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COMMON FIXED POINT RESULTS FOR QUASI-CONTRACTIONS OF CIRIC TYPE IN b-METRIC SPACES WITH Qt-FUNCTIONS

XIAO-YUE ZHAO, FEI HE*, YU-QI SUN

School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China

Abstract. In this paper, common fixed point theorems for Ciric type quasi-contractive mappings in b-metric spaces with Qt-functions are established. An example is also provided to support the common fixed point theorems. The main results presented in this paper improve and extend the corresponding results announced recently.

Keywords. b-metric space; Qt-function; Common fixed point; Quasi-contraction.

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1. Introduction and preliminaries

In 1974, Ciric [1] introduced the concept of quasi-contractions, which is one of the most general contractive type mappings and proved quasi-contractive mappings have a unique fixed point in the framework of complete metric spaces. Subsequently, many authors considered the generalizations of this type fixed point theorem; see, for example, [2, 3, 4, 5, 6] and the references therein.

The concept of b-metric spaces was introduced and studied by Bakhtin [7] and Czerwik [8]. Since then, many fixed point results in b-metric space have been established by researchers; see, for example, [9, 10, 11, 12, 13, 14] and the references therein. Recently, Hussain, Saadati and Agrawal [15] introduced the concept of the wt-distance in b-metric spaces and established some fixed point results with the wt-distance.

*Corresponding author.

E-mail address: hefei@imu.edu.cn (F. He).

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In this paper, we introduce the concept of a Qt-function defined on a b-metric space which generalizes the notion of a wt-distance. Some common fixed point theorems for Ciric quasi-contractive mappings in complete b-metric spaces with a Qt-function are established. These results obtained in this paper improve and unify the results of Ilic and Rakoevic [4] and Amini-Harandi [13]. Particularly, we show the condition of the Fatou property in the result of Amini-Harandi [13] may be removed. Meanwhile, we also establish some common fixed point theorems for four self-mappings in b-metric spaces with a Qt-function, which improve the result of Roshan *et al.* [16].

Now let us recall some basic definitions and facts about b-metric spaces.

Definition 1.1. [7, 8, 15] Let X be a nonempty set and let $K \geq 1$ be a given real number. A function $D : X \times X \to [0,\infty)$ is called a b-metric on X if the following conditions hold:

(b1) $D(x,y) = 0$ if and only if $x = y$;

(b2) $D(x,y) = D(y,x)$;

(b3) $D(x,y) \leq K(D(x,z) + D(z,y))$,

for all $x,y,z \in X$. In this case, (X,D) is called a b-metric space (or a metric type space).

It is clear that a metric space is a b-metric space with $K = 1$. However, the converse is not true; see [8, 13, 15] and the references therein.

Definition 1.2. [15] Let (X,D) be a b-metric space.

(1) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to\infty} D(x_n,x) = 0$;

(2) The sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} D(x_n,x_m) = 0$;

(3) (X,D) is complete if and only if every Cauchy sequence in (X,D) is convergent.

Next, we introduce the concept of a Qt-function on b-metric spaces.

Definition 1.3. Let (X,D) be a b-metric space with constant $K \geq 1$. A function $P : X \times X \to [0,\infty)$ is called a Qt-function on X if the following are satisfied:

(q1) $P(x,z) \leq K(P(x,y) + P(y,z))$, for any $x,y,z \in X$;

(q2) if $x \in X$ and $\{y_n\}$ is a sequence in X such that it converges to a point y and $P(x,y_n) \leq M$ for some $M = M(x) > 0$, then $P(x,y) \leq KM$;

(q3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P(z,x) \leq \delta$ and $P(z,y) \leq \delta$ imply $D(x,y) \leq \varepsilon$.

Remark 1.4. (1) It is obvious that the Q-function in [17] is coincident with the Qt-function with $K = 1$.

(2) If condition (q2) in Definition is replaced by the following stronger condition:

(q2$'$) for any $x \in X$, $P(x,\cdot) : X \to [0,\infty)$ is K-lower semi-continuous,

then the Qt-function is called a wt-distance on X; see [15]. It is easy to see that every b-metric is a wt-distance and every wt-distance is a Qt-function; see [15] and [17].

Now, we give some properties of a Qt-function which are similar to the properties of a wt-distance, see, for example, [15].

Lemma 1.5. *Let (X,D) be a b-metric space with constant $K \geq 1$ and let P be a Qt-function on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero. Let $x,y,z \in X$. Then the following hold:*

(i) *If $P(x_n,y) \leq \alpha_n$ and $P(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $P(x,y) = 0$ and $P(x,z) = 0$, then $y = z$.*

(ii) *If $P(x_n,y_n) \leq \alpha_n$ and $P(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $D(y_n,z) \to 0$.*

(iii) *If $P(x_n,x_m) \leq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

(iv) *If $P(y,x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Definition 1.6. [18] Let f and g be two self-maps on a nonempty set X. If $w = fx = gx$ for some $x \in X$, then x is called the coincidence point of f and g. If f and g commute at every coincidence point, then they are said to be weakly compatible.

2. Common fixed points of two self-mappings

Let (X,D) be a b-metric space and let P be a Qt-function on X. For $E \subseteq X$, we denote $\delta_P(E) = \sup\{P(x,y) : x,y \in E\}$. If $f(X) \subseteq g(X)$ and $x_0 \in X$, we define $x_1 \in X$ such that $fx_0 = gx_1$. In view of $x_n \in X$, let $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$. Denote

$$\mathcal{O}(x_0,n) = \{fx_0, fx_1, \cdots, fx_n\},$$

$$\mathcal{O}(x_0,\infty) = \{fx_0, fx_1, \cdots\},$$

$$\mathcal{O}(x_n,\infty) = \{fx_n, fx_{n+1}, \cdots\}.$$

Lemma 2.1. *Let (X,D) be a b-metric space with constant $K \geq 1$ and let P a Qt-function on X. Let $f,g : X \to X$ such that $f(X) \subseteq g(X)$. Suppose that there exists a constant $\lambda \in [0,\frac{1}{K})$ such that for every $x,y \in X$*

$$\begin{aligned} P(fx,fy) \leq \lambda \max\{&P(gx,gy),P(gy,gx),P(gx,fx),P(fx,gx),P(gy,fy),P(fy,gy),\\ &P(gx,fy),P(fy,gx),P(gy,fx),P(fx,gy),P(gx,gx),P(gy,gy)\}. \end{aligned} \tag{2.1}$$

For $x_0 \in X$, let $x_1 \in X$ such that $fx_0 = gx_1$. Let $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$.

