6, 17, 18, 19] and the references therein. However, in general, the implicit iteration has no strong convergence.

In this paper, we continue to discuss the iterative algorithm (1.3) in Banach space. Inspired by the results in [3, 12, 17], we prove that the iterative algorithm (1.3) still has the strong convergence in 2-uniformly smooth Banach spaces. An implicit iterative scheme for strict pseudo-contraction in 2-uniformly smooth Banach space is also introduced and the strong convergence of the implicit iterative algorithm is proved. Our result improves the corresponding results of Marino, Scardamaglia and Karapinar [12] from Hilbert spaces to 2-uniformly smooth Banach spaces. This is a positive answer to the open question of them. Finally, we give an example to illustrate the main result presented in this paper.

2. Preliminaries

A Banach space E is said to be strictly convex iff $\frac{\|x+y\|}{2} < 1$ for any $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is said to be uniformly convex iff for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $|x\|, \|y\| < 1$ and $\|x-y\| \leq 2(1-\delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\|, \|y\| \leq 1, \|x-y\| \geq \varepsilon\},$$

for all $\varepsilon \in [0,2]$. E is said to be uniformly convex iff $\delta_E(0) = 0$, and $\delta(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex.

Let E be a real Banach space with norm $\|\cdot\|$. The dual of E is defined by E^*, the value of $f \in E^*$ at $x \in E$ by $\langle x, f \rangle$. The duality mapping J of E into 2^{E^*} is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $S(E) = \{x \in E : \|x\| = 1\}$. The Banach space E is said to be smooth iff the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. It is known that if E is smooth, then duality mapping J is single-valued.

Let $\rho_E(t) : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E), \|y\| \leq t\}.$$

A Banach space E is said to be uniformly smooth iff $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. If E is q-uniformly smooth, then $q \leq 2$. Typical examples of uniformly smooth Banach space is L^p, where $p > 1$. In fact, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Let C be a nonempty closed and convex subset of E, and let K be a nonempty subset of C. Let $Q : C \to K$ be a mapping. Q is said to be:

1. sunny iff for each $x \in C$ and $t \in [0,1]$ we have $Q(tx + (1-t)Qx) = Qx$;
2. a retraction of C onto K iff $Qx = x$, $\forall x \in K$;
3. a sunny nonexpansive retraction iff Q is sunny, nonexpansive and a retraction onto K.

It it known that the following conclusions are equivalent [20, 21, 22]:

(a) Q is sunny and expansive.

(b) $\|Qx - Qy\|^2 \leq \langle x-y, J(Qx-Qy)\rangle$, $\forall x, y \in E$.

(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0$, $\forall x \in E, y \in K$.

In the sequel, we will use the following lemmas for our main results.

Lemma 2.1. [16] *Let E be a 2-uniformly smooth Banach space with best smooth constant K. Then for any $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2,$$

where $j(x-y) \in J(x-y)$.

Lemma 2.2. [3] *Let C be a nonempty subset of a real 2-uniformly smooth Banach space E with best smooth constant K and let $T : C \to C$ be a κ-strict pseudo-contraction. For $\alpha \in (0, \frac{\kappa}{K^2})$, define $T_\alpha x = (1-\alpha)x + \alpha Tx$ for all $x \in C$. Then T_α is nonexpansive and $Fix(T_\alpha) = Fix(T)$.*

Lemma 2.3. [23] *Let $\{a_n\} \subset [0,1]$ be a real sequence. Let $\{\sigma_n\}$ be a nonnegative sequence of real numbers and let $\{\gamma_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \geq 0.$$

If the following conditions are satisfied:

(i) $\alpha_n \in [0,1], \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$,

then we have $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [14] *Let C be a closed convex subset of a real uniformly smooth Banach space E, and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set $Fix(T)$. Then there exists a unique sunny nonexpansive retraction $Q_{Fix(T)} : C \to Fix(T)$ such that*

$$\limsup_{n\to\infty} \langle u - Q_{Fix(T)}u, J(x_n - Q_{Fix(T)}u)\rangle \leq 0,$$

for any given $u \in C$ and $\{x_n\} \subset C$ with $x_n - Tx_n \to 0$.

