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# SOME METRIC CHARACTERIZATIONS OF WELL-POSEDNESS FOR HEMIVARIATIONAL-LIKE INEQUALITIES

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**Abstract.** In this paper, some conditions for the well-posedness of the hemivariational-like inequalities in Banach spaces are investigated. Under different monotonicity assumptions, some equivalent formulations of the hemivariational-like inequality are presented. In addition, some metric characterizations of the well-posed hemivariational-like inequalities are presented. A class of hemivariational-like inequalities which are well-posed are also obtained.

**Keywords.** Approximating sequence; Clarke's generalized gradient; Hemivariational-like inequality; Well-posedness.

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## 1. Introduction

Let $[0,T]$ be a real interval and let $X$ be a reflexive Banach space with topological dual $X^*$, where $X \subset C^0[0,T]$ with norm $\|x\| = \|x\|_\infty$ for all $x \in X$ (for simplicity, write $x$ for $x(t)$, where $x : [0,T] \to R$). Assume that $A : [0,T] \times X \to X^*$ is a single-valued operator, $J : [0,T] \times X \to \mathbb{R}$ is a locally Lipschitz function with respect to the second component, $\eta : X \times X \to X$ is a vector-valued function, $f$ is some given element in $X^*$ and each functions $t \mapsto A(t,x)$ is integrable for all $x \in X$.

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A hemivariational-like inequality associated with $(A, f, J)$ is:

**HVLI(A, f, J):** Find $x \in X$ such that

$$\int_0^T [\langle A(t,x) - f, \eta(y,x)\rangle + J^\circ(t,x;\eta(y,x))]dt \geq 0, \quad \forall y \in X,$$

where $\langle .,. \rangle$ denotes the dual pair between $X$ and $X^*$ and $J^\circ(t,x;\eta(y,x))$ denotes the generalized directional derivative of function $J(t,.)$ at $x$ in direction $\eta(y,x)$; see [1] and the references therein.

Variational inequality theory was presented by Stampacchia [2] in 1960. By using this theory, Hartman and Stampacchia [3] studied some partial differential equations with their applications in mechanics. Panagiotopoulos [4] in 1983, as a useful generalization of the variational inequality, introduced the hemivariational inequality to formulate the variational principles connected to nonconvex and nonsmooth energy functions. The hemivariational inequalities by using the concepts of Clarke's generalized directional derivative and Clarke's generalized gradient were studied in [5].

The concept of the well-posedness plays a crucial role in the theory of optimization problems and is closely related to the variational inequalities. In 1966, Tykhonov [6] introduced the classical concept of the well-posedness for a minimization problem and it is known as the Tykhonov well-posedness. A minimization problem is Tykhonov well-posed if it has a unique solution and every minimizing sequence of the problem converges to the unique solution. Various kinds of the well-posedness for optimization problems, such as Levitin-Polyak well-posedness, $\alpha$-well-posedness and well-posedness by perturbations have been introduced and studied by many authors; see [7, 8, 9] and the references therein. The concept of the well-posedness has been generalized to some other problems such as, variational inequality problems [10, 11, 12, 13], inclusion and fixed point problems [14, 15], Nash equilibrium problems [16, 17] and saddle point problems [18].

The well-posedness for the hemivariational inequalities has been scarcely studied by comparison with variational inequalities. Goeleven and Mentagui [19] considered the following hemivariational inequality in Banach space $V$:

Find $u \in K \subset V$ such that

$$\textbf{(GM)}: \langle Au + Tu, v - u\rangle + \int_\Omega J^\circ(x, u(x); v(x) - u(x))d\Omega \geq \langle f, v - u\rangle, \quad \forall v \in K.$$

Also, they introduced the concept of the well-posedness for hemivariational inequality $(GM)$ as follows:

