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ON A NONLOCAL NEUMANN PROBLEM IN ORLICZ-SOBOLEV SPACES

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Abstract. In this paper, we consider the existence of solutions to a class of nonlocal quasilinear elliptic equations with the Neumann boundary condition. The problem is settled in Orlicz-Sobolev spaces and the main tool used is the Ekeland variational principle.

Keywords. Nonlocal Neumann problem; Ekeland variational principle; Orlicz-Sobolev space.

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1. Introduction

In this article, we are concerned with a class of nonlocal problems in Orlicz-Sobolev spaces of the form

$$\begin{cases} -M(\int_\Omega \Phi(|\nabla u|)dx)\mathrm{div}(a(|\nabla u|)\nabla u) = f(u) + h(x) \text{ in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, η is the unit exterior vector on $\partial\Omega$. The function $\varphi(t) := a(|t|)t$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. We remark that if we let $a(t) = |t|^{p-2}$, problem (1.1) turns into the well-known p-Kirchhoff equation and if $p = p(x)$, i.e. $a(t) = |t|^{p(x)-2}$, problem (1.1) becomes the $p(x)$-Kirchhoff equation, the generalization of p-Kirchhoff equation. If we additionally consider $M(t) = 1$, then equation (1.1) becomes the $p(x)$-Laplace equation, a generalization of p-Laplace equation given by

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$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,u)$, $1 < p < N$. Therefore, equation (1.1) has the capacity particularly to generalize the problems involving variable exponents. This kind of problems have been extensively studied by many authors over the past twenty years due to its significant role in many fields of mathematics; see, e.g., [1]-[17] and the references therein.

The study of variational problems in the classical Sobolev and Orlicz-Sobolev spaces is an interesting topic of research due to its significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, non-linear potential theory, the theory of quasiconformal mappings, non-Newtonian fluids, image processing, differential geometry, geometric function theory, and probability theory; see, e.g., [18]-[23] and the references therein. Moreover, problem (1.1) posses more complicated nonlinearities, for example, it is inhomogeneous, so in the discussions, some special techniques will be needed. However, the inhomogeneous nonlinearities have important physical background. Therefore, equation (1.1) may represent a variety of mathematical models corresponding to certain phenomenons; see, e.g., [21] and the references therein.

(1) Nonlinear elasticity: $\varphi(t) = \left(1+t^2\right)^{\alpha} - 1$, $\alpha > \frac{1}{2}$,
(2) Plasticity: $\varphi(t) = t^{\alpha}\left(\log(1+t)\right)^{\beta}$, $\alpha \geq 1, \beta > 0$,
(3) Generalized Newtonian fluids: $\varphi(t) = \int_0^t s^{1-\alpha}\left(\sinh^{-1} s\right)^{\beta} ds$, $0 \leq \alpha \leq 1, \beta > 0$.

In this work, we study the existence of solutions of (1.1). The problem is settled in Orlicz-Sobolev spaces and treated by variational approach and the main medium is the Ekeland variational principle. We generalize the problem and results obtained in [24] to the Orlicz-Sobolev spaces. Therefore, we had to apply a slightly complicated analysis carried on Orlicz-Sobolev spaces, such as compact embeddings between Orlicz-Sobolev spaces and the Orlicz spaces. We notice here that since problem (1.1) contains function a, by which the differential operator appears in equation (1.1) can be particularized to some well-known operators as mentioned in the previous page, and (1.1) represents a more general problem than the one studied in [24]. The main difficulty about problem (1.1) was the noncoerciveness of the corresponding functional. Instead, we decomposed the space and obtained a Poincaré-type inequality to obtain the boundedness of the corresponding functional as well as some other boundedness results. To the authors best knowledge, the results obtained in the present papers are not covered in the literature, and therefore, it has potential to contribute it.

2. Preliminaries

To attack to problem (1.1), we use the theory of the Orlicz-Sobolev spaces since equation (1.1) contains a nonhomogeneous function φ in the differential operator. Therefore, we start with some basic concepts of the Orlicz-Sobolev spaces. For more details we refer the readers to the monographs [25]-[31] and the papers [21, 23]

The function $a:(0,\infty)\to\mathbb{R}$ is a function such that the mapping defined by

$$\varphi(t):=\begin{cases} a(|t|)t & \text{for } t\neq 0,\\ 0, & \text{for } t=0, \end{cases} \tag{2.1}$$

is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

For function φ above, let us define

$$\Phi(t)=\int_0^t \varphi(s)ds, \quad \bar{\Phi}(t)=\int_0^t \varphi^{-1}(s)ds, \quad t\in\mathbb{R}. \tag{2.2}$$

Then functions Φ and $\bar{\Phi}$ are complementary N-functions, i.e. Young functions satisfying some specific conditions (see e.g., [25, 30, 32]), whereas $\bar{\Phi}$ satisfies the following

$$\bar{\Phi}(t)=\sup\{st-\Phi(s):\ s\geq 0\}, \quad t\geq 0.$$

These functions allow us to define the Orlicz spaces $L_\Phi(\Omega)$ and $L_{\bar{\Phi}}(\Omega)$, respectively.