3. Main results

In this section, we consider two iterative algorithms for finding fixed points of strict pseudo-contractions defined on a nonempty closed cone of 2-uniformly smooth Banach spaces.

First, we give the following explicit iterative algorithm.

Theorem 3.1. *Let C be a nonempty closed cone of a real 2-uniformly smooth Banach space E with best smooth constant K. Let $T : C \to C$ be a κ-strict pseudo-contraction with $0 \leq \kappa < 1$ and assume that $Fix(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (1 - \frac{\kappa}{3K^2}, 1)$ and $\{\mu_n\} \subset (0,1)$ be two real sequences satisfying the following conditions:*

(1) $1 - \frac{\kappa}{3K^2} < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;

(2) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\mu_n - \mu_{n-1}| < \infty$;

(3) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=1}^{\infty} \mu_n = \infty$.

Define a sequence $\{x_n\}$ by

$$x_1 \in C,\ x_{n+1} = \alpha_n(1-\mu_n)x_n + (1-\alpha_n)Tx_n,\ n \in \mathbb{N}. \tag{3.1}$$

Then $\{x_n\}$ converges strongly to some $p \in Fix(T)$.

Proof. Define a new mapping $T_\lambda : C \to C$ by

$$T_\lambda x = (1-\lambda)x + \lambda Tx,\ \forall x \in C, \tag{3.2}$$

where $\lambda = \frac{\kappa}{3K^2}$. From Lemma 2.2, one sees that T_λ is a nonexpansive mapping from C into itself and $Fix(T) = Fix(T_\lambda)$. Let $\beta_n = \frac{\alpha_n+\lambda-1}{\lambda}$ and $\gamma_n = \frac{\alpha_n\mu_n}{\beta_n}$ for each $n \in \mathbb{N}$. We rewrite (3.1) as

$$x_1 \in C,\ x_{n+1} = \beta_n(1-\gamma_n)x_n + (1-\beta_n)T_\lambda x_n,\ n \in \mathbb{N}. \tag{3.3}$$

It is obvious that $\{\beta_n\} \subset (0,1)$. Since $\mu_n \to 0$, one has $\gamma_n \to 0$. Hence we can assume that $\{\gamma_n\} \subset (0,1)$.

Now we prove that $\{x_n\}$ is bounded. For $p = Q_{Fix(T_\lambda)}0 \in F(T_\lambda)$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n(1-\gamma_n)x_n + (1-\beta_n)T_\lambda x_n - p\| \\
&= \|\beta_n(1-\gamma_n)(x_n - p) + (1-\beta_n)(T_\lambda x_n - p) - \beta_n\gamma_n p\| \\
&\le \beta_n(1-\gamma_n)\|x_n - p\| + (1-\beta_n)\|x_n - p\| + \beta_n\gamma_n\|p\| \\
&= (1-\beta_n\gamma_n)\|x_n - p\| + \beta_n\gamma_n\|p\| \\
&\le \max\{\|x_n - p\|, \|p\|\}
\end{aligned}$$

for each $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded. We show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. From (3.3) we have

