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EXISTENCE OF THREE SOLUTIONS FOR MULTI-POINT BOUNDARY VALUE PROBLEMS

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Abstract. This paper is devoted to the study of the existence of at least three non-negative classical solutions for a second-order multi-point boundary value system. We use variational methods for smooth functionals defined on reflexive Banach spaces in order to achieve our results. An example is provided to illustrate the applicability of our results.

Keywords. Multi-point boundary value problem; Classical solution; Three solutions; Variational method.

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1. Introduction

In this paper, we study the existence of at least three non-negative classical solutions to the multi-point boundary value system

$$(P^{F,G}_{\lambda,\mu}) \qquad \begin{cases} -(\phi_{p_i}(u_i'))' = \lambda F_{u_i}(x,u_1,\dots,u_n) + \mu G_{u_i}(x,u_1,\dots,u_n), \quad \text{in } (0,1), \\ u_i(0) = \sum_{j=1}^m a_j u_i(x_j), \quad u_i(1) = \sum_{j=1}^m b_j u_i(x_j), \end{cases}$$

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for $i = 1, \ldots, n$, where $p_i > 1$, $\phi_{p_i}(x) = |x|^{p_i-2}x$ for $i = 1, \ldots, n$, $\lambda > 0, \mu \geq 0$ are parameters, $m, n \in \mathbb{N}$, $F, G : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are measurable with respect to x, for every $(t_1, \ldots, t_n) \in \mathbb{R}^n$, continuously differentiable in $(x_1, \ldots, x_n)$, for almost every $t \in [0,1]$, $F(x, 0, \ldots, 0) = G(x, 0, \ldots, 0) = 0$ for all $x \in [0,1]$ and satisfy the standard summability condition

$$\sup_{|\xi| \leq \rho_1} \left\{ \max\{|F(\cdot,\xi)|,\ |G(\cdot,\xi)|,\ |F_{\xi_i}(\cdot,\xi)|,\ |G_{\xi_i}(\cdot,\xi)|,\ i = 1, \ldots, n\} \right\} \in \mathrm{L}^1([0,1]) \qquad (1.1)$$

for any $\rho_1 > 0$ with $\xi = (\xi_1, \ldots, \xi_n)$ and $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$ and $a_j, b_j \in \mathbb{R}$ for $j = 1, \ldots, m$.

Multi-point boundary value problems (BVPs) that arise from different areas of applied mathematics, physics, finance and economics have received a lot of attention in the literature in the last decades; see, for example, [1, 2] and the references therein. Indeed, a number of problems in the theory of elastic stability can be treated as a multi-point problem [3] and also the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be handled as a multi-point boundary value problem [4].

Recently, multi-point BVPs have been extensively studied by many researchers, see [5, 6, 7] and the references therein. In [8], Ma, based on Guo-Krasnoselskii fixed point theorem, studied the existence and multiplicity of positive solutions for following m-point boundary value problem

$$\begin{cases} (p(t)u')' - q(t)u + \lambda h(t)f(u) = 0, \quad 0 < t < 1, \\ au(0) - bp(0)u'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(1) - dp(1)u'(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

where $a, b, c, d \in [0, +\infty)$, $\xi_i \in (0,1)$, $\alpha_i, \beta_i \in [0, +\infty)$ (for $i \in \{1, \ldots, m-2\}$) are given constants, $p, q \in C([0,1], (0, +\infty))$, $h \in C([0,1], [0, +\infty))$, and $f \in C([0, +\infty), [0, +\infty))$ satisfying some suitable conditions. In [5], Feng and Ge studied the existence of at least three positive solutions to the following boundary value problem based on a fixed point theorem due to Avery and Peterson

$$\begin{cases} (\phi_p(u'))' + q(t)f(t, u, u') = 0, \qquad \text{in } (0,1), \\ u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \ldots < \xi_{m-2} < 1$ and $a_i \in [0,1)$, $0 \leq \sum_{i=1}^{m-2} a_i < 1$. Liu [9] established the existence of at least three solutions for a non-homogeneous multi-point boundary value problem of second order differential equation with one-dimensional p-Laplacian by using multiple fixed point theorems. In [6], the existence and multiplicity of classical solutions to parametric version of system $(P^{F,G}_{\lambda,\mu})$ was discussed based on a critical point theorem due to Ricceri. Also, in [10], the existence of at least one nontrivial solution to parametric version of the system $(P^{F,G}_{\lambda,\mu})$ was studied based on variational methods and critical point theory. In [11], the existence of at least three classical solutions for a second-order multi-point boundary value problem with impulsive effects was discussed based on variational methods for smooth functionals defined on reflexive Banach spaces.

In this paper, by using a three critical points theorem proved in [12], we ensure the existence of at least three non-negative classical solutions for system $(P^{F,G}_{\lambda,\mu})$ for appropriate values of parameters λ and μ belonging to real intervals. In particular, we require that there is a growth of F which is greater than quadratic in a suitable interval, and which is less than quadratic in a following suitable interval. An example and some remarks are also provided to illustrate our main results.

The organization of this article is as follows. In Section 2, we recall some definitions and notations. In Section 3, we state and prove the main results of the paper.

We assume throughout and without further mention that the following conditions hold:

(H1) Either $\underline{p} \geq 2$ or $\overline{p} < 2$, where $\underline{p} = \min\{p_1, \ldots, p_n\}$ and $\overline{p} = \max\{p_1, \ldots, p_n\}$.

(H2) $\sum_{j=1}^m a_j \neq 1$ and $\sum_{j=1}^m b_j \neq 1$.

