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ORDERED θ-CONTRACTIONS AND SOME FIXED POINT RESULTS

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Abstract. Recently, Jleli and Samet proved a fixed point result that is a proper generalization of the celebrated Banach contraction principle on complete metric spaces. By considering both θ-contractions and fixed point results on ordered metric spaces, we introduce a new concept of ordered θ-contractions on ordered metric spaces. Some fixed point theorems are obtained and an example is provided to support our main result.

Keywords. Fixed point; θ-contraction; Ordered θ-contraction; Complete metric space.

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1. Introduction and preliminaries

Recently, combining the ideas of the Tarski's fixed point theorem on ordered sets and the famous Banach contraction principle on complete metric spaces, Ran and Reurings [1] obtained a fixed point result on ordered complete metric spaces as follows.

Theorem 1.1. *Let $(X,\preceq)$ be an ordered set and let d be a metric on X such that (X,d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists*

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$x_0 \in X$ with $x_0 \preceq Tx_0$. Suppose that there exists $L \in (0,1)$ such that $d(Tx,Ty) \leq Ld(x,y)$ for all $x,y \in X$ with $x \preceq y$. If T continuous, then T has a fixed point in X.

In this theorem, the usual contraction of the Banach contraction principle is weakened and the mapping is also extended to be monotone. After this remarkable contribution, many researchers focused on this interesting result and presented some new results for contractions in partially ordered metric spaces; see [2, 3, 4, 5, 6, 7] and the references therein. For example, taking the regularity of the space instead of continuity of T, Nieto and Rodriguez- López [8] obtained similar results. There are several applications of the theorems in this direction to linear and nonlinear matrix equations, differential equations and integral equations; see [1, 9] and the references therein.

In 2014, one of the most interesting fixed point theorems on complete metric spaces was given by Jleli and Samet [10]. In this paper, we call the contraction defined in [10] as θ-contraction, which is a proper generalization of usual contraction. For the sake of completeness, we recall this concept.

Let Θ be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ_1) θ is nondecreasing,

(Θ_2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$,

(Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Some examples of the functions belonging Θ are $\theta_1(t) = e^{\sqrt{t}}$ and $\theta_2(t) = e^{\sqrt{te^t}}$.

Definition 1.2. [10] Let (X,d) be a metric space and let $T : X \to X$ be a mapping. Given $\theta \in \Theta$, we say that T is θ-contraction if there exists $k \in (0,1)$ such that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k, \tag{1.1}$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$.

If we consider the different type of mapping θ in Definition 1.2, we obtain some of variety of contractions. For example, let $\theta : (0,\infty) \to (1,\infty)$ be given by $\theta(t) = e^{\sqrt{t}}$. It is clear that $\theta \in \Theta$. Then (1.1) is reduced to, for all $x,y \in X$, $Tx \neq Ty$, $d(Tx,Ty) \leq k^2 d(x,y)$.

It is clear that for $x,y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) \leq k^2 d(x,y)$ also holds. Therefore, T is an usual contraction. Similarly, let $\theta : (0,\infty) \to (1,\infty)$ be given by $\theta(t) = e^{\sqrt{te^t}}$.

It is clear that $\theta \in \Theta$. Then (1.1) turns to, for all $x, y \in X, Tx \neq Ty$,

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \leq k^2. \tag{1.2}$$

In addition, note that every θ-contraction T is a contractive mapping, i.e., for all $x, y \in X$, $x \neq y. d(Tx,Ty) < d(x,y)$. Thus, every θ-contraction is a continuous mapping. On the other side, The example in [10] shows that mapping T is not a usual contraction , but it is a θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$. Thus, the following theorem, which was given as a corollary by Jleli and Samet [10], is a proper generalization of the Banach contraction Principle.

Theorem 1.3. (Corollary 2.1 of [10]) *Let (X,d) be a complete metric space and $T : X \to X$ be a θ-contraction. Then T has a unique fixed point in X.*

We can find some generalizations of Theorem 1.3 in [11, 12, 13]. The aim of this paper is to introduce the concept of ordered θ-contractions on ordered metric spaces, by taking into account the ideas of Ran and Reurings [1] and Jleli and Samet [10]. Throughout this article, we assume that $\mathbb{N}$ denotes the set of all positive integers.