Then the following statements hold:

(i) *For each $x_0 \in X, n \in \mathbb{N}$ and $i, j \in \mathbb{N}$ with $i, j \le n$, we have*

$$P(fx_i, fx_j) \le \lambda \delta_P(\mathscr{O}(x_0, n)). \tag{2.2}$$

(ii) *For each $x_0 \in X$ and $n \in \mathbb{N}$, there exist $l, k \in \mathbb{N}$ with $l, k \le n$ such that*

$$\delta_P(\mathscr{O}(x_0, n)) = \max\{P(fx_0, fx_0), P(fx_0, fx_k), P(fx_l, fx_0)\}.$$

(iii) *For each $x_0 \in X$, $\delta_P(\mathscr{O}(x_0, n)) \le \frac{K}{1-K\lambda} \cdot a(x_0)$, where*

$$a(x_0) = P(fx_0, fx_0) + P(fx_0, fx_1) + P(fx_1, fx_0).$$

(iv) $\delta_P(\mathscr{O}(x_0, \infty)) \le \frac{K}{1-K\lambda} \cdot a(x_0)$.

(v) $\delta_P(\mathscr{O}(x_n, \infty)) \to 0$ *as $n \to \infty$.*

(vi) *For each $x \in X$, $\{fx_n\}_{n=1}^{\infty}$ is a Cauchy sequence. If $\{fx_n\}_{n=1}^{\infty}$ converges to $y \in X$, then*

$$P(fx_n, y) \le \lambda^n \frac{K^2}{1-K\lambda} \cdot a(x_0). \tag{2.3}$$

Proof. (i) Let $x_0 \in X, n \in \mathbb{N}$ and $i, j \in \mathbb{N}$ with $i, j \le n$. Using (2.1), we get

$$\begin{aligned}
P(fx_i, fx_j) &\le \lambda \max\{P(gx_i, gx_j), P(gx_j, gx_i), P(gx_i, fx_i), P(fx_i, gx_i), P(gx_j, fx_j), P(fx_j, gx_j),\\
&\quad P(gx_i, fx_j), P(fx_j, gx_i), P(gx_j, fx_i), P(fx_i, gx_j), P(gx_i, gx_i), P(gx_j, gx_j)\}\\
&= \lambda \max\{P(fx_{i-1}, fx_{j-1}), P(fx_{j-1}, fx_{i-1}), P(fx_{i-1}, fx_i), P(fx_i, fx_{i-1}),\\
&\quad P(fx_{j-1}, fx_j), P(fx_j, fx_{j-1}), P(fx_{i-1}, fx_j), P(fx_j, fx_{i-1}),\\
&\quad P(fx_{j-1}, fx_i), P(fx_i, fx_{j-1}), P(fx_{i-1}, fx_{i-1}), P(fx_{j-1}, fx_{j-1})\}\\
&\le \lambda \delta_P(\mathscr{O}(x_0, n)) < \delta_P(\mathscr{O}(x_0, n)).
\end{aligned}$$

(ii) Clearly, (i) implies (ii).

(iii) From (ii), it follows that there exist $k, l \in \mathbb{N}$ with $1 \le k, l \le n$ such that

$$\delta_P(\mathscr{O}(x_0, n)) = \max\{P(fx_0, fx_0), P(fx_0, fx_k), P(fx_l, fx_0)\}.$$

If $\delta_P(\mathscr{O}(x_0,n)) = P(fx_0,fx_0)$, we have $\delta_P(\mathscr{O}(x_0,n)) \leq \frac{K}{1-K\lambda}P(fx_0,fx_0)$. If $\delta_P(\mathscr{O}(x_0,n)) = P(fx_0,fx_k)$, then

$$\begin{aligned}\delta_P(\mathscr{O}(x_0,n)) = P(fx_0,fx_k) &\leq K(P(fx_0,fx_1)+P(fx_1,fx_k))\\ &\leq K[P(fx_0,fx_1)+\lambda\,\delta_P(\mathscr{O}(x_0,n))],\end{aligned}$$

which implies $\delta_P(\mathscr{O}(x_0,n)) \leq \frac{K}{1-K\lambda}P(fx_0,fx_1)$. If $\delta_P(\mathscr{O}(x_0,n)) = P(fx_l,fx_0)$, then

$$\begin{aligned}\delta_P(\mathscr{O}(x_0,n)) = P(fx_l,fx_0) &\leq K(P(fx_l,fx_1)+P(fx_1,fx_0))\\ &\leq K[P(fx_1,fx_0)+\lambda\,\delta_P(\mathscr{O}(x_0,n))],\end{aligned}$$

which implies $\delta_P(\mathscr{O}(x_0,n)) \leq \frac{K}{1-K\lambda}P(fx_1,fx_0)$. Thus we proved (iii).