$$\begin{aligned}
x_{n+1} - x_n &= \beta_n(1-\gamma_n)x_n + (1-\beta_n)T_\lambda x_n - \beta_{n-1}(1-\gamma_{n-1})x_{n-1} \\
&\quad - (1-\beta_{n-1})T_\lambda x_{n-1} \\
&= \beta_n(1-\gamma_n)(x_n - x_{n-1}) + \beta_n(1-\gamma_n)x_{n-1} + (1-\beta_n)(T_\lambda x_n - T_\lambda x_{n-1}) \\
&\quad + (1-\beta_n)T_\lambda x_{n-1} - \beta_{n-1}(1-\gamma_{n-1})x_{n-1} - (1-\beta_{n-1})T_\lambda x_{n-1} \\
&= \beta_n(1-\gamma_n)(x_n - x_{n-1}) + \big(\beta_n(1-\gamma_n) - \beta_{n-1}(1-\gamma_{n-1})\big)x_{n-1} \\
&\quad + (1-\beta_n)(T_\lambda x_n - T_\lambda x_{n-1}) + (\beta_{n-1} - \beta_n)T_\lambda x_{n-1} \\
&= \beta_n(1-\gamma_n)(x_n - x_{n-1}) + \big((\beta_n - \beta_{n-1})(1-\gamma_{n-1}) + \beta_n(\gamma_{n-1} - \gamma_n)\big)x_{n-1} \\
&\quad + (1-\beta_n)(T_\lambda x_n - T_\lambda x_{n-1}) + (\beta_{n-1} - \beta_n)T_\lambda x_{n-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \beta_n(1-\gamma_n)\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\gamma_{n-1} - \gamma_n|)\|x_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n|\|T_\lambda x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&= (1-\beta_n\gamma_n)\|x_n - x_{n-1}\| + (2|\beta_n - \beta_{n-1}| + |\gamma_{n-1} - \gamma_n|)M \\
&\le (1-b\gamma_n)\|x_n - x_{n-1}\| + (2|\beta_n - \beta_{n-1}| + |\gamma_{n-1} - \gamma_n|)M,
\end{aligned} \tag{3.4}$$

where $b = \inf_{n\geq 1} \beta_n$ and $M = \max\{\sup_{n\geq 1} \|x_n\|, \sup_{n\geq 1} \|T_\lambda x_n\|\}$. Note that

$$\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| = \frac{\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|}{\lambda} < \infty$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| &= \Big| \sum_{n=1}^{\infty} \frac{\alpha_n \mu_n}{\beta_n} - \frac{\alpha_{n-1}\mu_{n-1}}{\beta_{n-1}} \Big| \\
&= \sum_{n=1}^{\infty} \frac{|\beta_{n-1}\alpha_n\mu_n - \beta_n\alpha_{n-1}\mu_{n-1}|}{\beta_n\beta_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{|\beta_{n-1}\alpha_n(\mu_n - \mu_{n-1}) + (\beta_{n-1}(\alpha_n - \alpha_{n-1}) + (\beta_{n-1} - \beta_n)\alpha_{n-1})\mu_{n-1}|}{\beta_n\beta_{n-1}} \\
&< \sum_{n=1}^{\infty} \frac{|\mu_n - \mu_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_{n-1} - \beta_n|}{b^2} \\
&< \infty.
\end{aligned}$$

Using Lemma 2.3, we get from (3.4) that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{3.5}$$

In view (3.3), we have

$$T_\lambda x_n - x_n = \frac{x_{n+1} - x_n + \beta_n\gamma_n x_n}{1 - \beta_n}.$$

Hence,

$$\|T_\lambda x_n - x_n\| \leq \frac{\|x_{n+1} - x_n\| + \beta_n\gamma_n\|x_n\|}{1 - \beta_n} \to 0. \tag{3.6}$$

Putting $z_n = \beta_n x_n + (1-\beta_n)T_\lambda x_n$, $\forall n \in \mathbb{N}$, we have $z_n - x_n = (1-\beta_n)(T_\lambda x_n - x_n)$. Using (3.3), we have

$$\begin{aligned}
x_{n+1} &= z_n - \beta_n\gamma_n x_n \\
&= (1-\beta_n\gamma_n)z_n + \beta_n\gamma_n(z_n - x_n) \\
&= (1-\beta_n\gamma_n)z_n + \beta_n\gamma_n(1-\beta_n)(T_\lambda x_n - x_n).
\end{aligned} \tag{3.7}$$