2. Preliminaries

In this section, we discuss the existence of at least three solutions for system $(P^{F,G}_{\lambda,\mu})$. Here, X^* denotes the dual space of X. Let X be a nonempty set and let $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r,\ r_1,\ r_2 > \inf_X \Phi,\ r_2 > r_1,\ r_3 > 0$, we define

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{(\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1,r_2) := \inf_{u\in\Phi^{-1}(]-\infty,r_1[)} \sup_{v\in\Phi^{-1}([r_1,r_2[)} \frac{\Psi(v)-\Psi(u)}{\Phi(v)-\Phi(u)},$$

$$\gamma(r_2,r_3) := \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2+r_3[)} \Psi(u)}{r_3},$$

$$\alpha(r_1,r_2,r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2,r_3)\}.$$

Theorem 2.1. [12, Theorem 3.3] *Let X be a reflexive real Banach space. Let $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative* *admits a continuous inverse on X^* and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

(a_1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(a_2) *for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

(a_3) $\varphi(r_1) < \beta(r_1, r_2)$;

(a_4) $\varphi(r_2) < \beta(r_1, r_2)$;

(a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}[$ the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(]-\infty, r_1[)$, $u_2 \in \Phi^{-1}([r_1, r_2[)$ and $u_3 \in \Phi^{-1}(]-\infty, r_2 + r_3[)$.

We remark here that Theorem 2.1, which extends previous results of Pucci and Serrin [13, 14], is a counter-part of a general result (three critical point theorem) of Ricceri [15, 16]. Theorem 2.1 has been successfully employed to obtain the existence of at least three solutions for boundary value problems; see [17, 18, 19] and the references therein. In order to investigate the existence of solutions for system $(P^{F,G}_{\lambda,\mu})$, we introduce the following basic notations and results which will be used in the proofs of our main results.

Throughout this paper, we let X be the Cartesian product of n spaces

$$X_i = \left\{ \xi \in W^{1,p_i}([0,1]) \ : \ \xi(0) = \sum_{j=1}^m a_j \xi(x_j), \quad \xi(1) = \sum_{j=1}^m b_j \xi(x_j) \right\}$$

for $i = 1, \dots, n$, i.e., $X = X_1 \times \cdots \times X_n$, endowed with the norm

$$\|(u_1, \dots, u_n)\| = \sum_{i=1}^n \|u_i\|_{p_i},$$

where

$$\|u_i\|_{p_i} = \left(\int_0^1 |u_i'(x)|^{p_i} dx \right)^{1/p_i}, \quad i = 1, \dots, n.$$

By a *classical solution* of $(P^{F,G}_{\lambda,\mu})$, we mean a function $u = (u_1, \dots, u_n) \in X$ such that, for $i = 1, \dots, n$, $u_i \in C^1[0,1]$, $\phi_{p_i}(u_i^p) \in C^1[0,1]$, and u satisfies $(P^{F,G}_{\lambda,\mu})$. We say that a function $u = (u_1, \dots, u_n) \in X$ is a *weak solution* of $(P^{F,G}_{\lambda,\mu})$ if

$$\int_0^1 \sum_{i=1}^n |u_i'(x)|^{p_i - 2} u_i'(x) v_i'(x) dx - \lambda \int_0^1 \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx$$

$$-\mu\int_0^1 G_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)dx = 0$$

for any $v=(v_1,\dots,v_n)\in X$.

From Lemma 2.4 below, we see that a weak solution of $(P^{F,G}_{\lambda,\mu})$ is indeed a classical solution.

Let $\phi_{p_i}^{-1}$ denote the inverse of ϕ_{p_i} for each $i=1,\dots,n$. Then, $\phi_{p_i}^{-1}(t)=\phi_{q_i}(t)$ where $\frac{1}{p_i}+\frac{1}{q_i}=1$. It is clear that ϕ_{p_i} is increasing on R,

$$\lim_{t\to-\infty}\phi_{p_i}(t)=-\infty \text{ and } \lim_{t\to+\infty}\phi_{p_i}(t)=+\infty. \tag{2.1}$$

Lemma 2.2. *For fixed λ, $\mu\in\mathbb{R}$, $u=(u_1,\dots,u_n)\in(C([0,1]))^n$, and $i=1,\dots,n$, define $\alpha_i(t;u):\mathbb{R}\to\mathbb{R}$ by*

$$\alpha_i(t;u)=\int_0^1\phi_{p_i}^{-1}\Bigg(t-\lambda\int_0^\delta F_{u_i}(\xi,u_1(\xi),\dots,u_n(\xi))d\xi$$
$$-\mu\int_0^\delta G_{u_i}(\xi,u_1(\xi),\dots,u_n(\xi))d\xi\Bigg)d\delta+\sum_{j=1}^m a_ju_i(x_j)-\sum_{j=1}^m b_ju_i(x_j).$$

Then, equation

$$\alpha_i(t;u)=0 \tag{2.2}$$

has a unique solution $t_{u,i}$.

Proof. Taking (2.1) into account, we have

$$\lim_{t\to-\infty}\alpha_i(t,u)=-\infty \text{ and } \lim_{t\to+\infty}\alpha_i(t,u)=+\infty.$$

Since $\alpha_i(\cdot;u)$ is continuous and increasing on $\mathbb{R}$, we can obtain the desired conclusion immediately.

Lemma 2.3. *The function $u=(u_1,\dots,u_n)$ is a solution of system $(P^{F,G}_{\lambda,\mu})$ if and only if $u_i(x)$ is a solution of the equation*

$$u_i(x)=\sum_{j=1}^m a_ju_i(x_j)+\int_0^x\phi_{p_i}^{-1}\Bigg(t_{u,i}-\lambda\int_0^\delta F_{u_i}(\xi,u_1(\xi),\dots,u_n(\xi))d\xi$$
$$-\mu\int_0^\delta G_{u_i}(\xi,u_1(\xi),\dots,u_n(\xi))d\xi\Bigg)d\delta$$

for $i=1,\dots,n$, where $t_{u,i}$ is the unique solution of (2.2).