Hemivariational inequality $(GM)$ is well-posed if $G(\varepsilon) \neq \phi$ for all $\varepsilon > 0$ and $diam(G(\varepsilon)) \to 0$ as $\varepsilon \to 0$, where

$$G(\varepsilon) = \{u \in K : \langle Au + Tu - f, v - u \rangle + \int_\Omega J^\circ(x, u(x); v(x) - u(x)) d\Omega \geq -\varepsilon \|v - u\| \quad \forall v \in V\}.$$

In this paper, inspired by [19, 20, 21], we present a notion of the well-posedness for the hemivariational-like inequalities. We establish some metric characterizations for the well-posedness of the hemivariational-like inequalities and give some conditions under which the hemivariational-like inequality is well-posed. Finally, we obtain a class of the hemivariational-like inequalities which is well-posed.

## 2. Preliminaries and notations

Let $g : X \to \mathbb{R}$ be a nondifferentiable function. Clarke's generalized directional derivative of $g$ at $x$ in the direction of a given $d \in X$ is denoted by $g^\circ(x;d)$ and defined by

$$g^\circ(x;d) := \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{g(y + \lambda d) - g(y)}{\lambda}.$$

Clarke's generalized gradient of $g$ at $x$ is denoted by $\partial_C g(x)$ and defined by

$$\partial_C g(x) := \{x^* \in X^* : \langle x^*, d \rangle \leq g^\circ(x;d), \quad \forall d \in X\}.$$

The following proposition provides some properties for Clarke's generalized gradient and Clarke's generalized directional derivative.

**Proposition 2.1.** [1] *Let $X$ be a Banach space, $g : X \to \mathbb{R}$ a locally Lipschitz functional and $x, d \in X$. Then*

*(1) The function $d \mapsto g^\circ(x;d)$ is finite, positively homogeneous, subadditive and then convex on $X$.*

*(2) $g^\circ(x;d)$ is upper semicontinuous on $X \times X$ as a function of $(x,d)$, i.e., for all $x, d \in X$, $\{x_n\} \subset X$, $\{d_n\} \subset X$ such that $x_n \to x$ and $d_n \to d$ in $X$. Then*

$$\limsup g^\circ(x_n; d_n) \leq g^\circ(x;d).$$

*(3)* $g^\circ(x;-d) = (-g)^\circ(x;d)$.

*(4)* *For all* $x \in X$, $\partial_C g(x)$ *is a nonempty, convex, bounded and weak*$^*$*-compact subset of* $X^*$.

*(5)* *For every* $d \in X$, *one has*

$$g^\circ(x;d) = \max\{\langle x^*, d\rangle : x^* \in \partial_C g(x)\}.$$

*(6)* *The graph of clarke's generalized gradient* $\partial_C g(x)$ *is closed in* $X \times (w^* - X^*)$ *topology, where* $(w^* - X^*)$ *denotes the space* $X^*$ *equipped with weak*$^*$ *topology, i.e., if* $\{x_n\} \subset X$ *and* $\{x_n{}^*\} \subset X^*$ *are sequences such that* $x_n{}^* \in \partial_C g(x_n)$, $x_n \to x$ *in* $X$ *and* $x_n{}^* \to x^*$ *weakly*$^*$ *in* $X^*$. *Then* $x^* \in \partial_C g(x)$.

Now, we introduce the condition $C$ as follows:

It is said function $\eta : X \times X \to X$ satisfies the condition $C$, if, for any $x, y \in X$ and $\lambda \in [0,1]$

$$\eta(y, y + \lambda\eta(x,y)) = -\lambda\eta(x,y), \quad \eta(x, y + \lambda\eta(x,y)) = (1-\lambda)\eta(x,y).$$

**Remark 2.2.** If $\eta : X \times X \to X$ satisfies condition $C$, then

$$\eta(y + \lambda\eta(x,y), y) = \lambda\eta(x,y).$$

**Definition 2.3.** Let $F : [0,T] \times X \to 2^{X^*}$ be a set-valued mapping. $F$ is said to be invariant monotone with respect to $\eta$, if, for any $x, y \in X$, $t \in [0,T]$, $\xi \in F(t,x)$ and $\gamma \in F(t,y)$, one has

$$\int_0^T [\langle \xi, \eta(y,x)\rangle + \langle \gamma, \eta(x,y)\rangle]\, dt \le 0.$$