2. Main results

Let $(X, \preceq)$ be an ordered set and let d be a metric on X. Then we say that the tripled $(X, \preceq, d)$ is an ordered metric space. If (X,d) is complete, then $(X, \preceq, d)$ is called a ordered complete metric space. Recall that $T : X \to X$ is said to be a nondecreasing mapping if

$$x \preceq y \Rightarrow Tx \preceq Ty$$

for all $x, y \in X$. we say that X is regular if ordered metric space $(X, \preceq, d)$ has the following condition:

$$\begin{cases} \text{If } \{x_n\} \subset X \text{ is a non-decreasing sequence with } x_n \to x \text{ in } X, \\ \text{then } x_n \preceq x \text{ for all } n. \end{cases}$$

Definition 2.1. Let $(X, \preceq, d)$ be an ordered metric space. Let $T : X \to X$ be a mapping and $\theta \in \Theta$. We say that T is an ordered θ-contraction if there exists $k \in (0,1)$ such that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k, \tag{2.1}$$

for all $(x,y) \in S$, where

$$S = \{(x,y) \in X \times X : x \preceq y,\ d(Tx,Ty) > 0\}.$$

Theorem 2.2. *Let $(X,\preceq,d)$ be an ordered complete metric space and let $T : X \to X$ be an ordered θ-contraction. Suppose that T is a nondecreasing mapping and there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. If T is continuous or X is regular, then T has a fixed point.*

Proof. Let $x_0 \in X$ be as mentioned in the hypotheses. Define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and so the proof is completed. Thus, suppose that for every $n \in \mathbb{N}$, $x_{n+1} \neq x_n$. Since $x_0 \preceq Tx_0$ and T is nondecreasing, we obtain

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots.$$

Since $x_n \preceq x_{n+1}$ and $d(Tx_n,Tx_{n-1}) > 0$ for every $n \in \mathbb{N}$, one sees that $(x_n,x_{n+1}) \in S$. So we can use inequality (2.1) for the consecutive terms of $\{x_n\}$. It follows that

$$\theta(d(x_{n+1},x_n)) = \theta(d(Tx_n,Tx_{n-1})) \leq [\theta(d(x_n,x_{n-1}))]^k . \tag{2.2}$$

Denote $\gamma_n = d(x_n,x_{n+1})$ for $n \in \mathbb{N}$. We obtain from (2.2) that

$$1 < \theta(\gamma_n) \leq [\theta(d(x_0,x_1))]^{k^n} . \tag{2.3}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.3), we obtain

$$\lim_{n\to\infty} \theta(\gamma_n) = 1. \tag{2.4}$$

In view of (Θ_2), we have $\lim_{n\to\infty} \gamma_n = 0^+$. Therefore, from (Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\gamma_n) - 1}{[\gamma_n]^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\left| \frac{\theta(\gamma_n) - 1}{(\gamma_n)^r} - l \right| \leq B.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta(\gamma_n) - 1}{(\gamma_n)^r} \geq l - B = B.$$

Then, for all $n \geq n_0$, $n(\gamma_n)^r \leq An\left[\theta(\gamma_n)-1\right]$, where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\frac{\theta(\gamma_n)-1}{(\gamma_n)^r} \geq B.$$

This implies that, for all $n \geq n_0$, $n(\gamma_n)^r \leq An\left[\theta(\gamma_n)-1\right]$, where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,$n(\gamma_n)^r \leq An\left[\theta(\gamma_n)-1\right]$. Using (2.3), we obtain, for all $n \geq n_0$,

$$n(\gamma_n)^r \leq An\left[\left[\theta(a_0)\right]^{k^n}-1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain that $\lim_{n\to\infty} n\gamma_n^r = 0$. Thus, there exits $n_1 \in \mathbb{N}$ such that $n\gamma_n^r \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$\gamma_n \leq \frac{1}{n^{1/r}}. \tag{2.5}$$

Let $m > n > n_1$. From (2.5), we have

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m) \\ &= \sum_{i=n}^{m-1}\gamma_i \leq \sum_{i=n}^{m-1}\frac{1}{i^{1/r}} \leq \sum_{i=n}^{\infty}\frac{1}{i^{1/r}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty}\frac{1}{i^{1/r}}$ is convergent, we get $d(x_n,x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. If T is continuous, then we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tz.$$

So z is a fixed point of T.