(iv) For any $i,j \geq 0$, we take $n = i+j$. Then we have $i,j \leq n$ and $fx_i, fx_j \in \mathscr{O}(x_0,n)$. From (iii), we see that

$$P(fx_i,fx_j) \leq \delta_P(\mathscr{O}(x_0,n)) \leq \frac{K}{1-K\lambda}\cdot a(x_0),$$

which implies

$$\delta_P(\mathscr{O}(x_0,\infty)) \leq \frac{K}{1-K\lambda}\cdot a(x_0).$$

(v) For any $i,j \geq n \geq 1$, $P(fx_i,fx_j) \leq \lambda\,\delta_P(\mathscr{O}(x_{n-1},\infty))$ implies

$$\delta_P(\mathscr{O}(x_n,\infty)) \leq \lambda\,\delta_P(\mathscr{O}(x_{n-1},\infty)).$$

Therefore, we have

$$\begin{aligned}\delta_P(\mathscr{O}(x_n,\infty)) &\leq \lambda\,\delta_P(\mathscr{O}(x_{n-1},\infty))\\ &\leq \lambda^2\delta_P(\mathscr{O}(x_{n-2},\infty))\\ &\leq \cdots\\ &\leq \lambda^n\delta_P(\mathscr{O}(x_0,\infty))\\ &\leq \lambda^n\frac{K}{1-K\lambda}\cdot a(x_0).\end{aligned}$$

Since $\lambda < 1$, we have $\lim_{n\to\infty}\delta_P(\mathscr{O}(x_n,\infty)) = 0$, that is, (v) holds.

(vi) For any $m,n \in \mathbb{N}$ with $m > n$,

$$P(fx_n,fx_m) \leq \delta_P(\mathscr{O}(x_n,\infty)) \leq \lambda^n\frac{K}{1-K\lambda}\cdot a(x_0) \to 0.$$

From Lemma 1.5 (iii), we find that $\{fx_n\}$ is a Cauchy sequence. If it converges to $y \in X$, then we get from (q2) that

$$P(fx_n,y) \leq K\cdot\lambda^n\frac{K}{1-K\lambda}\cdot a(x_0) = \lambda^n\frac{K^2}{1-K\lambda}\cdot a(x_0).$$

This completes the proof.

Theorem 2.2. *Let (X,D) be a complete b-metric space with constant $K \geq 1$ and let P be a Qt-function on X. Let $f,g : X \to X$ such that $f(X) \subseteq g(X)$. Suppose that there exists $\lambda \in [0, \frac{1}{K})$ satisfying condition (2.1) and, for every $y \in X$ with $fy \neq gy$,*

$$\inf\{P(gx,y)+P(gx,fx) : x \in X\} > 0. \tag{2.4}$$

If f and g are weakly compatible, then f and g have a unique common fixed point $u \in X$ and $P(u,u) = 0$.

Proof. Let $x_0 \in X$ be a given point. Since $f(X) \subseteq g(X)$, we choose $x_1 \in X$ such that $fx_0 = gx_1$. If $x_n \in X$ is well defined, we can choose $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$. By induction, we construct a sequence $\{x_n\}$ in X such that $fx_n = gx_{n+1}$, $n = 0,1,2\cdots$. Set $y_n = fx_n$. Using Lemma 2.1 (vi), we see that $\{y_n\}$ is a Cauchy sequence. Since X is a complete b-metric space, we find that there exists $y \in X$ such that $\{y_n\}$ converges to y. Let us prove that $fy = gy$. If $fy \neq gy$, then we deduce that

$$\begin{aligned} 0 &< \inf\{P(gx,y)+P(gx,fx) : x \in X\} \\ &\leq \inf\{P(gx_n,y)+P(gx_n,fx_n) : n \in \mathbb{N}\} \\ &= \inf\{P(fx_{n-1},y)+P(fx_{n-1},fx_n) : n \in \mathbb{N}\} \\ &\leq \frac{2K^2a(x_0)}{1-K\lambda} \cdot \inf\{\lambda^n : n \in \mathbb{N}\} = 0. \end{aligned}$$

This is a contradiction. Hence $fy = gy$. Since f and g are weakly compatible, we have $fgy = gfy$. If we denote $u = fy = gy$, then $fu = gu$.

Next, we claim that u is a common fixed point of f and g. Using (2.1), we get

$$P(u,u) = P(fy,fy) \leq \lambda P(u,u),$$

which implies $P(u,u) = 0$. Similarly, $P(fu,fu) = 0$. It also from (2.1) that

$$\begin{aligned} P(u,fu) = P(fy,fu) &\leq \lambda \max\{P(u,fu),P(fu,u),P(fu,fu),P(u,u)\} \\ &= \lambda \max\{P(u,fu),P(fu,u)\}. \end{aligned}$$

In the same way, we can prove

$$P(fu,u) \leq \lambda \max\{P(u,fu),P(fu,u)\}.$$

This implies $P(u,fu) = 0$ and $P(fu,u) = 0$. Thus, by Lemma 1.5 (i), we get $u = fu = gu$, that is, u is a fixed point of f and g.

Now, we are in a position to prove u is a unique common fixed point of f and g. Suppose that there exists another point $v \in X$ such that $fv = gv = v$. In view of (2.1), we have

$$P(v,v) = P(fv, fv) \leq \lambda P(v,v),$$

which implies $P(v,v) = 0$. It also follows from (2.1) that

$$\begin{aligned} P(u,v) = P(fu, fv) &\leq \lambda \max\{P(u,v), P(v,u), P(u,u), P(v,v)\} \\ &= \lambda \max\{P(u,v), P(v,u)\}. \end{aligned}$$

Similarly, we have

$$P(v,u) \leq \lambda \max\{P(u,v), P(v,u)\}.$$

Thus $P(u,v) = P(v,u) = 0$. In view of Lemma 1.5 (i), we get $u = v$. This completes the proof.

Theorem 2.3. *Let (X,D) be a b-metric space with constant $K \geq 1$ and let P be a Qt-function on X. Let $f,g : X \to X$ be two weakly compatible self-mappings and $f(X) \subseteq g(X)$. Suppose that there exists $\lambda \in [0, \frac{1}{K})$ satisfying condition (2.1). Suppose also that, for every $z \in X$ with $fz \neq gz$,*

$$\inf\{P(gx, gz) + P(gx, fx) : x \in X\} > 0. \tag{2.5}$$

If $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique common fixed point $u \in X$ and $P(u,u) = 0$.