**Definition 2.4.** Let $X$ be a Banach space with its dual $X^*$, and let $G : [0,T] \times X \to X^*$ be an operator from $X$ to $X^*$. $G$ is said to be hemicontinuous if, for any $x, y \in X$ and $t \in [0,T]$, the function

$$\lambda \mapsto \langle G(t, x + \lambda y), y\rangle$$

from $[0,1]$ into $(-\infty, \infty)$ is continuous at $0_+$.

**Definition 2.5.** A sequence $\{x_n\} \subseteq X$ is said to be an approximating sequence with respect to $\eta$ for the $HVLI(A, f, J)$, if there exists a nonnegative sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$\int_0^T [\langle A(t,x_n) - f, \eta(y,x_n)\rangle + J^\circ(t, x_n; \eta(y,x_n))]dt \ge -\varepsilon_n \|\eta(y,x_n)\|, \quad \forall y \in X.$$

Next, we present the concept of well-posedness for the $HVLI(A, f, J)$.

**Definition 2.6.** The $HVLI(A,f,J)$ is said to be well-posed, if it has a unique solution on $X$ and every approximating sequence converges to the unique solution.

## 3. Main results

In this section, we present an equivalent formulation of the $HVLI(A,f,J)$ under the assumption of the invariant monotonicity for single-valued operator $A$. Also, we establish some metric characterizations for the well-posedness of the hemivariational-like inequalities and give some conditions under which the hemivariational-like inequality is well-posed. Finally, we obtain a class of the hemivariational-like inequalities which is well-posed.

**Theorem 3.1.** *Let $A:[0,T]\times X\to X^*$ be an operator such that $A(t,.)$ is hemicontinuous and $\eta: X\times X\to X$ satisfies condition C. Assume that one of the following conditions is satisfied:*

*(1) $x\in X$ is a solution to the following associated hemivariational-like inequality:*

*AHVLI(A, f, J): Find $x\in X$ such that*

$$\int_0^T [\langle A(t,y)-f,\eta(y,x)\rangle + J^\circ(t,y;\eta(y,x))]dt \geq 0, \quad \forall y\in X,$$

*(2) $x\in X$ is a solution to the following multi-valued variational-like inequality:*

*MVLI(A, f, J): Find $x\in X$ such that for all $y\in X$ there exists $\gamma\in\partial_C J(t,y)$ satisfying*

$$\int_0^T \langle A(t,y)+\gamma-f,\eta(y,x)\rangle dt \geq 0.$$

*Then $x\in X$ is a solution to the $HVLI(A,f,J)$.*

**Proof.** Let $x$ be a solution to the $AHVLI(A,f,J)$ which means that

$$\int_0^T [\langle A(t,y)-f,\eta(y,x)\rangle + J^\circ(t,y;\eta(y,x))]dt \geq 0, \quad \forall y\in X.$$

For any $z\in X$ and $\lambda\in[0,1]$, set $y=x+\lambda\eta(z,x)\in X$. Then

$$\int_0^T [\langle A(t,x+\lambda\eta(z,x))-f,\eta(x+\lambda\eta(z,x),x)\rangle + J^\circ(t,x+\lambda\eta(z,x);\eta(x+\lambda\eta(z,x),x))]dt \geq 0.$$

The condition $C$ implies $\eta(x+\lambda\eta(z,x),x)=\lambda\eta(z,x)$ and Proposition 2.1 shows $J^\circ(t,x;y)$ is positively homogeneous with respect to $y$. Thus

$$\int_0^T [\langle A(t,x+\lambda\eta(z,x))-f,\eta(z,x)\rangle + J^\circ(t,x+\lambda\eta(z,x);\eta(z,x))]dt \geq 0. \tag{2.1}$$