Now we suppose that X is regular. Then $x_n \preceq z$ for all $n \in \mathbb{N}$. We consider the following two cases:

Case 1. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = z$, we obtain

$$Tz = Tx_{n_0} = x_{n_0+1} \preceq z.$$

Since $x_{n_0} \preceq x_{n_0+1}$, one finds that $z \preceq Tz$ and hence $z = Tz$.

Case 2. Now, we suppose that $x_n \neq z$ for every $n \in \mathbb{N}$ and $d(z,Tz) > 0$. Since $\lim_{n\to\infty} x_n = z$, one sees that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n+1},Tz) > 0$ and $d(x_n,z) < \frac{d(z,Tz)}{2}$ for all $n \geq n_0$. It follows that $(x_n,z) \in S$. From (Θ_1), we have, for all $n \geq n_0$,

$$\theta(d(x_{n+1},Tz)) = \theta(d(Tx_n,Tz)) \leq [\theta(d(x_n,z))]^k < \theta(d(x_n,z)) \leq \theta(\frac{d(z,Tz)}{2}),$$

which implies

$$d(x_{n+1},Tz) < \frac{d(z,Tz)}{2}. \tag{2.6}$$

Taking limit as $n \to \infty$, we deduce that$d(z,Tz) \leq \frac{d(z,Tz)}{2}$, a contradiction. Therefore, we conclude that $d(z,Tz) = 0$, i.e. $z = Tz$.

Remark 2.3. It is clear that Theorem 1.3 is a special cases of Theorem 2.2. Also, Theorem 1.1 is a special cases of Theorem 2.2 with $\theta(t) = e^{\sqrt{t}}$ and $L = k^2$.

The following example shows that Theorem 2.2 is a proper generalization of both Theorem 1.1 and Theorem 1.3.

Example 2.4. *Let $X = \{0,1,2,\cdots\}$ and let $d : X \times X \to [0,\infty)$ be given by*

$$d(x,y) = \begin{cases} 0, & x = y, \\ \\ x+y, & x \neq y. \end{cases}$$

Then (X,d) is a complete metric spaces. Define an order relation $\preceq$ on X as

$$x \preceq y \Leftrightarrow [x,y \in \mathbb{N} \text{ and } x < y] \text{ or } [x = y].$$

Note that 0 is not comparable with another element of X. Obviously, $(X,\preceq,d)$ is ordered complete metric space. Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x, & x \in \{0,1\}, \\ \\ x-1, & x \geq 2. \end{cases}$$

It is easy to see that T is nondecreasing. Also, for $x_0 = 0$ we have $x_0 \preceq Tx_0$. Since $2 \preceq x$ for all $x \geq 3$ and

$$\lim_{x\to\infty} \frac{d(T2,Tx)}{d(2,x)} = \lim_{x\to\infty} \frac{x}{x+2} = 1,$$

we can not find $L \in (0,1)$ satisfying $d(Tx,Ty) \le Ld(x,y)$. Thus, Theorem 1.1 , which is main result of [1], *is not applicable to this example. Also, T is not θ-contraction. Indeed, for $x=0$ and $y=1$, since $d(T0,T1) = 1 = d(0,1)$, we have $\theta(d(T0,T1)) > [\theta(d(0,1))]^k$, for all $\theta \in \Theta$ and $k \in (0,1)$. Therefore, Theorem 1.3, which is main result of* [10], *is not applicable to this example.*

Now, we claim that T is a ordered θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$ and $k = e^{-\frac{1}{2}}$. To see this, we first show that

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \le e^{-1}. \tag{2.7}$$

for all $(x,y) \in S$. Observe that

$$\begin{aligned} S &= \{(x,y) \in X \times X : x \preceq y \text{ and } d(Tx,Ty) > 0\} \\ &= \preceq \setminus [\triangle \cup (1,2)], \end{aligned}$$

where $\triangle$ is the diagonal of $X \times X$, i.e.,$\triangle = \{(x,x) \in X \times X : x \in X\}$. Letting $(x,y) \in S$, one finds that

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \le \frac{x+y-1}{x+y} e^{-1} \le e^{-1}.$$

This shows that (2.7) is true. Also, since τ_d is discrete topology, T is continuous (and X is regular). Therefore all conditions of Theorem 2.2 are satisfied and so T has a fixed point in X. Here 0 and 1 are fixed point of T.