Proof. Let $x_0 \in X$ be fixed. Similar to the proof of Theorem 2.2, we can construct a sequence $\{x_n\}$ such that $fx_n = gx_{n+1}$, $n = 0, 1, \cdots$. Using Lemma 2.1 (vi), we see that $\{fx_n\}$ is a Cauchy sequence. Since $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is complete, there exists $y \in g(X)$ such that $\{fx_n\}$ converges to y. Let $z \in X$ such that $y = gz$. Next, we prove $fz = gz$. If $fz \neq gz$, then

$$\begin{aligned} 0 &< \inf\{P(gx, gz) + P(gx, fx) : x \in X\} \\ &\leq \inf\{P(gx_n, gz) + P(gx_n, fx_n) : n \in \mathbb{N}\} \\ &= \inf\{P(fx_{n-1}, gz) + P(fx_{n-1}, fx_n) : n \in \mathbb{N}\} \\ &\leq \frac{2K^2 a(x_0)}{1 - K\lambda} \cdot \inf\{\lambda^n : n \in \mathbb{N}\} = 0. \end{aligned}$$

This is a contradiction. Hence $fz = gz$. Similar to the proof of Theorem 2.2, we can prove $u = fz = gz$, which is a unique common fixed point of f and g and $P(u,u) = 0$. This completes the proof.

From Theorem 2.2 and Theorem 2.3, we can get the following results which generalize the result of Ilić and Rakŏević [4].

Corollary 2.4. *Let (X,d) be a complete metric space and let p be a Q-function (or a w-distance) on X. Let $f,g: X\to X$ such that $f(X)\subseteq g(X)$. Suppose that there exists $\lambda\in[0,1)$ such that, for every $x,y\in X$,*

$$p(fx,fy)\le\lambda\max\{p(gx,gy),p(gy,gx),p(gx,fx),p(fx,gx),p(gy,fy),p(fy,gy),\\ p(gx,fy),p(fy,gx),p(gy,fx),p(fx,gy),p(gx,gx),p(gy,gy)\}.$$

and, for every $y\in X$ with $fy\neq gy$,

$$\inf\{p(gx,y)+p(gx,fx):x\in X\}>0.$$

If f and g are weakly compatible, then f and g have a unique common fixed point $u\in X$ and $p(u,u)=0$.

Corollary 2.5. *Let (X,d) be a metric space and let p be a Q-function (or a w-distance) on X. Let $f,g: X\to X$ such that $f(X)\subseteq g(X)$. Suppose that there exists $\lambda\in[0,1)$ such that, for every $x,y\in X$,*

$$p(fx,fy)\le\lambda\max\{p(gx,gy),p(gy,gx),p(gx,fx),p(fx,gx),p(gy,fy),p(fy,gy),\\ p(gx,fy),p(fy,gx),p(gy,fx),p(fx,gy),p(gx,gx),p(gy,gy)\}.$$

Suppose also that $f(X)$ or $g(X)$ is a complete subspace of X and, for every $z\in X$ with $fz\neq gz$,

$$\inf\{p(gx,gz)+p(gx,fx):x\in X\}>0.$$

If f and g are weakly compatible, then f and g have a unique common fixed point $u\in X$ and $p(u,u)=0$.

Next, we consider a special case of Theorem 2.3 by replacing Qt-function P in the condition (2.1) with b-metric D. To this end, we need the following lemma.

Lemma 2.6. *Let (X,D) be a b-metric space with constant $K\ge1$ and let $f,g: X\to X$ be two self-mappings with $f(X)\subseteq g(X)$. If there exists $\lambda\in[0,\frac{1}{K})$ such that, for every $x,y\in X$,*

$$D(fx,fy)\le\lambda\max\{D(gx,gy),D(gx,fx),D(gy,fy),D(gx,fy),D(gy,fx)\},\tag{2.6}$$

then, for every $z\in X$ with $fz\neq gz$,

$$\inf\{D(gx,gz)+D(gx,fx):x\in X\}>0.$$

Proof. Suppose that there exists $z\in X$ with $fz\neq gz$ such that

$$\inf\{D(gx,gz)+D(gx,fx):x\in X\}=0.$$

Then there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty}[D(gx_n, gz) + D(gx_n, fx_n)] = 0.$$

From this, we see that $D(gx_n, gz) \to 0$ and $D(gx_n, fx_n) \to 0$. In view of

$$D(gz, fx_n) \le K(D(gz, gx_n) + D(gx_n, fx_n)),$$

we have $D(gz, fx_n) \to 0$ as $n \to \infty$. Next, we prove $D(fx_n, fz) \to 0$. Using (2.6), we have

$$D(fx_n, fz) \le \lambda \max\{D(gx_n, gz), D(gx_n, fx_n), D(gz, fz), D(gx_n, fz), D(gz, fx_n)\}.$$

Notice that the following two facts.

(i) $D(fx_n, fz) \le \lambda D(gz, fz) \le K\lambda D(gz, fx_n) + K\lambda D(fx_n, fz)$ implies

$$D(fx_n, fz) \le \frac{K\lambda}{1-K\lambda} D(gz, fx_n).$$

(ii) $D(fx_n, fz) \le \lambda D(gx_n, fz) \le K\lambda D(gx_n, fx_n) + K\lambda D(fx_n, fz)$ implies

$$D(fx_n, fz) \le \frac{K\lambda}{1-K\lambda} D(gx_n, fx_n).$$

It follows that

$$\begin{aligned} D(fx_n, fz) &\le \lambda \max\{D(gx_n, gz), D(gx_n, fx_n), \frac{K}{1-K\lambda} D(gz, fx_n), \\ &\quad \frac{K}{1-K\lambda} D(gx_n, fx_n), D(gz, fx_n)\} \\ &= \lambda \max\{D(gx_n, gz), \frac{K}{1-K\lambda} D(gz, fx_n), \frac{K}{1-K\lambda} D(gx_n, fx_n)\}, \end{aligned}$$

which implies $D(fx_n, fz) \to 0$. Since $D(gz, fz) \le K(D(gz, fx_n) + D(fx_n, fz))$, we find that $D(gz, fz) = 0$ and $fz = gz$. This is a contradiction. The proof is complete.

From Theorem 2.3 and Lemma 2.6, we obtain the following result.