Since $x+\lambda\eta(z,x) \to x$ as $\lambda \to 0$ and $A$ is hemicontinuous, we obtain

$$\langle A(t,x+\lambda\eta(z,x)) - f, \eta(z,x)\rangle \to \langle A(t,x) - f, \eta(z,x)\rangle.$$

Note that $J^{\circ}(t,x;y)$ is upper semicontinuous. Taking upper limit when $\lambda \to 0^+$ at both sides of inequality (2.1), we deduce that

$$\begin{aligned} 0 &\leq \limsup \int_0^T [\langle A(t,x+\lambda\eta(z,x)) - f, \eta(z,x)\rangle + J^{\circ}(t,x+\lambda\eta(z,x);\eta(z,x))]dt \\ &\leq \int_0^T [\langle A(t,x) - f, \eta(z,x)\rangle + J^{\circ}(t,x;\eta(z,x))]dt. \end{aligned}$$

So $x$ is a solution to the $HVLI(A,f,J)$.

Now, we suppose that $x$ is a solution to the $MVLI(A,f,J)$. For any $z \in X$ and $\lambda \in [0,1]$, set $y_\lambda = x+\lambda\eta(z,x) \in X$. Then there exists $\gamma_\lambda \in \partial_C J(t,y_\lambda)$ such that

$$\int_0^T \langle A(t,y_\lambda) + \gamma_\lambda - f, \eta(y_\lambda,x)\rangle dt \geq 0.$$

The condition $C$ shows

$$\int_0^T \langle A(t,y_\lambda) + \gamma_\lambda - f, \eta(z,x)\rangle dt \geq 0. \tag{3.2}$$

Since operator $A$ is hemicontinuous, one has $\langle A(t,y_\lambda), \eta(z,x)\rangle \to \langle A(t,x), \eta(z,x)\rangle$. It is obvious that $y_\lambda \to x$ as $\lambda \to 0$. Therefore $\{y_\lambda\}$ is bounded on $X$. By boundedness of $\partial_C J(t,.)$ and the reflexivity of space $X$, there exists a $\gamma \in X^*$ such that $\gamma_\lambda \rightharpoonup \gamma$ in the weak$^*$-topology. Moreover, the closedness of graph $\partial_C J(t,.)$ implies that $\gamma \in \partial_C J(t,x)$ (Proposition 2.1). In view of (3.2), one has

$$\int_0^T \langle f - A(t,x), \eta(z,x)\rangle dt \leq \int_0^T \langle \gamma, \eta(z,x)\rangle dt \leq \int_0^T J^{\circ}(t,x;\eta(z,x))dt.$$

So $x$ is a solution to the $HVLI(A,f,J)$.

**Remark 3.2.** Let $A : [0,T] \times X \to X^*$ be an operator and let $\eta$ satisfy the condition $C$. If $A$ is invariant monotone with respect to $\eta$, then we can easily deduce that $x \in X$ is a solution to the $AHVLI(A,f,J)$ when it solves $HVLI(A,f,J)$.

**Corollary 3.3.** *Let $A : [0,T] \times X \to X^*$ be hemicontinuous and invariant monotone with respect to $\eta$ and let $\eta$ satisfy the condition $C$. Then $x \in X$ is a solution to the $HVLI(A,f,J)$ if and only if $x$ is a solution to the $AHVLI(A,f,J)$.*

**Example 3.4.** Assume that $f \equiv 0$, $A(t,x) = e^t x$ and $J : [0,1] \times X \to \mathbb{R}$ is defined by

$$yJ(t,x) = \begin{cases} tx, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Furthermore, let $\eta : X \times X \to X$ be defined by

$$\eta(x,y) = \begin{cases} x - y, & \text{if } x \geq 0,\ y \geq 0 \text{ or } x \leq 0,\ y \leq 0, \\ -y, & \text{otherwise.} \end{cases}$$