Proof. This can be verified from direct computations.

Lemma 2.4. [20, Lemma 2.5] *A weak solution to* $(P^{F,G}_{\lambda,\mu})$ *coincides with a classical solution to* $(P^{F,G}_{\lambda,\mu})$.

In order to prove our main result, we need the following result.

Lemma 2.5. [20, Lemma 2.6] *Let* $T : X \to X^*$ *be the operator defined by*

$$T(u_1,\dots,u_n)(h_1,\dots,h_n) = \int_0^1 \sum_{i=1}^n |u_i'(x)|^{p_i-2}u_i'(x)h_i'(x)dx$$

for every $(u_1,\dots,u_n),(h_1,\dots,h_n) \in X$. *Then T admits a continuous inverse on* X.

Let

$$c = \max\left\{ \sup_{u_i\in X_i\setminus\{0\}} \frac{\max_{x\in[0,1]}|u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} \ : \ \text{for } 1\le i\le n\right\}. \tag{2.3}$$

Since $p_i > 1$ for $i = 1,\dots,n$, the embedding $X = X_1\times\cdots\times X_n \hookrightarrow C^0([0,1])\times\cdots\times C^0([0,1])$ is compact. So, $c < +\infty$. Moreover, if (H2) holds, from [21, Lemma 3.1],

$$\sup_{v\in X_i\setminus\{0\}} \frac{\max_{x\in[0,1]}|v(x)|}{\|v\|_{p_i}} \le \frac{1}{2}\left(1+\frac{\sum_{j=1}^m |a_j|}{|1-\sum_{j=1}^m a_j|}+\frac{\sum_{j=1}^m |b_j|}{|1-\sum_{j=1}^m b_j|}\right)$$

for $i = 1,\dots,n$.

For any $\gamma > 0$, we denote by $K(\gamma)$ the set

$$\left\{(t_1,\dots,t_n)\in R^n : \ \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \le \gamma\right\}.$$

This set will be used in some of our hypotheses with appropriate choices of γ. For positive constants θ and τ, set

$$G^\theta := \int_0^1 \sup_{(t_1,\dots,t_n)\in K\left(\frac{\theta^{p_n}}{\prod_{i=1}^n p_i}\right)} G(x,t_1,\dots,t_n)dx$$

and

$$G_\tau := \inf_{[0,1]\times[0,\tau]\times\cdots\times[0,\tau]} G.$$

3. Main results

We are now in a position to present our main results.

From now on, for positive constant τ, let $\bar{\tau}$ be the vectors in $\mathbb{R}^n$ defined by $\bar{\tau} = (0,\ldots,0,\tau)$. Let

$$\sigma_n = \left[2^{p_n-1}\left(x_1^{1-p_n}\left|1-\sum_{j=1}^{m} a_j\right|^{p_n} + (1-x_m)^{1-p_n}\left|1-\sum_{j=1}^{m} b_j\right|^{p_n}\right)\right]^{1/p_n}. \tag{3.1}$$

We fix four positive constants θ_1, θ_2, θ_3 and τ and put

$$\begin{aligned}\delta_{\lambda,G} := \min\Bigg\{ & \frac{1}{c\prod_{i=1}^n p_i}\min\Bigg\{\frac{\theta_1^{p_n} - \lambda c\prod_{i=1}^n p_i\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{G^{\theta_1}},\\ & \frac{\theta_2^{p_n} - \lambda c\prod_{i=1}^n p_i\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_2^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{G^{\theta_2}},\\ & \frac{(\theta_3^{p_n}-\theta_2^{p_n}) - \lambda c\prod_{i=1}^n p_i\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_3^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{G^{\theta_3}}\Bigg\},\\ & \frac{\frac{(\sigma_n\tau)^{p_n}}{p_n} - \lambda\Big(\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx - \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx\Big)}{G_\tau - G^{\theta_1}}\Bigg\}.\end{aligned} \tag{3.2}$$

Theorem 3.1. *Let $F:[0,1]\times\mathbb{R}^n \to \mathbb{R}$ satisfies the condition $F(x,t_1,\ldots,t_n)\geq 0$ for all $(x,t_1,\ldots,t_n)\in [0,1]\times(\mathbb{R}^+\cup\{0\})^n$ for $1\leq i\leq n$. Assume that there exist positive constants θ_1, θ_2, θ_3 and τ with $\theta_1 < \sqrt[p_n]{\frac{c\prod_{i=1}^n p_i}{p_n}}\sigma_n\tau < \theta_2$ and $\theta_2 < \theta_3$ such that*

(A_1)

$$\begin{aligned}\max\Bigg\{ & \frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\theta_1^{p_n}},\\ & \frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_2^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\theta_2^{p_n}},\\ & \frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_3^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\theta_3^{p_n}-\theta_2^{p_n}}\Bigg\}\\ & < \frac{p_n}{\sigma_n^{p_n} c\prod_{i=1}^n p_i}\,\frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx - \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\tau^{p_n}}.\end{aligned}$$