It follows from Lemma 2.1 and (3.2) that

$$\begin{aligned}
\|z_n-p\|^2 &\le \|x_n-p\|^2-2(1-\beta_n)\langle x_n-T_\lambda x_n, j(x_n-p)\rangle+2K^2(1-\beta_n)^2\|x_n-T_\lambda x_n)\|^2\\
&=\|x_n-p\|^2-2\lambda(1-\beta_n)\|x_n-p\|^2+2\lambda(1-\beta_n)\langle Tx_n-Tp, j(x_n-p)\rangle\\
&+2K^2(1-\beta_n)^2\|x_n-T_\lambda x_n\|^2\\
&\le \|x_n-p\|^2-2\lambda(1-\beta_n)\kappa\|Tx_n-x_n\|^2+2K^2(1-\beta_n)\|x_n-T_\lambda x_n\|^2\\
&=\|x_n-p\|^2-2\lambda(1-\beta_n)\kappa\|Tx_n-x_n\|^2+2K^2(1-\beta_n)\lambda^2\|x_n-Tx_n\|^2\\
&=\|x_n-p\|^2-2\lambda(1-\beta_n)[\kappa-K^2\lambda]\|x_n-Tx_n\|^2\\
&\le \|x_n-p\|^2.
\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8), we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\beta_n\gamma_n)z_n+\beta_n\gamma_n(1-\beta_n)(T_\lambda x_n-x_n)-p\|^2\\
&=\|(1-\beta_n\gamma_n)(z_n-p)+\beta_n\gamma_n[(1-\beta_n)(T_\lambda x_n-x_n)-p]\|^2\\
&\le (1-\beta_n\gamma_n)\|z_n-p\|^2+2\beta_n\gamma_n(1-\beta_n)\langle T_\lambda x_n-x_n, j(x_{n+1}-p)\rangle\\
&+2\beta_n\gamma_n\langle -p, j(x_{n+1}-p)\rangle\\
&\le (1-\beta_n\gamma_n)\|x_n-p\|^2+2\beta_n\gamma_n(1-\beta_n)\langle T_\lambda x_n-x_n, j(x_{n+1}-p)\rangle\\
&+2\beta_n\gamma_n\langle -p, j(x_{n+1}-p)\rangle.
\end{aligned} \tag{3.9}$$

Using Lemma 2.4, we find from (3.6) that

$$\lim_{n\to\infty}\langle T_\lambda x_n-x_n, j(x_{n+1}-p)\rangle=0 \tag{3.10}$$

and

$$\limsup_{n\to\infty}\langle -p, j(x_{n+1}-p)\rangle\le 0. \tag{3.11}$$

Therefore, from (3.9)-(3.11) and Lemma 2.3, we get that $\lim_{n\to\infty}\|x_n-p\|=0$. The proof is complete.

In 1974, Deimling [24] proved that each continuous strong pseudo-contraction defined on a nonempty closed convex subset of a real Banach space has a unique fixed point. Let $T: C\to C$ where C is a nonempty closed cone of a real Banach space E, be a strict pseudo-contraction. Let $u\in C$ and $t\in(0,1)$. Define a mapping $S: C\to C$ by

$$Sx=tu+tTx,\ \ \forall x\in C.$$

Then S has a unique fixed point in C. In fact, for each $x, y \in C$, we have

$$\langle Sx - Sy, j(x-y)\rangle = t\langle Tx - Ty, j(x-y)\rangle \leq t\|x-y\|^2$$

for some $j(x-y) \in J(x-y)$. Hence S is a strong pseudo-contraction.

Next, we give an implicit iterative algorithm for strict pseudo-contractions in a real 2-uniformly smooth Banach space.