Then

$$\partial_C J(t,x) = \begin{cases} t, & \text{if } x > 0, \\ [-1,t], & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Notice that $\eta$ satisfies condition $C$ and

$$\int_0^1 [\langle A(t,y), \eta(y,0)\rangle + J^\circ\big(t,y;\eta(y,0)\big)]dt \geq 0, \quad \forall y \in X.$$

Therefore $x = 0$ solves $AHVLI(A,f,J)$ and Theorem 3.1 implies that $x = 0$ is a solution to the $HVLI(A,f,J)$. Also, $A$ is invariant monotone with respect to $\eta$. Hence, all assumptions of Corollary 3.3 are fulfilled and therefore the solution sets of $HVLI(A,f,J)$ and $AHVLI(A,f,J)$ are equivalent.

Suppose that $\varepsilon > 0$ is given and the following two sets are defined

$$G(\varepsilon) := \{x \in X : \int_0^T [\langle A(t,x) - f, \eta(y,x)\rangle + J^\circ(t,x;\eta(y,x))]dt \\ \geq -\varepsilon\|\eta(y,x)\|, y \in X\},$$

and

$$B(\varepsilon) := \{x \in X : \int_0^T [\langle A(t,y) - f, \eta(y,x)\rangle + J^\circ(t,x;\eta(y,x))]dt \\ \geq -\varepsilon\|\eta(y,x)\|, y \in X\}.$$

**Definition 3.5.** The function $\eta : X \times X \to X$ is said to be skew if

$$\eta(x,y) + \eta(y,x) = 0, \quad \forall x,y \in X.$$

In the sequel, we suppose that $\eta : X \times X \to X$ is a skew function.

**Lemma 3.6.** *Suppose that $A : [0,T] \times X \to X^*$ is hemicontinuous and invariant monotone with respect to $\eta$, where $\eta$ satisfies the condition C. Then $G(\varepsilon) = B(\varepsilon)$ for all $\varepsilon > 0$.*

**Proof.** Let $\varepsilon > 0$ be given and $x \in G(\varepsilon)$. Then

$$\int_0^T [\langle A(t,x) - f, \eta(y,x)\rangle + J^\circ(t,x;\eta(y,x))]dt \geq -\varepsilon\|\eta(y,x)\|, \quad \forall y \in X. \tag{3.3}$$

Since $A$ is invariant monotone with respect to $\eta$, one has

$$\int_0^T [\langle A(t,x), \eta(y,x)\rangle + \langle A(t,y), \eta(x,y)\rangle]dt \leq 0.$$

Moreover, since $\eta$ is skew, from this inequality and (3.3), we obtain

$$\int_0^T [\langle A(t,y) - f, \eta(y,x)\rangle + J^\circ(t,x;\eta(y,x))]dt \geq -\varepsilon\|\eta(y,x)\|, \quad \forall - y \in X. \tag{3.4}$$

Therefore $x \in B(\varepsilon)$.

Now, let $x \in B(\varepsilon)$. For any $z \in X$ and $\lambda \in [0,1]$, set $y = x + \lambda\eta(z,x)$ in (3.4). By the positive homogeneousness of $J^\circ(t,x;y)$ with respect to $y$ and the condition $C$, one has

$$\int_0^T [\langle A(t,x+\lambda\eta(z,x)) - f, \eta(z,x)\rangle + J^\circ(t,x;\eta(z,x))]dt \geq -\varepsilon\|\eta(z,x)\|, \quad \forall z \in X. \tag{3.5}$$

If $\lambda \to 0_+$ in (3.5), the hemicontinuity of mapping $A$ implies

$$\int_0^T [\langle A(t,x) - f, \eta(z,x)\rangle + J^\circ(t,x;\eta(z,x))]dt \geq -\varepsilon\|\eta(z,x)\|, \quad \forall z \in X$$

It follows that $x \in G(\varepsilon)$. Thus $B(\varepsilon) = G(\varepsilon)$. This completes the proof.