Then, for every

$$\lambda \in \Bigg(\frac{\frac{(\sigma_n \tau)^{p_n}}{p_n}}{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx - \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx},$$

$$\frac{1}{c\prod_{i=1}^n p_i} \min\Bigg\{ \frac{\theta_1^{p_n}}{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx},$$

$$\frac{\theta_2^{p_n}}{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_2^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx},$$

$$\frac{\theta_3^{p_n} - \theta_2^{p_n}}{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_3^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx} \Bigg\}\Bigg)$$

for every non-negative function $G:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ there exists $\delta_{\lambda,G}>0$ given by (3.2) such that, for each $\mu\in[0,\delta_{\lambda,G})$, system $(P_{\lambda,\mu}^{F,G})$ possesses at least three non-negative classical solutions $u^1=(u_1^1,\ldots,u_n^1)$, $u^2=(u_1^2,\ldots,u_n^2)$, and $u^3=(u_1^3,\ldots,u_n^3)$ such that

$$\max_{x\in[0,1]}\sum_{i=1}^n |u_i^1(x)|^{p_i} \le \frac{\underline{p}\theta_1^{p_n}}{\prod_{i=1}^n p_i},\ \max_{x\in[0,1]}\sum_{i=1}^n |u_i^2(x)|^{p_i} \le \frac{\underline{p}\theta_2^{p_n}}{\prod_{i=1}^n p_i}$$

and

$$\max_{x\in[0,1]}\sum_{i=1}^n |u_i^3(x)|^{p_i} \le \frac{\underline{p}\theta_3^{p_n}}{\prod_{i=1}^n p_i}.$$

Proof. In order to apply Theorem 2.1 to this problem, we introduce functionals Φ, $\Psi: X\to R$ for each $u=(u_1,...,u_n)\in X$ as follows

$$\Phi(u)=\sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \tag{3.3}$$

and

$$\Psi(u)=\int_0^1 F(x,u_1(x),\ldots,u_n(x))dx+\frac{\mu}{\lambda}\int_0^1 G(x,u_1(x),\ldots,u_n(x))dx. \tag{3.4}$$

Since $p_i>1$ for $1\le i\le n$, X is compactly embedded in $C^0([0,1])\times\cdots\times C^0([0,1])$ and it is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at point $u=(u_1,\ldots,u_n)\in X$ are the functionals $\Phi'(u)$, $\Psi'(u)\in X^*$ given by

$$\Phi'(u)(v)=\int_0^1\sum_{i=1}^n |u_i'(x)|^{p_i-2}u_i'(x)v_i'(x)dx$$

and

$$\Psi'(u)(v) = \int_0^1 \sum_{i=1}^n F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx + \frac{\mu}{\lambda} \int_0^1 G_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx$$

for every $v = (v_1, \ldots, v_n) \in X$. Moreover, Ψ is sequentially weakly upper semicontinuous. From Lemma 2.5, Φ' admits a continuous inverse on X^*, and since Φ' is monotone, Φ is sequentially weakly lower semi continuous (see [22, Proposition 25.20]). Put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$. Note that the solutions of $(P^{F,G}_{\lambda,\mu})$ are exactly the critical points of I_λ. Put $r_1 = \frac{\theta_1^{p_n}}{c\prod_{i=1}^n p_i}$, $r_2 = \frac{\theta_2^{p_n}}{c\prod_{i=1}^n p_i}$, $r_3 = \frac{\theta_3^{p_n} - \theta_2^{p_n}}{c\prod_{i=1}^n p_i}$ and $w = (0, \ldots, 0, w_n(x))$ with

$$w_n(x) = \begin{cases} \tau\left(\sum_{j=1}^m a_j + \frac{2(1-\sum_{j=1}^m a_j)}{x_1} x\right), & \text{if } x \in [0, \frac{x_1}{2}), \\ \tau, & \text{if } x \in [\frac{x_1}{2}, \frac{1+x_m}{2}], \\ \tau\left(\frac{2-\sum_{j=1}^m b_j - x_m \sum_{j=1}^m b_j}{1-x_m} - \frac{2(1-\sum_{j=1}^m b_j)}{1-x_m} x\right), & \text{if } x \in (\frac{1+x_m}{2}, 1]. \end{cases}$$

It is easy to see that $w = (0, \ldots, 0, w_n) \in X$, in particular, $\|w_n\|_{p_n}^{p_n} = (\sigma_n \tau)^{p_n}$. So $\Phi(w) = \frac{(\sigma_n\tau)^{p_n}}{p_n}$. From conditions $\theta_3 > \theta_2$ and $\theta_1 < \sqrt[p_n]{\frac{c\prod_{i=1}^n p_i}{p_n}}\sigma_n\tau < \theta_2$, we get that $r_3 > 0$ and $r_1 < \Phi(w) < r_2$. From (2.3), for each $(u_1, \ldots, u_n) \in X$,

$$\sup_{x\in[0,1]} |u_i(x)|^{p_i} \le c\|u_i\|_{p_i}^{p_i}$$

for $i = 1, \ldots, n$, we

$$\sup_{x\in[0,1]} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \le c \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i}. \tag{3.5}$$

It follows that

$$\begin{aligned} \Phi^{-1}(-\infty, r_1) &= \{u = (u_1, \ldots, u_n) \in X;\ \Phi(u) < r_1\} \\ &= \left\{u \in X;\ \sum_{i=1}^n \frac{\|u_i\|^{p_i}}{p_i} < r_1\right\} \\ &\subseteq \left\{u \in X;\ \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \le cr_1 \text{ for each } x \in [0,1]\right\} \\ &= \left\{u \in X;\ \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \le \frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i} \text{ for each } x \in [0,1]\right\}. \end{aligned}$$

Hence

$$\begin{aligned}\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(-\infty,r_1)}\Psi(u) &= \sup_{(u_1,\dots,u_n)\in\Phi^{-1}(-\infty,r_1)}\int_0^1 F(x,u_1(x),\dots,u_n(x))dx\\ &\leq \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{pn}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx.\end{aligned}$$

In a similar way, we have

$$\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(-\infty,r_2)}\Psi(u)\leq\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_2^{pn}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx$$

and

$$\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(-\infty,r_3)}\Psi(u)\leq\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_3^{pn}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx.$$