Corollary 2.5. *Let $(X, \preceq, d)$ be an ordered complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Suppose that there exists $L \in (0,1)$ such that*

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \le L < 1,$$

for all $(x,y) \in S$. If T is continuous or X is regular, then T has a fixed point in X.

Proof. Let $\theta(t) = e^{\sqrt{te^t}}$. Setting $k = \sqrt{L}$ in Theorem 2.2, one obtains the desired conclusion immediately.

Corollary 2.6. *Let $(X, \preceq, d)$ be an ordered complete metric space and let $T : X \to X$ be a nondecreasing mapping such that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Suppose that there exists $L \in (0,1)$ such that*

$$\frac{d(Tx,Ty)(d(Tx,Ty)+1)}{d(x,y)(d(x,y)+1)} \le L < 1,$$

for all $(x,y) \in S$. If T is continuous or X is regular, then T has a fixed point in X.

Proof. Letting $\theta(t) = e^{\sqrt{t^2+t}}$, one sees that $\theta \in \Theta$. Setting $k = \sqrt{L}$ in Theorem 2.2, one obtains the desired conclusion immediately.

Theorem 2.7. *In Theorem 2.2, if we assume the following condition:*

$$\text{every pair of elements has a lower bound and upper bound,} \tag{2.8}$$

then T has a unique fixed point in X.

Proof. For the proof, it is sufficient to show that for every $x \in X$,$\lim_{n\to\infty} T^n x = z$, where z is the fixed point of T such that$z = \lim_{n\to\infty} T^n x_0$. For this we will consider the following cases. Let $x \in X$ and x_0 be as in Theorem 2.2.

Case 1. If $x \preceq x_0$ or $x_0 \preceq x$, then $T^n x \preceq T^n x_0$ or $T^n x_0 \preceq T^n x$ for all $n \in \mathbb{N}$. If $T^{n_0} x = T^{n_0} x_0$ for some $n_0 \in \mathbb{N}$, then $T^n x \to z$. Now let $T^n x_0 \neq T^n x$ for all $n \in \mathbb{N}$. It follows that $d(T^n x_0, T^n x) > 0$ and so $(T^n x_0, T^n x) \in S$ for all $n \in \mathbb{N}$. In view of (2.1), we have

$$\begin{aligned} \theta(d(T^n x_0, T^n x)) \leq & \left[\theta(d(T^{n-1}x_0, T^{n-1}x))\right]^k \\ \leq & \left[\theta(d(T^{n-2}x_0, T^{n-2}x))\right]^{k^2} \\ & \vdots \\ \leq & \left[\theta(d(x_0, x))\right]^{k^n}. \end{aligned} \tag{2.9}$$

Taking into account (Θ_2) we find from (2.9) that $\lim_{n\to\infty} d(T^n x_0, T^n x) = 0$ and so $\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} T^n x = z$.

Case 2. If $x \npreceq x_0$ or $x_0 \npreceq x$, then we find from (2.8) that there exist $x_1, x_2 \in X$ such that

$$x_2 \preceq x \preceq x_1 \text{ and } x_2 \preceq x_0 \preceq x_1.$$

Therefore, as in the Case 1, we can show that

$$\lim_{n\to\infty} T^n x_1 = \lim_{n\to\infty} T^n x_2 = \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^n x_0 = z.$$

Remark 2.8. In Example 2.4, since pair $\{0,1\}$ has neither a lower bound nor an upper bound, one sees that condition (2.8) is not satisfied. Therefore T may not has a unique fixed.

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