Corollary 2.7. *Let (X, D) be a b-metric space with constant $K \ge 1$ and let $f, g : X \to X$ be two self-mappings with $f(X) \subseteq g(X)$. Suppose that there exists $\lambda \in [0, \frac{1}{K})$ satisfying the condition (2.6). If $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Setting $g = I$ in Corollary 2.7, where $I : X \to X$ is a identity mapping, we get the following result.

Corollary 2.8. *Let (X,D) be a b-metric space with constant $K \geq 1$ and let $f : X \to X$ be a self-mapping. Suppose that there exists $\lambda \in [0, \frac{1}{K})$ such that for every $x,y \in X$,*

$$D(fx,fy) \leq \lambda \max\{D(x,y), D(x,fx), D(y,fy), D(x,fy), D(y,fx)\}. \tag{2.7}$$

Then f have a unique fixed point in X.

Remark 2.9. In [13], Amini-Harandi first gave the fixed point theorem of Ćirić type in b-metric spaces, where the b-metric satisfies the Fatou property. From Corollary 2.8, we see that the condition of Fatou property can be removed.

3. Common fixed points of four self-mappings

Now we give a common fixed point result for four self-mappings in b-metric spaces with a Qt-function.

Theorem 3.1. *Let (X,D) be a b-metric space with constant $K \geq 1$ and let P be a Qt-function on X satisfying $P(x,x) = 0$ for all $x \in X$. Let F,T,S and H be four self maps on X such that $F(X) \subseteq H(X)$ and $T(X) \subseteq S(X)$. Suppose that there exists $\lambda \in [0, \frac{1}{K})$ such that, for every $x,y \in X$,*

$$\begin{aligned}
&\max\{P(Fx,Ty), P(Ty,Fx)\} \\
&\leq \lambda \max\{P(Sx,Hy), P(Hy,Sx), P(Sx,Fx), P(Fx,Sx), P(Hy,Ty), P(Ty,Hy), \\
&\quad \frac{P(Sx,Ty)+P(Fx,Hy)}{2}, \frac{P(Ty,Sx)+P(Hy,Fx)}{2}\}.
\end{aligned} \tag{3.1}$$

Suppose also that

(i) *for every $z \in X$ with $Tz \neq Hz$,* $\inf\{P(Hx,Hz) + P(Hx,Tx) : x \in X\} > 0$;

(ii) *for every $z \in X$ with $Sz \neq Fz$,* $\inf\{P(Sx,Sz) + P(Sx,Fx) : x \in X\} > 0$.

If the range of one of F,T,S and H is a complete subspace of X, then

(1) *F and S have a coincidence point,*

(2) *T and H have a coincidence point.*

If, moreover, $\{F,S\}$ and $\{T,H\}$ are weakly compatible, then F,T,S and H have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $F(X) \subseteq H(X)$, we see that there exists $x_1 \in X$ such that $Hx_1 = Fx_0$. Since $T(X) \subseteq S(X)$, we find that there exists $x_2 \in X$ such that $Sx_2 = Tx_1$.

Continuing this process, we can construct $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Hx_{2n+1} = Fx_{2n}, \;\; y_{2n+1} = Sx_{2n+2} = Tx_{2n+1}, \;\; n = 0, 1, 2, \cdots . \tag{3.2}$$

Denote

$$D_n = \max\{P(y_i, y_j) : 0 \le i, j \le n, i, j \in \mathbb{N}\}, D_\infty = \sup\{P(y_i, y_j) : i, j \in \mathbb{N}\},$$

$$\delta_n = \sup\{P(y_i, y_j) : i, j \ge n, i, j \in \mathbb{N}\}.$$

It is obvious that $\delta_0 = D_\infty$. Next, we complete our proof in the following six steps.

Step 1. We prove that $\max\{P(y_n, y_{n+1}), P(y_{n+1}, y_n)\} \to 0$ as $n \to \infty$.

Using (3.1), we obtain

$$\begin{aligned}
& \max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\} \\
= \;& \max\{P(Fx_{2n}, Tx_{2n+1}), P(Tx_{2n+1}, Fx_{2n})\} \\
\le \;& \lambda \max\{P(Sx_{2n}, Hx_{2n+1}), P(Hx_{2n+1}, Sx_{2n}), P(Sx_{2n}, Fx_{2n}), \\
& P(Fx_{2n}, Sx_{2n}), P(Hx_{2n+1}, Tx_{2n+1}), P(Tx_{2n+1}, Hx_{2n+1}), \\
& \frac{P(Sx_{2n}, Tx_{2n+1}) + P(Fx_{2n}, Hx_{2n+1})}{2}, \frac{P(Tx_{2n+1}, Sx_{2n}) + P(Hx_{2n+1}, Fx_{2n})}{2}\} \\
= \;& \lambda \max\{P(y_{2n-1}, y_{2n}), P(y_{2n}, y_{2n-1}), P(y_{2n}, y_{2n+1}), \\
& P(y_{2n+1}, y_{2n}), \frac{1}{2}P(y_{2n-1}, y_{2n+1}), \frac{1}{2}P(y_{2n+1}, y_{2n-1})\}.
\end{aligned}$$

Hence, we get the following three cases.

Case i. $\max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\} \le \lambda \max\{P(y_{2n-1}, y_{2n}), P(y_{2n}, y_{2n-1})\}$.

Case ii. $\max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\} \le \lambda \max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\}$.