Since $0\in\Phi^{-1}(-\infty,r_1)$ and $\Phi(0)=\Psi(0)=0$, one has

$$\begin{aligned}\varphi(r_1) &= \inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{(\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u))-\Psi(u)}{r_1-\Phi(u)}\\ &\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1}\\ &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\left[\int_0^1 F(x,u(x))dx+\frac{\mu}{\lambda}\int_0^1 G(x,u(x))dx\right]}{r_1}\\ &\leq \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{pn}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx+\frac{\mu}{\lambda}G^{\theta_1}}{\frac{\theta_1^{pn}}{c\prod_{i=1}^n p_i}},\end{aligned}$$

$$\begin{aligned}\varphi(r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2}\\ &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\left[\int_0^1 F(x,u(x))dx+\frac{\mu}{\lambda}\int_0^1 G(x,u(x))dx\right]}{r_2}\\ &\leq \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_2^{pn}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx+\frac{\mu}{\lambda}G^{\theta_2}}{\frac{\theta_2^{pn}}{c\prod_{i=1}^n p_i}}\end{aligned}$$

and

$$\begin{aligned}\gamma(r_2,r_3) &\leq \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3} \\ &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\left[\int_0^1 F(x,u(x))dx+\frac{\mu}{\lambda}\int_0^1 G(x,u(x))dx\right]}{r_3} \\ &\leq \frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_3^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx+\frac{\mu}{\lambda}G^{\theta_3}}{\frac{\theta_3^{p_n}-\theta_2^{p_n}}{c\prod_{i=1}^n p_i}}.\end{aligned}$$

Note that

$$\tau\sum_{j=1}^m a_j \leq w_n(x)\leq \tau \ \text{ for each } \ x\in[0,\frac{x_1}{2}] \ \text{ if } \ \sum_{j=1}^m a_j<1,$$

$$\tau\leq w_n(x)\leq \tau\sum_{j=1}^m a_j \ \text{ for each } \ x\in[0,\frac{x_1}{2}] \ \text{ if } \ \sum_{j=1}^m a_j>1,$$

$$\tau\sum_{j=1}^m b_j \leq w_n(x)\leq \tau \ \text{ for each } \ x\in[\frac{1+x_m}{2},1] \ \text{ if } \ \sum_{j=1}^m b_j<1,$$

and

$$\tau\leq w_n(x)\leq \tau\sum_{j=1}^m b_j \ \text{ for each } \ x\in[\frac{1+x_m}{2},1] \ \text{ if } \ \sum_{j=1}^m b_j>1.$$

Since F is non-negative, we have

$$\int_0^1 F(x,w_1(x),\ldots,w_n(x))dx \geq \int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,w_1(x),\ldots,w_n(x))dx. \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}\Psi(w) &= \int_0^1 F(x,w_1(x),\ldots,w_n(x))dx+\frac{\mu}{\lambda}\int_0^1 G(x,w_1(x),\ldots,w_n(x))dx \\ &\geq \int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx+\frac{\mu}{\lambda}G_\tau.\end{aligned}$$

For each $u\in\Phi^{-1}(-\infty,r_1)$, one has

$$\begin{aligned}&\beta(r_1,r_2) \\ &\geq \frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx-\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx+\frac{\mu}{\lambda}(G_\tau-G^{\theta_1})}{\Phi(w)-\Phi(u)} \\ &\geq \frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx-\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx+\frac{\mu}{\lambda}(G_\tau-G^{\theta_1})}{\frac{(\sigma_n\tau)^{p_n}}{p_n}}.\end{aligned}$$

In view of (A_2), we get $\alpha(r_1,r_2,r_3) < \beta(r_1,r_2)$. Now, we show that functional I_λ satisfies the assumption (a_2) of Theorem 2.1. Let $u^\star = (u_1^\star,\dots,u_n^\star)$ and $u^{\star\star} = (u_1^{\star\star},\dots,u_n^{\star\star})$ be two local minima for $\Phi-\lambda\Psi$. Then $u^\star$ and $u^{\star\star}$ are critical points for $\Phi-\lambda\Psi$, and so, they are weak solutions for the system $(P^{F,G}_{\lambda,\mu})$. Since $F(x,t_1,\dots,t_n)\geq 0$ for all $(x,t_1,\dots,t_n)\in[0,1]\times(\mathbb{R}^+\cup\{0\})^n$ for $1\leq i\leq n$ and G is a non-negative function, from the Weak Maximum Principle (see for instance [23]) we deduce $u_i^\star(x)\geq 0$ and $u_i^{\star\star}(x)\geq 0$ for every $x\in[0,1]$ for $1\leq i\leq n$. So, it follows that $su_i^\star+(1-s)u_i^{\star\star}\geq 0$ for all $s\in[0,1]$ for $1\leq i\leq n$, and that

$$F(su^\star+(1-s)u^{\star\star},t_1,\dots,t_n)+\frac{\mu}{\lambda}G(su^\star+(1-s)u^{\star\star},t_1,\dots,t_n)\geq 0$$

for $1\leq i\leq n$, and consequently, $\Psi(su^\star+(1-s)u^{\star\star})\geq 0$ for all $s\in[0,1]$. Hence, Theorem 2.1 implies that

$$\lambda\in\Bigg(\frac{\frac{(\sigma_n\tau)^{p_n}}{p_n}}{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}}F(x,\bar{\tau})dx-\int_0^1\sup_{(t_1,\dots,t_n)\in K\left(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i}\right)}F(x,t_1,\dots,t_n)dx},$$

$$\frac{1}{c\prod_{i=1}^n p_i}\min\Bigg\{\frac{\theta_1^{p_n}}{\int_0^1\sup_{(t_1,\dots,t_n)\in K\left(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i}\right)}F(x,t_1,\dots,t_n)dx},$$

$$\frac{\theta_2^{p_n}}{\int_0^1\sup_{(t_1,\dots,t_n)\in K\left(\frac{\theta_2^{p_n}}{\prod_{i=1}^n p_i}\right)}F(x,t_1,\dots,t_n)dx},$$