In the sequel, we use the following assumption:

$$1<\varphi_0:=\inf_{t>0}\frac{t\varphi(t)}{\Phi(t)}\leq\frac{t\varphi(t)}{\Phi(t)}\leq\varphi^0:=\sup_{t>0}\frac{t\varphi(t)}{\Phi(t)}<\infty, \quad t\geq 0. \tag{2.3}$$

With the help of condition (2.3), Orlicz space $L_\Phi(\Omega)$ coincides the equivalence classes of measurable functions $u:\Omega\to\mathbb{R}$ such that

$$\int_\Omega \Phi(|u(x)|)dx<\infty, \tag{2.4}$$

and is equipped with the Luxemburg norm

$$|u|_\Phi:=\inf\left\{k>0:\ \int_\Omega \Phi\left(\frac{|u(x)|}{k}\right)dx\leq 1\right\}. \tag{2.5}$$

For Orlicz spaces, Hölder inequality reads as follows (see, e.g., [25, 32])

$$\int_{\Omega} uv\,dx \leq 2\|u\|_{L_{\Phi}(\Omega)}\|u\|_{L_{\bar{\Phi}}(\Omega)}, \quad \forall u \in L_{\Phi}(\Omega), v \in L_{\bar{\Phi}}(\Omega).$$

Orlicz-Sobolev space $W^1L_{\Phi}(\Omega)$ building upon $L_{\Phi}(\Omega)$ is the space defined by

$$W^1L_{\Phi}(\Omega) := \left\{ u \in L_{\Phi}(\Omega) : \ \frac{\partial u}{\partial x_i} \in L_{\Phi}(\Omega), \ i = 1, 2, ..., N \right\}.$$

under the norm

$$\|u\|_{1,\Phi} := |u|_{\Phi} + \sum_{i=1}^{N} || \frac{\partial u}{\partial x_i} ||_{\Phi}. \tag{2.6}$$

The spaces $L_{\Phi}(\Omega)$ and $W^1L_{\Phi}(\Omega)$ generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively, where the role played by the convex mapping $t \longmapsto \frac{|t|^p}{p}$ is assumed by a more general convex function $\Phi(t)$.

The Orlicz-Sobolev conjugate Φ_* of Φ is defined by

$$\Phi_*^{-1}(t) := \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds. \tag{2.7}$$

Let Φ_1 and Φ_2 be two N-functions. If

$$\lim_{t \to \infty} \frac{\Phi_1(t)}{\Phi_2(\lambda t)} = 0, \quad \forall \lambda > 0,$$

we write

$$\Phi_1 \preccurlyeq \Phi_2 \tag{2.8}$$

and say that Φ_1 *grows essentially more slowly than* Φ_2. For example, $t^p \preccurlyeq t^q$ for $1 < p < q$.

Lemma 2.1. [21, 25] *If (2.3) and (2.10) hold, then $L_{\Phi}(\Omega)$ and $W^1L_{\Phi}(\Omega)$ are separable and reflexive Banach spaces.*

Lemma 2.2. [25, 29] *Let Φ be a N-function and let Φ_* be its Orlicz-Sobolev conjugate. If*

$$\int_1^{\infty} \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds = +\infty$$

holds, then embedding $W^1L_{\Phi}(\Omega) \hookrightarrow L_{\Psi}(\Omega)$ is compact for every N-function Ψ provided that $\Psi \preccurlyeq \Phi_*$.

Lemma 2.3. [23, 33] *Define the modular $\rho(u) := \int_{\Omega} \Phi(|u|)dx : L_{\Phi}(\Omega) \to \mathbb{R}$. Then, for every $u_n, u \in L_{\Phi}(\Omega)$, we have*

(i) $|u|_{\Phi}^{\varphi^0} \leq \rho(u) \leq |u|_{\Phi}^{\varphi_0}$ *if* $|u| < 1$,

(ii) $|u|_{\Phi}^{\varphi_0} \leq \rho(u) \leq |u|_{\Phi}^{\varphi^0}$ *if* $|u| > 1$, *(iii)* $|u_n - u|_\Phi \to 0 \Leftrightarrow \rho(u_n - u) \to 0$,

(iv) $|u_n - u|_\Phi \to \infty \Leftrightarrow \rho(u_n - u) \to \infty$.