Theorem 3.2. *Let C be a nonempty closed cone of a real 2-uniformly smooth Banach space E with best smooth constant K. Let $T : C \to C$ be a κ-strict pseudo-contraction with $0 \leq \kappa < 1$ and assume that $Fix(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1)$ and $\{\mu_n\} \subset (0,1)$ be two real sequences satisfying the following conditions:*

(1) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;
(2) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\mu_n - \mu_{n-1}| < \infty$;
(3) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=1}^{\infty} \mu_n = \infty$.

Define a sequence $\{x_n\}$ by

$$x_0 \in C,\ x_n = \alpha_n(1-\mu_n)x_{n-1} + (1-\alpha_n)Tx_n,\ n \in \mathbb{N}. \tag{3.12}$$

Then $\{x_n\}$ converges strongly to some $p \in Fix(T)$.

Proof. First, it is not hard to find that (3.12) is well defined. Define a mapping $T_\lambda : C \to C$ by

$$T_\lambda x = (1-\lambda)x + \lambda Tx,\ \forall x \in C, \tag{3.13}$$

where $\lambda = \frac{\kappa}{3K^2}$. From [3], we see that T_λ is a nonexpansive mapping from C into itself and $Fix(T) = Fix(T_\lambda)$. Let $\beta_n = \frac{\alpha_n\lambda}{1-\alpha_n(1-\lambda)}$ and $\gamma_n = \mu_n$ for each $n \in \mathbb{N}$. We rewrite (3.12) as

$$x_1 \in C,\ x_n = \beta_n(1-\gamma_n)x_{n-1} + (1-\beta_n)T_\lambda x_n,\ n \in \mathbb{N}. \tag{3.14}$$

It is obvious that $\{\beta_n\} \subset (0,1)$. Now we prove that $\{x_n\}$ is bounded. For $p = Q_{Fix(T_\lambda)}0 \in F(T_\lambda)$, we have

$$\begin{aligned}\|x_n - p\| &= \|\beta_n(1-\gamma_n)x_{n-1} + (1-\beta_n)T_\lambda x_n - p\| \\ &= \|\beta_n(1-\gamma_n)(x_{n-1}-p) + (1-\beta_n)(T_\lambda x_n - p) - \beta_n\gamma_n p\| \\ &\leq \beta_n(1-\gamma_n)\|x_{n-1}-p\| + (1-\beta_n)\|x_n - p\| + \beta_n\gamma_n\|p\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_n - p\| &\leq (1-\gamma_n)\|x_{n-1} - p\| + \gamma_n\|p\| \\ &\leq \max\{\|x_{n-1} - p\|, \|p\|\}\end{aligned}$$

for each $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. From (3.14), we have

$$\begin{aligned}x_{n+1} - x_n &= \beta_{n+1}(1-\gamma_{n+1})x_n + (1-\beta_{n+1})T_\lambda x_{n+1} - \beta_n(1-\gamma_n)x_{n-1} - (1-\beta_n)T_\lambda x_n \\ &= \beta_{n+1}(1-\gamma_{n+1})(x_n - x_{n-1}) + \beta_{n+1}(1-\gamma_{n+1})x_{n-1} + (1-\beta_{n+1})(T_\lambda x_{n+1} - T_\lambda x_n) \\ &\quad + (1-\beta_{n+1})T_\lambda x_n - \beta_n(1-\gamma_n)x_{n-1} - (1-\beta_n)T_\lambda x_n \\ &= \beta_{n+1}(1-\gamma_{n+1})(x_n - x_{n-1}) + \big(\beta_{n+1}(1-\gamma_{n+1}) - \beta_n(1-\gamma_n)\big)x_{n-1} \\ &\quad + (1-\beta_{n+1})(T_\lambda x_{n+1} - T_\lambda x_n) + (\beta_n - \beta_{n+1})T_\lambda x_n \\ &= \beta_{n+1}(1-\gamma_{n+1})(x_n - x_{n-1}) + \big((\beta_{n+1} - \beta_n)(1-\gamma_n) + \beta_{n+1}(\gamma_n - \gamma_{n+1})\big)x_{n-1} \\ &\quad + (1-\beta_{n+1})(T_\lambda x_{n+1} - T_\lambda x_n) + (\beta_n - \beta_{n+1})T_\lambda x_n.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\beta_{n+1}(1-\gamma_{n+1})(x_n - x_{n-1}) + \big((\beta_{n+1} - \beta_n)(1-\gamma_n) + \beta_{n+1}(\gamma_n - \gamma_{n+1})\big)x_{n-1} \\ &\qquad + (1-\beta_{n+1})(T_\lambda x_{n+1} - T_\lambda x_n) + (\beta_n - \beta_{n+1})T_\lambda x_n\| \\ &\leq \beta_{n+1}(1-\gamma_{n+1})\|x_n - x_{n-1}\| + (|\beta_{n+1} - \beta_n| + |\gamma_n - \gamma_{n+1}|)\|x_{n-1}\| \\ &\qquad + (1-\beta_{n+1})\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|T_\lambda x_n\|.\end{aligned}$$