$$\frac{\theta_3^{p_n}-\theta_2^{p_n}}{\int_0^1\sup_{(t_1,\dots,t_n)\in K\left(\frac{\theta_3^{p_n}}{\prod_{i=1}^n p_i}\right)}F(x,t_1,\dots,t_n)dx}\Bigg\}\Bigg)$$

and $\mu\in[0,\delta_{\lambda,G})$, functional I_λ has three critical points $u^1=(u_1^1,\dots,u_n^1)$, $u^2=(u_1^2,\dots,u_n^2)$, and $u^3=(u_1^3,\dots,u_n^3)$ such that

$$\max_{x\in[0,1]}\sum_{i=1}^n|u_i^1(x)|^{p_i}\leq\frac{\underline{p}\theta_1^{p_n}}{\prod_{i=1}^n p_i},\ \max_{x\in[0,1]}\sum_{i=1}^n|u_i^2(x)|^{p_i}\leq\frac{\underline{p}\theta_2^{p_n}}{\prod_{i=1}^n p_i}$$

and

$$\max_{x\in[0,1]}\sum_{i=1}^n|u_i^3(x)|^{p_i}\leq\frac{\underline{p}\theta_3^{p_n}}{\prod_{i=1}^n p_i}.$$

Then, taking into account the fact that the weak solutions of system $(P^{F,G}_{\lambda,\mu})$ are exactly critical points of functional I_λ, we find from Lemma 2.4 the desired conclusion immediately.

Remark 3.2. There are no asymptotic conditions on F and G are needed and only algebraic conditions on F are imposed to guarantee the existence of the classical solutions in Theorem 3.1.

For positive constants θ_1, θ_4 and τ, set

$$\begin{aligned}\delta'_{\lambda,G} := \min\Bigg\{ &\frac{1}{c\prod_{i=1}^n p_i} \min\Bigg\{ \frac{\theta_1^{p_n} - \lambda c \prod_{i=1}^n p_i \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx}{G^{\theta_1}},\\
&\frac{\theta_4^{p_n} - 2\lambda c \prod_{i=1}^n p_i \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_4^{p_n}}{2\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx}{2G^{\frac{1}{\sqrt[p_n]{2}}\theta_4}},\\
&\frac{\theta_4^{p_n} - 2\lambda c \prod_{i=1}^n p_i \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_4^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx}{2G^{\theta_4}}\Bigg\},\\
&\frac{\frac{(\sigma_n\tau)^{p_n}}{p_n} - \lambda\Big(\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx - \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx\Big)}{G_\tau - G^{\theta_1}}\Bigg\}.\end{aligned} \tag{3.7}$$

Now, we deduce the following straightforward consequence of Theorem 3.1.

Theorem 3.3. *Let $F:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ satisfy condition $F(x,t_1,\dots,t_n)\geq 0$ for all $(x,t_1,\dots,t_n)\in[0,1]\times(\mathbb{R}^+\cup\{0\})^n$ for $1\leq i\leq n$. Assume that there exist positive constants θ_1, θ_4 and τ with $\theta_1<\tau$ and $\theta_1<\sqrt[p_n]{\frac{c\prod_{i=1}^n p_i}{p_n}}\sigma_n\tau<\frac{\theta_4}{\sqrt[p_n]{2}}$ such that*

(A$_2$)

$$\max\Bigg\{\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx}{\theta_1^{p_n}},\ \frac{2\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_4^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx}{\theta_4^{p_n}}\Bigg\}$$

$$< \frac{p_n}{p_n+\sigma_n^{p_n}c\prod_{i=1}^n p_i}\frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}{\tau^{p_n}}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{p_n+\sigma_n^{p_n} c\prod_{i=1}^n p_i}{p_n c\prod_{i=1}^n p_i}\tau^{p_n}}{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}, \frac{1}{c\prod_{i=1}^n p_i} \min\left\{ \frac{\theta_1^{p_n}}{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}, \right.\right.$$

$$\left.\left. \frac{\theta_4^{p_n}}{2\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_4^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx} \right\}\right)$$

and for every non-negative function $G : [0,1]\times\mathbb{R}^n \to \mathbb{R}$*, there exists* $\delta'_{\lambda,G} > 0$ *given by (3.7) such that, for each* $\mu \in [0,\delta'_{\lambda,G})$*, system* $(P^{F,G}_{\lambda,\mu})$ *possesses at least three non-negative classical solutions* $u^1 = (u_1^1,\ldots,u_n^1)$*,* $u^2 = (u_1^2,\ldots,u_n^2)$*, and* $u^3 = (u_1^3,\ldots,u_n^3)$ *such that*

$$\max_{x\in[0,1]}\sum_{i=1}^n |u_i^1(x)|^{p_i} \le \frac{\underline{p}\theta_1^{p_n}}{\prod_{i=1}^n p_i},\ \max_{x\in[0,1]}\sum_{i=1}^n |u_i^2(x)|^{p_i} \le \frac{\underline{p}\theta_4^{p_n}}{2\prod_{i=1}^n p_i}$$

and

$$\max_{x\in[0,1]}\sum_{i=1}^n |u_i^3(x)|^{p_i} \le \frac{\underline{p}\theta_4^{p_n}}{\prod_{i=1}^n p_i}.$$

Proof. Choose $\theta_2 = \frac{1}{\sqrt[p_n]{2}}\theta_4$ and $\theta_3 = \theta_4$. From (A_2), one has

$$\begin{aligned}\frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_2^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\theta_2^{p_n}} &= \frac{2\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_4^{p_n}}{2\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\theta_4^{p_n}} \\ &\le \frac{2\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_4^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\theta_4^{p_n}} \\ &< \frac{p_n}{p_n+\sigma_n^{p_n} c\prod_{i=1}^n p_i}\frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}{\tau^{p_n}}\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_3^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\theta_3^{p_n}-\theta_2^{p_n}} &= \frac{2\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta_4^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx}{\theta_4^{p_n}} \\ &< \frac{p_n}{p_n+\sigma_n^{p_n} c\prod_{i=1}^n p_i}\frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}{\tau^{p_n}}.\end{aligned} \tag{3.9}$$