Lemma 2.3 ((iii)-(iv)) means that norm and modular topology coincide on $L_\Phi(\Omega)$ provided that Φ satisfies (2.3), which enables that the well-known Δ_2-*condition* holds, i.e.,

$$\Phi(2t) \leq k\Phi(t), \ \ \forall t \geq 0, \tag{2.9}$$

where k is a positive constant; see, e.g., [23] and the references therein. We also assume that the following condition holds:

$$\text{the function } t \to \Phi(\sqrt{t}) \text{ is convex, } \forall t \geq 0. \tag{2.10}$$

The main tool we apply is the following, which is a version of the Ekeland variational principle; see [34] and the references therein.

Lemma 2.4. *Let X be a Banach space and let $\Lambda : X \to \mathbb{R}$ be a C^1-functional which is bounded from below. Then, for any $\varepsilon > 0$, there exists $\psi_\varepsilon \in X$ such that*

$$\Lambda(\psi_\varepsilon) \leq \inf_X \Lambda + \varepsilon \ \textit{and} \ \|\Lambda'(\psi_\varepsilon)\|_{X^*} \leq \varepsilon.$$

3. Main results

Definition 3.1. *We say that $u \in W^1 L_\Phi(\Omega)$ is a weak solution of (1.1) if*

$$M\left(\int_\Omega \Phi(|\nabla u|)dx\right)\int_\Omega a(|\nabla u|)\nabla u \nabla v dx = \int_\Omega (f(u)+h)v dx, \quad \forall v \in W^1 L_\Phi(\Omega).$$

The energy functional corresponding to problem (1.1) is defined as $J : W^1 L_\Phi(\Omega) \to \mathbb{R}$,

$$J(u) := \hat{M}\left(\int_\Omega \Phi(|\nabla u|)dx\right) - \int_\Omega F(u)dx - \int_\Omega hu dx,$$

where $\hat{M}(t) = \int_0^t M(s)ds$.

The standard arguments with some simple calculations show that J is of class $C^1(W^1 L_\Phi(\Omega), \mathbb{R})$, and the derivative of J is given by

$$\langle J'(u), v\rangle = M\left(\int_\Omega \Phi(|\nabla u|)dx\right)\int_\Omega a(|\nabla u|)\nabla u \nabla v dx - \int_\Omega (f(u)+h)v dx, \quad \forall v \in W^1 L_\Phi(\Omega).$$

From the information given above, it is obvious that $u \in W^1L_\Phi(\Omega)$ is a weak solution of (1.1) if and only if u is a nontrivial critical point of J.

We will assume the following assumptions.

(M) $M : (0,\infty) \to (0,\infty)$ is a continuous function and there are positive constants m, β with $\beta \geq 1$ such that $M(t) \geq mt^{\beta-1}$ for all $t > 0$.

(F1) There is a C^1-function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(u) = f(u)$ for all $u \in \mathbb{R}$.

(F2) F is periodic, i.e., there is $k > 0$ such that $F(u+k) = F(u)$ for all $u \in \mathbb{R}$.

(H) $h \in L_{\bar{\Phi}}(\Omega)$ such that $\int_\Omega h dx = 0$, where $\bar{\Phi}$ is the complementary function of Φ.

The main result of the present paper is as follows.

Theorem 3.2. *Suppose (M), (F1), (F2) and (H) hold. Then, problem (1.1) has a weak solution in* $W^1L_\Phi(\Omega)$.

By the same idea developed in [24], we can split space $W^1L_\Phi(\Omega)$ as follows. Define

$$W_0 := \left\{ w \in W^1L_\Phi(\Omega) : \quad \int_\Omega w dx = 0 \right\}.$$

For $w \in W^1L_\Phi(\Omega)$, denote $\bar{w} = \frac{1}{|\Omega|}\int_\Omega w dx$ and $\tilde{w} = w - \bar{w}$. Then $w = \tilde{w} + \bar{w}$, where $\bar{w} \in \mathbb{R}$ and $\tilde{w} \in W_0$. Therefore, $W^1L_\Phi(\Omega) = W_0 \oplus \mathbb{R}$. W_0 is a closed linear subspace of $W^1L_\Phi(\Omega)$ with codimension 1.