Case iii.

$$\begin{aligned}
& \max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\} \\
\le \;& \frac{\lambda}{2} \max\{P(y_{2n-1}, y_{2n+1}), P(y_{2n+1}, y_{2n-1})\} \\
\le \;& \frac{\lambda K}{2} \max\{P(y_{2n-1}, y_{2n}) + P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n}) + P(y_{2n}, y_{2n-1})\} \\
\le \;& \frac{\lambda K}{2} [\max\{P(y_{2n-1}, y_{2n}), P(y_{2n}, y_{2n-1})\} + \max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\}],
\end{aligned}$$

which in turn implies

$$\max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\} \le \frac{\lambda K}{2 - \lambda K} \max\{P(y_{2n-1}, y_{2n}), P(y_{2n}, y_{2n-1})\}.$$

Letting $\alpha = \max\{\lambda, \frac{\lambda K}{2-\lambda K}\}$, we find that $0 < \alpha < 1$ and

$$\max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\} \leq \alpha \max\{P(y_{2n-1}, y_{2n}), P(y_{2n}, y_{2n-1})\}.$$

Similarly, we also have

$$\max\{P(y_{2n+1}, y_{2n+2}), P(y_{2n+2}, y_{2n+1})\} \leq \alpha \max\{P(y_{2n}, y_{2n+1}), P(y_{2n+1}, y_{2n})\}.$$

It follows that

$$\max\{P(y_n, y_{n+1}), P(y_{n+1}, y_n)\} \leq \alpha \max\{P(y_{n-1}, y_n), P(y_n, y_{n-1})\}, \tag{3.3}$$

for all $n \geq 1$. From $0 < \alpha < 1$, we see that

$$\max\{P(y_n, y_{n+1}), P(y_{n+1}, y_n)\} \leq \max\{P(y_{n-1}, y_n), P(y_n, y_{n-1})\}. \tag{3.4}$$

Using (3.3), we have

$$\begin{aligned}\max\{P(y_n, y_{n+1}), P(y_{n+1}, y_n)\} \leq& \alpha \max\{P(y_{n-1}, y_n), P(y_n, y_{n-1})\}\\ \leq& \cdots\\ \leq& \alpha^n \max\{P(y_0, y_1), P(y_1, y_0)\}.\end{aligned}$$

Since $0 < \alpha < 1$, $\max\{P(y_n, y_{n+1}), P(y_{n+1}, y_n)\} \to 0 (n \to \infty)$.

Step 2. We prove that $\{D_n\}$ is bounded and $D_\infty < +\infty$.

Let $n \in \mathbb{N}$ be given. For any $i, j \in \mathbb{N}$ with $0 \leq i < j \leq n$, we consider the following three cases.

Case i. If $i - j \equiv 1 \mod 2$ and $i, j \geq 1$, then $i = 2r$ and $j = 2s+1$ for some $r, s \in \mathbb{N}$. Using (3.1) and (3.2), we have

$$\begin{aligned}&\max\{P(y_i, y_j), P(y_j, y_i)\}\\ =&\max\{P(Fx_{2r}, Tx_{2s+1}), P(Tx_{2s+1}, Fx_{2r})\}\\ \leq&\lambda \max\{P(Sx_{2r}, Hx_{2s+1}), P(Hx_{2s+1}, Sx_{2r}), P(Sx_{2r}, Fx_{2r}), P(Fx_{2r}, Sx_{2r}),\\ &P(Hx_{2s+1}, Tx_{2s+1}), P(Tx_{2s+1}, Hx_{2s+1}), \frac{P(Sx_{2r}, Tx_{2s+1}) + P(Fx_{2r}, Hx_{2s+1})}{2},\\ &\frac{P(Tx_{2s+1}, Sx_{2r}) + P(Hx_{2s+1}, Fx_{2r})}{2}\}\\ =&\lambda \max\{P(y_{i-1}, y_{j-1}), P(y_{j-1}, y_{i-1}), P(y_{i-1}, y_i), P(y_i, y_{i-1}), P(y_{j-1}, y_j),\\ &P(y_j, y_{j-1}), \frac{P(y_{i-1}, y_j) + P(y_i, y_{j-1})}{2}, \frac{P(y_j, y_{i-1}) + P(y_{j-1}, y_i)}{2}\}\\ \leq&\lambda D_{n-1}.\end{aligned} \tag{3.5}$$

Case ii. If $i-j\equiv 0 \bmod 2$, we find from (3.4) and (3.5) that

$$\begin{aligned}
\max\{P(y_i,y_j),P(y_j,y_i)\} &\leq K\max\{P(y_i,y_{i+1})+P(y_{i+1},y_j),P(y_j,y_{i+1})+P(y_{i+1},y_i)\}\\
&\leq K\max\{P(y_i,y_{i+1}),P(y_{i+1},y_i\}+K\max\{P(y_{i+1},y_j),P(y_j,y_{i+1})\}\\
&\leq K\max\{P(y_0,y_1),P(y_1,y_0)\}+K\lambda D_n.
\end{aligned}$$

Case iii. If $i=0$ and $j=2r-1$ for some $r\in\mathbb{N}$, we find from (3.5) that

$$\begin{aligned}
\max\{P(y_i,y_j),P(y_j,y_i)\} &= \max\{P(y_0,y_{2r-1}),P(y_{2r-1},y_0)\}\\
&\leq K\max\{P(y_0,y_2)+P(y_2,y_{2r-1}),P(y_{2r-1},y_2)+P(y_2,y_0)\}\\
&\leq K\max\{P(y_0,y_2),P(y_2,y_0)\}+K\max\{P(y_2,y_{2r-1}),P(y_{2r-1},y_2)\}\\
&\leq K\max\{P(y_0,y_2),P(y_2,y_0)\}+K\lambda D_n.
\end{aligned}$$

Thus we obtain, for any $i,j\in\mathbb{N}$ with $0\leq i<j\leq n$,

$$\max\{P(y_i,y_j),P(y_j,y_i)\}\leq K\max\{P(y_0,y_1),P(y_1,y_0),P(y_0,y_2),P(y_2,y_0)\}+K\lambda D_n.$$

It follows that $D_n\leq K\max\{P(y_0,y_1),P(y_1,y_0),P(y_0,y_2),P(y_2,y_0)\}+K\lambda D_n$, which implies

$$D_n\leq\frac{K}{1-K\lambda}\max\{P(y_0,y_1),P(y_1,y_0),P(y_0,y_2),P(y_2,y_0)\}.$$

Therefore, we get that $\{D_n\}$ is bounded and $D_\infty<+\infty$.

Step 3. We prove that $\delta_n\to 0$ as $n\to\infty$.

Letting $n\in\mathbb{N}$ be given, for any $i,j\in\mathbb{N}$ with $i,j\geq n$ and $i\neq j$, we consider the following three cases.