Hence, one has

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1-\gamma_{n+1})\|x_n - x_{n-1}\| + \frac{1}{\beta_{n+1}}(2|\beta_{n+1} - \beta_n| + |\gamma_n - \gamma_{n+1}|)M \\ &\leq (1-\gamma_{n+1})\|x_n - x_{n-1}\| + \frac{1-a(1-\lambda)}{a\lambda}(2|\beta_{n+1} - \beta_n| + |\gamma_n - \gamma_{n+1}|)M,\end{aligned} \tag{3.15}$$

where $a = \inf_{n\geq 1} a_n$ and $M = \max\{\sup_{n\geq 1}\|x_{n-1}\|, \sup_{n\geq 1}\|T_\lambda x_n\|\}$. Note that $\sum_{n=1}^\infty |\gamma_n - \gamma_{n-1}| = \sum_{n=1}^\infty |\mu_n - \mu_{n-1}| < \infty$ and

$$\begin{aligned}\sum_{n=1}^\infty |\beta_n - \beta_{n-1}| &= \Big|\sum_{n=1}^\infty \frac{\alpha_n\lambda}{1-\alpha_n(1-\lambda)} - \frac{\alpha_{n-1}\lambda}{1-\alpha_{n-1}(1-\lambda)}\Big| \\ &= \sum_{n=1}^\infty \frac{|\alpha_n\lambda - \alpha_{n-1}\lambda|}{\big(1-\alpha_n(1-\lambda)\big)\big(1-\alpha_{n-1}(1-\lambda)\big)} \\ &\leq \sum_{n=1}^\infty \frac{\lambda|\alpha_n - \alpha_{n-1}|}{\big(1-a(1-\lambda)\big)^2} < \infty.\end{aligned}$$

Using Lemma 2.3, we get

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{3.16}$$

In view of (3.14), we have $T_\lambda x_n - x_n = \frac{\beta_n(x_n-x_{n-1})+\beta_n\gamma_n x_{n-1}}{1-\beta_n}$. Hence,

$$\|T_\lambda x_n - x_n\| \le \frac{\beta_n\|x_n - x_{n-1}\| + \beta_n\gamma_n\|x_{n-1}\|}{1-\beta_n} \to 0, \tag{3.17}$$