Moreover, taking into account that $\theta_1 < \tau$, by using (A_2), we have

$$\frac{p_n}{\sigma_n^{p_n} c \prod_{i=1}^n p_i} \frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx - \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx}{\tau^{p_n}}$$

$$> \frac{p_n}{\sigma_n^{p_n} c \prod_{i=1}^n p_i} \Bigg(\frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}{\tau^{p_n}} - \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)dx}{\theta_1^{p_n}} \Bigg)$$

$$> \frac{p_n}{\sigma_n^{p_n} c \prod_{i=1}^n p_i} \Bigg(\frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}{\tau^{p_n}} - \frac{p_n}{p_n + \sigma_n^{p_n} c \prod_{i=1}^n p_i} \frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}{\tau^{p_n}} \Bigg)$$

$$= \frac{p_n}{p_n + \sigma_n^{p_n} c \prod_{i=1}^n p_i} \frac{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}{\tau^{p_n}}.$$

Hence, from (A_2), (3.8) and (3.9), it is easy to see that the assumption (A_1) of Theorem 3.1 is satisfied, and it follows the conclusion.

We now present the following example to illustrate Theorem 3.3.

Example 3.4. *Let $n = 2$, $p_1 = p_2 = 2$, $m = 1$, $x_1 = \frac{1}{2}$ and $a_1 = b_1 = 2$. We consider the following problem*

$$\begin{cases} -u_1'' = \lambda F_{u_1}(u_1,u_2) + \mu G_{u_1}(u_1,u_2), & t \in (0,1), \\ -u_2'' = \lambda F_{u_2}(u_1,u_2) + \mu G_{u_2}(u_1,u_2), & t \in (0,1), \\ u_1(0) = 2u_1(\frac{1}{2}), \quad u_1(1) = 2u_1(\frac{1}{2}), \\ u_2(0) = 2u_2(\frac{1}{2}), \quad u_2(1) = 2u_2(\frac{1}{2}), \end{cases} \tag{3.10}$$

where

$$F(t_1,t_2) = \begin{cases} (t_1^2+t_2^2)^6, & \text{if } t_1^2+t_2^2 \le 1, \\ \frac{1}{(t_1^2+t_2^2)^2}, & \text{if } t_1^2+t_2^2 > 1. \end{cases}$$

By choosing $\tau = 1$, we have $w(t) = (0, w_1(t))$ with

$$w_1(t) = \begin{cases} 2(1-2t), & \text{if } t \in [0,\frac{1}{4}), \\ 1, & \text{if } t \in [\frac{1}{4},\frac{3}{4}], \\ -2(1-2t), & \text{if } t \in (\frac{3}{4},1]. \end{cases}$$

By some simple calculations, we obtain $c = \frac{5}{2}$ and $\sigma_2 = 2\sqrt{2}$. Taking $\theta_1 = 10^{-8}$ and $\theta_4 = 4\times 10^2$, we see that all conditions in Theorem 3.3 are satisfied. Therefore, it follows that for *each*

$$\lambda \in \Big(\frac{2+32c}{4c}, \frac{2\times 10^8}{c}\Big)$$

and for every non-negative function $G : \mathbb{R}^2 \to \mathbb{R}$*, there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta)$*, system (3.10) possesses at least three non-negative classical solutions* $u^1 = (u_1^1, u_2^1)$*,* $u^2 = (u_1^2, u_2^2)$*, and* $u^3 = (u_1^3, u_2^3)$ *such that*

$$\max_{x\in[0,1]} \sum_{i=1}^{2} |u_i^1(x)|^2 \le \frac{10^{-16}}{2}, \ \max_{x\in[0,1]} \sum_{i=1}^{2} |u_i^2(x)|^2 \le 4\times 10^4$$

and

$$\max_{x\in[0,1]} \sum_{i=1}^{2} |u_i^3(x)|^2 \le 8\times 10^4.$$

Following the idea in [24, Corollary 3.1] we present the below result as a consequence of Theorem 3.3.

Theorem 3.5. *Let* $F : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ *be a continuous function such that*

$$\exists \xi_i, \ F_{\xi_i}(x,\xi_1,\ldots,\xi_i,\ldots,\xi_n)\xi_i > 0, \ i = 1,\ldots,n$$

and

$$F(x,\xi_1,\ldots,0,\ldots,\xi_n) \ge 0$$

for all $(x,\xi_1,\ldots,\xi_n) \in [0,1]\times\mathbb{R}^n$*. Assume that*

(A$_3$) $$\lim_{(\xi_1,\ldots,\xi_n)\to(0,\ldots,0)} \frac{F(x,\xi_1,\ldots,\xi_n)}{\sum_{i=1}^n |\xi_i|^{p_i}} = \lim_{(|\xi_1|,\ldots,|\xi_n|)\to(\infty,\ldots,\infty)} \frac{F(x,\xi_1,\ldots,\xi_n)}{\sum_{i=1}^n |\xi_i|^{p_i}} = 0.$$

Then, for every $\lambda > \overline{\lambda}$*, where*

$$\overline{\lambda} = \frac{p_n + \sigma_n^{p_n} c \prod_{i=1}^n p_i}{p_n c \prod_{i=1}^n p_i} \max\Big\{ \inf_{\tau>0} \frac{\tau^{p_n}}{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}, \inf_{\tau<0} \frac{(-\tau)^{p_n}}{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx} \Big\},$$

system $(P_{\lambda,\mu}^{F,G})$*, in the case* $\mu = 0$*, possesses at least four distinct non-trivial and non-negative classical solutions.*