To obtain our main result, we need to prove the following lemmas.

Lemma 3.3. [Poincaré-type Inequality] *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be bounded and smooth. There exists a positive constant c independent of w such that*

$$|w|_\Phi \leq c|\nabla w|_\Phi, \quad \forall w \in W_0.$$

Proof. We argue by contradiction. Assume that there exists a sequence $(w_n) \subset W_0$ such that $|w_n|_\Phi \geq n|\nabla w_n|_\Phi$. Then, without loss of generality, we can assume that $|w_n|_\Phi = 1$. Hence $|\nabla w_n|_\Phi \leq \frac{1}{n}$. Since w_n is bounded, there is a subsequence (still denoted by w_n) such that $w_n \rightharpoonup w_0$ in $W^1L_\Phi(\Omega)$. Then, by the compact embedding, i.e., Lemma 2.2, $W^1L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$, we have

$$\begin{aligned} &w_n \to w_0 \text{ in } L_\Psi(\Omega), \\ &w_n(x) \to w_0(x) \text{ a.e.} x \in \Omega. \end{aligned} \tag{3.1}$$

If we consider $|w_n|_\Phi = 1$ and (3.1), we get $|w_0|_\Phi = 1$. Additionally, if we apply the Lebesgue dominated theorem and consider definition of W_0, we obtain that

$$\int_\Omega w_0 dx = \lim_{n\to\infty}\int_\Omega w_n dx = 0.$$

Moreover, by (3.1), we find that ∇w_n converges in measure to ∇w_0 in Ω. Therefore we can find a subsequence (still denoted by ∇w_n) such that $\nabla w_n(x) \to \nabla w_0(x)$ a.e. $x \in \Omega$. Using the Fatou lemma, we have

$$\int_\Omega |\nabla w_0| dx \le \liminf_{n\to\infty}\int_\Omega |\nabla w_n| dx = 0.$$

From the pieces of information obtained above, we obtain that $w_0 \in W_0$ and $\nabla w_0 = 0$ which mean $w_0 = 0$. This contradicts $|w_0|_\Phi = 1$. This completes the proof.

Lemma 3.4. *The functional J is bounded from below on $W^1L_\Phi(\Omega)$.*

Proof. By (F1) and (F2), it is obvious that $|F(u)| \le C$, for all $u \in \mathbb{R}$. Using (M), it reads

$$\hat{M}\left(\int_\Omega \Phi(|\nabla u|)dx\right) + \int_\Omega |F(u)|dx < \infty,$$

i.e., J is well-defined. Let $u \in W^1L_\Phi(\Omega)$. Then u can be rewritten as $u = u_0 + \alpha$, where $u_0 \in W_0$ with $\int_\Omega u_0 dx = 0$ and $\alpha \in \mathbb{R}$. Therefore, from Poincaré inequality, Hölder inequality and (M), we have

$$\begin{aligned}
J(u) &= \hat{M}\left(\int_\Omega \Phi(|\nabla u|)dx\right) - \int_\Omega F(u_0+\alpha)dx - \int_\Omega h(u_0+\alpha)dx \\
&\ge \hat{M}\left(\int_\Omega \Phi(|\nabla u|)dx\right) - C|\Omega| - \int_\Omega h u_0 dx - \alpha\int_\Omega h dx \\
&\ge \hat{M}\left(\int_\Omega \Phi(|\nabla u|)dx\right) - C|\Omega| - \int_\Omega h u_0 dx \\
&\ge m\int_\Omega^{\rho(\nabla u)} s^{\beta-1}ds - |h|_{\bar{\Phi}}|u_0|_\Phi - c_1 \\
&\ge \frac{m}{\beta}\left(\int_\Omega \Phi(|\nabla u|)dx\right)^\beta - |h|_{\bar{\Phi}}|u_0|_\Phi - c_1 \\
&\ge \frac{m}{\beta}(\rho(\nabla u))^\beta - c_2|\nabla u_0|_\Phi - c_1 \\
&\ge \frac{m}{\beta}(\rho(\nabla u))^\beta - c_3\max((\rho(\nabla u_0))^{\frac{1}{\varphi_0}}, (\rho(\nabla u_0))^{\frac{1}{\varphi^0}}) - c_1,
\end{aligned}$$

which means that functional J is bounded from below on $W^1L_\Phi(\Omega)$. This completes the proof.