From (3.13) and (3.17), we conclude that

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{3.18}$$

By (3.12) and (1.1), we get

$$\begin{aligned}
\|x_n - p\|^2 &= \langle \alpha_n(1-\mu_n)x_{n-1} + (1-\alpha_n)Tx_n - p, j(x_n - p)\rangle \\
&= \langle \alpha_n(1-\mu_n)(x_{n-1} - p) + (1-\alpha_n)(Tx_n - p) - \alpha_n\mu_n p, j(x_n - p)\rangle \\
&= \alpha_n(1-\mu_n)\langle (x_{n-1} - p), j(x_n - p)\rangle + (1-\alpha_n)\langle (Tx_n - p), j(x_n - p)\rangle \\
&\quad + \alpha_n\mu_n\langle -p, j(x_n - p)\rangle \\
&\le \alpha_n(1-\mu_n)\|x_{n-1} - p\|\|x_n - p\| + (1-\alpha_n)\left[\|x_n - p\|^2 - \kappa\|x_n - Tx_n\|^2\right] \\
&\quad + \alpha_n\mu_n\langle -p, j(x_n - p)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_n - p\|^2 &\le (1-\mu_n)\|x_{n-1} - p\|\|x_n - p\| - \frac{(1-\alpha_n)\kappa}{\alpha_n}\|x_n - Tx_n\|^2 + \mu_n\langle -p, j(x_n - p)\rangle \\
&\le \frac{(1-\mu_n)}{2}[\|x_{n-1} - p\|^2 + \|x_n - p\|^2] + \mu_n\|x_n - Tx_n\|^2 + \mu_n\langle -p, j(x_n - p)\rangle,
\end{aligned}$$

which implies that

$$\|x_n - p\|^2 \le (1-\mu_n)\|x_{n-1} - p\|^2 + 2\mu_n\|x_n - Tx_n\|^2 + 2\mu_n\langle -p, j(x_n - p)\rangle. \tag{3.19}$$

In view of (3.17) and Lemma 2.4, we find that

$$\limsup_{n\to\infty}\langle -p, j(x_n - p)\rangle \le 0. \tag{3.20}$$

From (3.18)-(3.20) and Lemma 2.3, we conclude that $\lim_{n\to\infty}\|x_n - p\| = 0$. The proof is complete.

Remark 3.3. Let T is a k-strict pseudo-contraction and best smooth constant is K. Let $\alpha \in (0,1)$ and $M \in \mathbb{N}$ such that $1 + \alpha - \frac{\kappa}{3K^2} + \frac{1}{M} \in (0,1)$. Sequences $\{\alpha_n\}$ and $\{\mu_n\}$ can be taken as

$a_n = 1 - \frac{\kappa}{3K^2} + \alpha + \frac{1}{M_n}$ and $\mu_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the conditions of Theorem 3.1.

Remark 3.4. Theorem 3.1 improves the result of Marino, Scardamaglia and Karapinar from Hilbert space to 2-uniformly smooth Banach space and gives a positive answer to the open question of authors.

Finally, we give an example to illustrate Theorem 3.1.

Example 3.5. Let $C = L^3[a,b]$. It is known that C is a 2-uniformly smooth Banach space. Let $T : C \to C$ defined by $Tx = -\frac{1}{3}x$ for each $x \in C$. For all $x, y \in C$, we have

$$\langle Tx - Ty, j(x-y)\rangle = -\frac{1}{3}\langle x-y, j(x-y)\rangle = -\frac{1}{3}\|x-y\|^2$$

and

$$\|x-y\|^2 - \frac{3}{4}\|(I-T)x - (I-T)y\|^2 = -\frac{1}{3}\|x-y\|^2.$$

Hence T is a $\frac{3}{4}$-strict pseudo-contraction. Put $\alpha_n = \frac{1}{2} + \frac{1}{4n}$ and $\mu_n = \frac{1}{2n}$ for each $n \in \mathbb{N}$. Then $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the conditions of Theorem 3.2. Let $x_0 \in L^3[a,b]$. We compute some $\{x_n\}$ by (3.12) as follows:

$$x_1 = \frac{9}{26}x_0, \ \ x_2 = \frac{15}{104}x_0, \ \ x_3 = \frac{525}{8528}x_0, \ \ x_4 = \frac{19845}{750464}x_0.$$

By Theorem 3.2, we conclude that $\{x_n\}$ strongly converges to the fixed point $x^* \in Fix(T)$. In fact, $x^*(t) = 0$ for all $t \in [a,b]$.

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