**Lemma 3.7.** *Suppose that $\eta$ is continuous with respect to the second component. Then $B(\varepsilon)$ is closed in $X$ for all $\varepsilon > 0$.*

**Proof.** Let $\{x_n\} \subset B(\varepsilon)$ and $x_n \to x$ in $X$. Then

$$\int_0^T [\langle A(t,y) - f, \eta(y,x_n)\rangle + J^\circ(t,x_n;\eta(y,x_n))]dt \geq -\varepsilon\|\eta(y,x_n)\|, \quad \forall y \in X. \tag{3.6}$$

From upper semicontinuity of $J^\circ(t,x;y)$ with respect to $(x;y)$ and continuity of $\eta$ with respect to the second component, we have

$$\limsup J^\circ(t,x_n,\eta(y,x_n)) \leq J^\circ(t,x;\eta(y,x)).$$

By taking $\limsup$ from the both sides of (3.6), we obtain

$$\int_0^T [\langle A(t,y) - f, \eta(y,x)\rangle + J^\circ(t,x;\eta(y,x))]dt \geq -\varepsilon\|\eta(y,x)\|, \quad \forall y \in X,$$

which implies $x \in B(\varepsilon)$. This completes the proof.

Here, some conditions are presented under which the well-posedness for the $HVLI(A,f,J)$ is equivalent to the uniqueness of its solution.

**Theorem 3.8.** *Let $A:[0,T]\times X\to X^*$ be invariant monotone with respect to $\eta$ and hemicontinuous, $\eta$ continuous with respect to the second component and satisfies condition C. Assume that there exists some $\varepsilon>0$ such that $G(\varepsilon)$ is compact. Then $HVLI(A,f,J)$ is well-posed if and only if it has a unique solution on $X$.*

**Proof.** The necessity follows immediately from the definition of the well-posedness for the $HVLI(A,f,J)$. Assume that the $HVLI(A,f,J)$ has a unique solution $x^*$ and the $HVLI(A,f,J)$ is not well-posed. Then there exists an approximating sequence $\{x_n\}$ for the $HVLI(A,f,J)$ which does not converge to $x^*$. So one can find a nonnegative sequence $\{\varepsilon_n\}$ converges to zero such that

$$\int_0^T [\langle A(t,x_n)-f,\eta(y,x_n)\rangle + J^\circ(t,x_n;\eta(y,x_n))]dt \ge -\varepsilon_n\|\eta(y,x_n)\|, \quad \forall y\in X.$$

Therefore $x_n\in G(\varepsilon_n)$. Assume that $\varepsilon>0$ is such that $G(\varepsilon)$ is nonempty and bounded. Then there exists $n_0$ such that $x_n\in G(\varepsilon)$ for all $n>n_0$. This implies that $\{x_n\}$ is bounded and for some subsequence, $\{x_n\}$ converges to a point $\hat{x}$ which solves the $HVLI(A,f,J)$. In fact, for all $y\in X$, one has

$$\begin{aligned}
&\int_0^T [\langle A(t,y)-f,\eta(y,\hat{x})\rangle + J^\circ(t,\hat{x};\eta(y,\hat{x}))]dt \\
&\qquad \ge \limsup \int_0^T [\langle A(t,y)-f,\eta(y,x_n)\rangle + J^\circ(t,x_n;\eta(y,x_n))]dt \\
&\qquad \ge \limsup \int_0^T [\langle A(t,x_n)-f,\eta(y,x_n)\rangle + J^\circ(t,x_n;\eta(y,x_n))]dt \\
&\qquad \ge \limsup \varepsilon_n\|\eta(y,x_n)\| = 0.
\end{aligned}$$

For any $z\in X$ and $\lambda\in[0,1]$, set $y=\hat{x}+\lambda\eta(z,\hat{x})$ in the last inequality. The positive homogeneousness of $J^\circ(t,x;y)$ with respect to $y$ and condition $C$ imply

$$\int_0^T [\langle A(t,\hat{x}+\lambda\eta(z,\hat{x}))-f,\eta(z,\hat{x})\rangle + J^\circ(t,\hat{x};\eta(z,\hat{x}))]dt \ge 0. \tag{3.7}$$

Letting $\lambda\to 0_+$ and using the hemicontinuity of mapping $A$, we obtain

$$\int_0^T [\langle A(t,\hat{x})-f,\eta(z,\hat{x})\rangle + J^\circ(t,\hat{x};\eta(z,\hat{x}))]dt \ge 0.$$

Since $z$ is arbitrary, it follows that $\hat{x}$ solves the $HVLI(A, f, J)$. So $\hat{x} = x^*$, which is a contradiction. This completes the proof.