Proof. Set

$$F_1(x,t_1,\ldots,t_n) = \begin{cases} F(x,t_1,\ldots,t_n), & \text{if } (x,t_1,\ldots,t_n) \in [0,1]\times[0,+\infty)^n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F_2(x,t_1,\ldots,t_n) = \begin{cases} F(x,-t_1,\ldots,-t_n), & \text{if } (x,t_1,\ldots,t_n) \in [0,1]\times[0,+\infty)^n, \\ 0, & \text{otherwise.} \end{cases}$$

Fix $\lambda > \lambda^*$, and let $\tau > 0$ such that

$$\lambda > \frac{\frac{p_n+\sigma_n^{p_n} c\prod_{i=1}^n p_i}{p_n c\prod_{i=1}^n p_i}\tau^{p_n}}{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F_1(x,\bar{\tau})dx}.$$

From

$$\lim_{(\xi_1,\dots,\xi_n)\to(0^+,\dots,0^+)} \frac{F_1(x,\xi_1,\dots,\xi_n)}{\sum_{i=1}^n |\xi_i|^{p_i}} = \lim_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)} \frac{F_1(x,\xi_1,\dots,\xi_n)}{\sum_{i=1}^n |\xi_i|^{p_i}} = 0,$$

there is $\theta_1 > 0$ such that $\theta_1 < \sqrt[p_n]{\frac{c\prod_{i=1}^n p_i}{p_n}}\sigma_n\tau$ and

$$\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F_1(x,t_1,\dots,t_n)dx}{\theta_1^{p_n}} < \frac{1}{\lambda\sigma_n^{p_n} c\prod_{i=1}^n p_i}$$

and there is $\theta_4 > 0$ such that $\sqrt[p_n]{\frac{c\prod_{i=1}^n p_i}{p_n}}\sigma_n\tau < \frac{\theta_4}{\sqrt[p_n]{2}}$ and

$$\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_4^{p_n}}{\prod_{i=1}^n p_i})} F_1(x,t_1,\dots,t_n)dx}{\theta_4^{p_n}} < \frac{1}{2\lambda\sigma_n^{p_n} c\prod_{i=1}^n p_i}$$

Then, (A_2) in Theorem 3.3 is fulfilled,

$$\lambda \in \Bigg(\frac{\frac{p_n+\sigma_n^{p_n} c\prod_{i=1}^n p_i}{p_n c\prod_{i=1}^n p_i}\tau^{p_n}}{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F_1(x,\bar{\tau})dx},$$

$$\frac{1}{c\prod_{i=1}^n p_i}\min\Bigg\{\frac{\theta_1^{p_n}}{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i})} F_1(x,t_1,\dots,t_n)dx},$$

$$\frac{\theta_4^{p_n}}{2\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\frac{\theta_4^{p_n}}{\prod_{i=1}^n p_i})} F_1(x,t_1,\dots,t_n)dx}\Bigg\}\Bigg).$$

Hence, system $(P_{\lambda,\mu}^{F_1,G})$, in the case $\mu = 0$ admits two solutions $u^1 = (u_1^1,\dots,u_n^1)$, $u^2 = (u_1^2,\dots,u_n^2)$, which are positive solutions of the system $(P_{\lambda,\mu}^{F,G})$, in the case $\mu = 0$. Next, by similar arguments, from

$$\lim_{(\xi_1,\dots,\xi_n)\to(0^+,\dots,0^+)} \frac{F_2(x,\xi_1,\dots,\xi_n)}{\sum_{i=1}^n |\xi_i|^{p_i}} = \lim_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)} \frac{F_2(x,\xi_1,\dots,\xi_n)}{\sum_{i=1}^n |\xi_i|^{p_i}} = 0,$$

we ensure the existence of two solutions $u^3 = (u_1^3,\dots,u_n^3)$, $u^4 = (u_1^4,\dots,u_n^4)$ for system $(P_{\lambda,\mu}^{F_2,G})$, in the case $\mu = 0$. Clearly, $-u^3 = (-u_1^3,\dots,-u_n^3)$, $-u^4 = (-u_1^4,\dots,-u_n^4)$ are solutions of system $(P_{\lambda,\mu}^{F,G})$, in the case $\mu = 0$ and the conclusion is achieved.

Remark 3.6. We explicitly observe that in Theorem 3.5 no symmetric condition on F is assumed. However, whenever F is an odd continuous non-zero function such that $F(x,t_1,\ldots,t_n) \geq 0$ for all $(x,t_1,\ldots,t_n) \in [0,1] \times [0,+\infty)^n$, ($A_3$) can be replaced by

$$(A_4) \quad \lim_{(\xi_1,\ldots,\xi_n)\to(0^+,\ldots,0^+)} \frac{F(x,\xi_1,\ldots,\xi_n)}{\sum_{i=1}^n \xi_i^{p_i}} = \lim_{(\xi_1,\ldots,\xi_n)\to(\infty,\ldots,\infty)} \frac{F(x,\xi_1,\ldots,\xi_n)}{\sum_{i=1}^n \xi_i^{p_i}} = 0,$$

ensuring the existence of at least four distinct non-trivial and non-negative classical solutions for system ($P_{\lambda,\mu}^{F,G}$), in the case $\mu = 0$ for every $\lambda > \lambda^*$, where

$$\lambda^* = \frac{\frac{p_n+\sigma_n^{p_n} c\prod_{i=1}^n p_i}{p_n c\prod_{i=1}^n p_i}\tau^{p_n}}{\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,\bar{\tau})dx}.$$

For more studies on the subject, we refer the reader to [25,26] and the references therein.

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