Case i. If $i-j\equiv 1 \bmod 2$, then $i=2r$ and $j=2s+1$ for some $r,s\in\mathbb{N}$. Using (3.1) and (3.2), we have

$$\begin{aligned}
&\max\{P(y_i,y_j),P(y_j,y_i)\}\\
\leq&\lambda\max\{P(Sx_{2r},Hx_{2s+1}),P(Hx_{2s+1},Sx_{2r}),P(Sx_{2r},Fx_{2r}),P(Fx_{2r},Sx_{2r}),\\
&P(Hx_{2s+1},Tx_{2s+1}),P(Tx_{2s+1},Hx_{2s+1}),\frac{P(Sx_{2r},Tx_{2s+1})+P(Fx_{2r},Hx_{2s+1})}{2},\\
&\frac{P(Tx_{2s+1},Sx_{2r})+P(Hx_{2s+1},Fx_{2r})}{2}\} \qquad (3.6)\\
=&\lambda\max\{P(y_{i-1},y_{j-1}),P(y_{j-1},y_{i-1}),P(y_{i-1},y_i),P(y_i,y_{i-1}),P(y_{j-1},y_j),\\
&P(y_j,y_{j-1}),\frac{P(y_{i-1},y_j)+P(y_i,y_{j-1})}{2},\frac{P(y_j,y_{i-1})+P(y_{j-1},y_i)}{2}\}\\
\leq&\lambda\delta_{n-1}.
\end{aligned}$$

It follows from (3.6) that

$$\begin{aligned}\max\{P(y_i,y_j),P(y_j,y_i)\} \leq & KP(y_i,y_{i+2})+KP(y_{i+2},y_j)\\ \leq & K^2P(y_i,y_{i+1})+K^2P(y_{i+1},y_{i+2})+KP(y_{i+2},y_j)\\ \leq & 2K^2\max\{P(y_n,y_{n+1}),P(y_{n+1},y_n)\}+K\lambda\delta_n.\end{aligned}$$

Case ii. If $i-j\equiv 0 \bmod 2$, we find from (3.6) that

$$\begin{aligned}& \max\{P(y_i,y_j),P(y_j,y_i)\}\\ \leq\; & K\max\{P(y_i,y_{i+1})+P(y_{i+1},y_j),P(y_j,y_{i+1})+P(y_{i+1},y_i)\}\\ \leq\; & K\max\{P(y_i,y_{i+1}),P(y_{i+1},y_i\}+K\max\{P(y_{i+1},y_j),P(y_j,y_{i+1})\}\\ \leq\; & 2K^2\max\{P(y_n,y_{n+1}),P(y_{n+1},y_n)\}+K\lambda\delta_n.\end{aligned}$$

Hence, we obtain

$$\max\{P(y_i,y_j),P(y_j,y_i)\}\leq 2K^2\max\{P(y_n,y_{n+1}),P(y_{n+1},y_n)\}+K\lambda\delta_n,$$

which implies $\delta_n\leq 2K^2\max\{P(y_n,y_{n+1}),P(y_{n+1},y_n)\}+K\lambda\delta_n$. It follows that

$$\delta_n\leq\frac{2K^2}{1-K\lambda}\max\{P(y_n,y_{n+1}),P(y_{n+1},y_n)\}\to 0 \text{ as } n\to\infty.$$

For any $m,n\in\mathbb{N}$ with $m>n$, $P(y_n,y_m)\leq\delta_n\to 0$ as $n\to 0$. By Lemma 1.5 (iii), we get that $\{y_n\}$ is a Cauchy sequence in X. This leads to $\{Fx_{2n}\}$, $\{Hx_{2n+1}\}$, $\{Tx_{2n+1}\}$ and $\{Sx_{2n+2}\}$ are also Cauchy sequences. We assume, without loss of generality, that $H(X)$ is a complete subspace of X. Then there exists $u\in X$ such that $\{Hx_{2n+1}\}$ converges to Hu. Hence, $\{y_n\},\{y_{2n}\}$ and $\{y_{2n+1}\}$ converge to Hu.

Step 4. We prove that T and H have a coincidence point and F and S have a coincidence point.

First, we prove $Tu=Hu$. Assume to the contrary that $Tu\neq Hu$. Using condition (i), one has

$$\begin{aligned}0&<\inf\{P(Hx,Hu)+P(Hx,Tx):x\in X\}\\ &\leq\inf\{P(Hx_{2n+1},Hu)+P(Hx_{2n+1},Tx_{2n+1}):n\in\mathbb{N}\}\\ &=\inf\{P(y_{2n},Hu)+P(y_{2n},y_{2n+1}):n\in\mathbb{N}\}\\ &=0.\end{aligned}$$

This is a contradiction. Hence, we get u is a coincidence point of T and H. Since $T(X)\subseteq S(X)$, we find that there exists $v\in X$ such that $Sv=Tu$. Now, we prove that $Sv=Fv$. If $Sv\neq Fv$, then we find from condition (ii) that

$$\begin{aligned}
0 &< \inf\{P(Sx,Sv)+P(Sx,Fx) : x \in X\} \\
&\leq \inf\{P(Sx_{2n+2},Sv)+P(Sx_{2n+2},Fx_{2n}) : n \in \mathbb{N}\} \\
&= \inf\{P(y_{2n+1},Sv)+P(y_{2n+1},y_{2n}) : n \in \mathbb{N}\} \\
&= 0.
\end{aligned}$$

This is a contradiction. Hence we get v is a coincidence point of F and S.

Step 5. Now, we assume that $\{F,S\}$ and $\{T,H\}$ are weakly compatible and we show that F,T,S and H have a common fixed point.