Lemma 3.5. *The functional J is weakly lower semi-continuous.*

Proof. Let $(u_n) \subset W^1L_\Phi(\Omega)$ be a sequence such that $u_n \rightharpoonup u \in W^1L_\Phi(\Omega)$. Since Φ is convex, we see that $\int_\Omega \Phi(|\nabla u|)dx$ is weakly lower semi-continuous, namely,

$$\int_\Omega \Phi(|\nabla u|)dx \leq \liminf_{n\to\infty} \int_\Omega \Phi(|\nabla u_n|)dx. \tag{3.2}$$

If we consider (M), which means that $\hat{M}$ is a continuous and monotone function, along with (3.2), it reads

$$\begin{aligned} \liminf_{n\to\infty} \hat{M}\left(\int_\Omega \Phi(|\nabla u_n|)dx\right) &\geq \hat{M}\left(\liminf_{n\to\infty} \int_\Omega \Phi(|\nabla u_n|)dx\right) \\ &\geq \hat{M}\left(\int_\Omega \Phi(|\nabla u|)dx\right). \end{aligned} \tag{3.3}$$

On the other hand, from the compact embedding $W^1L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$, we have

$$\begin{aligned} u_n &\to u \text{ in } L_\Psi(\Omega), \\ u_n(x) &\to u(x) \text{ a.e.} x \in \Omega. \end{aligned} \tag{3.4}$$

By (F1), (F2) and (3.4), we have $F(u_n(x)) \to F(u(x))$ a.e. $x \in \Omega$, and $|F(u_n)| \leq C$, for all $n \in \mathbb{N}$. These information with the Lebesgue dominated theorem imply

$$\int_\Omega F(u_n)dx \to \int_\Omega F(u)dx. \tag{3.5}$$

Since $h \in L_{\bar{\Phi}}(\Omega)$ and $u_n \rightharpoonup u$ in $W^1L_\Phi(\Omega)$, we have

$$\int_\Omega h u_n dx \to \int_\Omega h u dx. \tag{3.6}$$

From (3.2), (3.3), (3.5) and (3.6), we conclude that

$$J(u) \leq \liminf_{n\to\infty} J(u_n) \tag{3.7}$$

that is, functional J is weakly lower semi-continuous. This completes the proof.

Proof. [Proof of Theorem 3.2] Note that J is weakly lower semi-continuous and bounded from below. From Ekeland variational principle, we have $(u_n) \subset W^1L_\Phi(\Omega)$ such that

$$J(u_n) \to \inf_{W^1L_\Phi(\Omega)} J \text{ and } J'(u_n) \to 0. \tag{3.8}$$

By decomposition of $W^1L_\Phi(\Omega)$, for each $n\in\mathbb{N}$, we can rewrite $(u_n)\subset W^1L_\Phi(\Omega)$ as $u_n=u_0^n+\alpha$, where $u_0^n\in W_0$ with $\int_\Omega u_0^n dx=0$ and $\alpha\in\mathbb{R}$. Additionally, we find from (3.8) that $|J(u_n)|\leq c_4$. Therefore, by Lemma 3.3, it follows

$$c_5\leq J(u_n)\leq c_6$$

The last inequality above means $\rho(\nabla u_0^n)$ is bounded. Employing Lemma 3.3 and Lemma 2.3, it is obvious that sequence $\rho(u_0^n)$ is also bounded. Therefore, (u_0^n) is bounded in $W^1L_\Phi(\Omega)$, which means in turn that (u_n) is bounded in $W^1L_\Phi(\Omega)$. Hence, for a convenient subsequence (still denoted by u_n), we have $u_n\rightharpoonup\tilde{u}$ in $W^1L_\Phi(\Omega)$. Now, using the fact that functional J is weakly lower semi-continuous, we obtain

$$\begin{aligned}\inf_{W^1L_\Phi(\Omega)} J &= \liminf_{n\to\infty} J(u_n)\\ &= \liminf_{n\to\infty}\left(\hat{M}\left(\int_\Omega \Phi(|\nabla u_n|)dx\right)-\int_\Omega F(u_n)dx-\int_\Omega hu_n dx\right)\\ &\geq J(\tilde{u})=\hat{M}\left(\int_\Omega \Phi(|\nabla\tilde{u}|)dx\right)-\int_\Omega F(\tilde{u})dx-\int_\Omega h\tilde{u}dx,\end{aligned}$$

which means that

$$\inf_{W^1L_\Phi(\Omega)} J = J(\tilde{u}).$$

Thus, $\tilde{u}\in W^1L_\Phi(\Omega)$ is a weak solution of problem (1.1).

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