In the next theorem, a class of the hemivariational-like inequalities which is well-posed is obtained.

**Theorem 3.9.** *Let $A : [0,T] \times X \to X^*$ be a hemicontinuous mapping and invariant monotone with respect to $\eta$ that is continuous with respect to the second component and satisfy condition C. If*

$$G(\varepsilon) \neq \phi, \quad \forall \varepsilon > 0 \quad and \quad diam(G(\varepsilon)) \to 0 \quad as \quad \varepsilon \to 0, \tag{3.8}$$

*then $HVLI(A, f, J)$ is well-posed.*

**Proof.** Letting $\{x_n\} \subset X$ be an approximating sequence for $HVLI(A, f, J)$, one sees that there exists a nonnegative sequence $\varepsilon_n \to 0$ such that

$$\int_0^T [\langle A(t,x_n) - f, \eta(y,x_n)\rangle + J^\circ(t,x_n;\eta(y,x_n))]dt \geq -\varepsilon_n \|\eta(y,x_n)\|, \quad \forall y \in X, \tag{3.9}$$

which implies that $x_n \in G(\varepsilon_n)$. It follows from (3.8) that $\{x_n\}$ is a Cauchy sequence. So $\{x_n\}$ converges strongly to some point $x \in X$. Since mapping $A$ is invariant monotone with respect to $\eta$, Clarke's generalized directional derivative $J^\circ(t,x;y)$ is upper semicontinuous with respect to $(x;y)$ and $\eta$ is continuous with respect to the second component, we deduce that

$$\begin{aligned}
&\int_0^T [\langle A(t,y) - f, \eta(y,x)\rangle + J^\circ(t,x;\eta(y,x))]dt \\
&\qquad \geq \limsup \int_0^T [\langle A(t,y) - f, \eta(y,x_n)\rangle + J^\circ(t,x_n;\eta(y,x_n))]dt \\
&\qquad \geq \limsup \int_0^T [\langle A(t,x_n) - f, \eta(y,x_n)\rangle + J^\circ(t,x_n;\eta(y,x_n))]dt \\
&\qquad \geq \limsup -\varepsilon_n \|\eta(y,x_n)\| = 0, \quad \forall y \in X.
\end{aligned}$$

For any $z \in X$ and $\lambda \in [0,1]$, set $y = x + \lambda\eta(z,x)$, in the last inequality. Then

$$\int_0^T [\langle A(t,x+\lambda\eta(z,x)) - f, \eta(x+\lambda\eta(z,x),x)\rangle + J^\circ(t,x;\eta(x+\lambda\eta(z,x),x))]dt \geq 0.$$

It follows from condition $C$ and the positive homogeneousness of $J^\circ(t,x;.)$ that

$$\int_0^T [\langle A(t,x+\lambda\eta(z,x)) - f, \eta(z,x)\rangle + J^\circ(t,x;\eta(z,x))]dt \geq 0. \tag{3.10}$$

Letting $\lambda \to 0_+$ and using the hemicontinuity of mapping $A$, one finds that

$$\int_0^T [\langle A(t,x) - f, \eta(z,x)\rangle + J^\circ(t,x;\eta(z,x))]dt \geq 0, \quad \forall z \in X,$$

which shows that $x$ solves the $HVLI(A,f,J)$. Since $diamG(\varepsilon) \to 0$ when $\varepsilon \to 0$, $HVLI(A,f,J)$ has a unique solution. This completes the proof.

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