Letting $y = Tu = Hu = Sv = Fv$, we see that

$$Ty = TTu = THu = HTu = HHu = Hy, \tag{3.7}$$

and

$$Fy = FFv = FSv = SFv = SSv = Sy. \tag{3.8}$$

Next, we prove y is a common fixed point of F,T,S and H. Using (3.1), we get

$$\begin{aligned}
& \max\{P(y,Ty),P(Ty,y)\} \\
= & \max\{P(Fv,Ty),P(Ty,Fv)\} \\
\leq & \lambda \max\{P(Sv,Hy),P(Hy,Sv),P(Sv,Fv),P(Fv,Sv),P(Hy,Ty),P(Ty,Hy), \\
& \frac{P(Sv,Ty)+P(Fv,Hy)}{2}, \frac{P(Ty,Sv)+P(Hy,Fv)}{2}\} \\
= & \lambda \max\{P(y,Ty),P(Ty,y),P(y,y),P(Ty,Ty)\} \\
= & \lambda \max\{P(y,Ty),P(Ty,y)\}.
\end{aligned}$$

Since $\lambda < \frac{1}{K} \leq 1$, we have $P(y,Ty) = 0$ and $P(Ty,y) = 0$. By Lemma 1.5 (i) and $P(y,y) = 0$, we get $Ty = y$. From (3.7), one has $Ty = Hy = y$. In view of (3.1), we find

$$\begin{aligned}
\max\{P(Fy,y),P(y,Fy)\} =& \max\{P(Fy,Ty),P(Ty,Fy)\} \\
\leq& \lambda \max\{P(Sy,Hy),P(Hy,Sy),P(Sy,Fy),P(Fy,Sy),P(Hy,Ty), \\
& P(Ty,Hy), \frac{P(Sy,Ty)+P(Fy,Hy)}{2}, \frac{P(Ty,Sy)+P(Hy,Fy)}{2}\} \\
=& \lambda \max\{P(Fy,Ty),P(Ty,Fy),P(Fy,Fy),P(Ty,Ty)\} \\
=& \lambda \max\{P(Fy,y),P(y,Fy)\}.
\end{aligned}$$

Since $\lambda < \frac{1}{K} \leq 1$, we have $P(y,Fy) = 0$ and $P(Fy,y) = 0$. By Lemma 1.5 (i) and $P(Ty,y) = 0$, we get $Fy = y$. I follows from (3.8) that $Fy = Sy = y$. Therefore, $y = Fy = Ty = Sy = Hy$, that is, y is a common fixed point of F,T,S and H.

Step 6. To prove the uniqueness, we suppose that there exists another point $z \in X$ such that $Fz = Sz = Tz = Hz = z$. Using (3.1), we have

$$\begin{aligned}
\max\{P(y,z),P(z,y)\} =& \max\{P(Fy,Tz),P(Tz,Fy)\} \\
\leq& \lambda \max\{P(Sz,Hy),P(Hy,Sz),P(Sz,Fz),P(Fz,Sz),P(Hy,Ty), \\
& P(Ty,Hy), \frac{P(Sz,Ty)+P(Fz,Hy)}{2}, \frac{P(Ty,Sz)+P(Hy,Fz)}{2}\} \\
=& \lambda \max\{P(z,y),P(y,z),P(z,z),P(y,y),P(z,y),P(y,z)\} \\
=& \lambda \max\{P(z,y),P(y,z)\}.
\end{aligned}$$

Since $\lambda < \frac{1}{K} \leq 1$, we have $P(y,z) = 0$ and $P(z,y) = 0$. By Lemma 1.5 (i), we get $z = y$. Therefore, F,T,S and H have a unique common fixed point $y \in X$. This completes the proof.

Setting $P = D$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let (X,D) be a b-metric space with constant $K \geq 1$ and let F,T,S and H be four self maps on X such that $F(X) \subseteq H(X)$ and $T(X) \subseteq S(X)$. Suppose that there exists $\lambda \in [0, \frac{1}{K})$ such that, for every $x,y \in X$,*

$$(1) \qquad D(Fx,Ty) \leq \lambda \max\{D(Sx,Hy),D(Sx,Fx),D(Hy,Ty), \frac{D(Sx,Ty)+D(Fx,Hy)}{2}\}.$$

If the range of one of F,T,S and H is a complete subspace of X, then

(1) *F and S have a coincidence point,*

(2) *T and H have a coincidence point.*

If, moreover, $\{F,S\}$ and $\{T,H\}$ are weakly compatible, then F,T,S and H have a unique common fixed point in X.

Remark 3.3. Corollary 3.2 is a improvement of the result in the recent paper of Roshan *et al.*[16]. Particularly, the contractive coefficient $\frac{q}{K^4}(0 < q < 1)$ is enlarged to $\lambda(0 < \lambda < \frac{1}{K})$.

Next, we give an example which can apply to Corollary 3.2 but not the Theorem 2.1 in [16].

Example 3.4. Let $X = [0,1]$ and D be a b-metric on X defined as $D(x,y) = (x-y)^2$, where $K = 2$. Define F,S,T and H on X by

$$F(x) = (\frac{x}{2})^6, T(x) = (\frac{x}{2})^4,$$

$$H(x) = (\frac{x}{2})^3, S(x) = (\frac{x}{2})^2.$$

Obviously, $F(X) \subseteq H(X)$ and $T(X) \subseteq S(X)$. Furthermore, $\{F,S\}$ and $\{T,H\}$ are weakly compatible. For each $x, y \in X$, we have

$$\begin{aligned}
D(Fx,Ty) &= (Fx - Ty)^2 = ((\frac{x}{2})^6 - (\frac{y}{2})^4)^2 \\
&= ((\frac{x}{2})^3 + (\frac{y}{2})^2)^2 \cdot ((\frac{x}{2})^3 - (\frac{y}{2})^2)^2 \\
&\leq (\frac{1}{8} + \frac{1}{4})^2 D(Hx,Sy) \\
&\leq \frac{9}{64} \max\{D(Sx,Hy), D(Sx,Fx), D(Hy,Ty), \frac{D(Sx,Ty)+D(Fx,Hy)}{2}\},
\end{aligned}$$

where $\frac{1}{K^4} < \frac{9}{64} < \frac{1}{K}$. Thus, all conditions of Corollary 3.2 are satisfied. Moreover, 0 is the unique common fixed point of F, T, S and H, however, we can not apply to the Theorem 2.1 of Roshan *et al.* [16].

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