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POSITIVE SOLUTIONS FOR A THIRD ORDER TWO-POINT BOUNDARY VALUE PROBLEM AT RESONANCE ON THE POSITIVE HALF LINE

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Abstract. In this paper, we establish sufficient conditions for the existence of positive solutions for a two-point boundary value problem of a third order differential equation at resonance on the positive half line. Our results are based on the Leggett-Williams norm-type theorem of coincidence. As an application, an example is given to demonstrate our results.

Keywords. Third order boundary value problem; Resonance; Cone; Positive solution; Leggett-Williams norm-type theorem.

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1. Introduction

This paper is devoted to the existence of positive solutions for the following third-order differential equation

$$x'''(t) + f(t, x(t)) = 0, \quad t \geq 0, \tag{1.1}$$

subject to the boundary conditions

$$x'(0) = 0,\ x''(\infty) = x''(0) = 2x(0), \tag{1.2}$$

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where $f : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a continuous function. We say that the boundary value problem (bvp for short) (1.1)-(1.2) is at resonance if the linear equation $Lx = -x'''$ with boundary value conditions (1.2) admits a nontrivial solution i.e., if $\dim \mathrm{Ker} L \geq 1$.

The existence of positive solutions for multi-point boundary value problems at resonance on finite or infinite intervals has been studied by several authors using various methods; see, e.g. [1] [2], [3], [4], [5] and references therein. However, most of the results on the third-order bvps only concern the existence of positive solutions in the non-resonant case. For instance, the following second-order bvp at resonance and set on the half-line is discussed in [6], where sufficient conditions for the existence of positive solutions were obtained by using the Leggett-Williams norm-type theorem

$$\begin{cases} (p'(t)x(t))' + g(t) f(t,x(t)) = 0, \text{a.e. } t\in(0,\infty), \\ x(0) = \int_0^\infty g(s)x(s)\,ds, \quad \lim_{t\to\infty} p'(t)x(t) = p'(0)x(0), \end{cases}$$

where $g\in L^1[0,\infty)$, $g(t)>0$ on $[0,\infty)$, $\int_0^\infty g(s)\,ds = 1$, $p\in C[0,\infty)\cap C^1(0,\infty)$, $\frac{1}{p}\in L^1[0,\infty)$, $p(t) > 0$ on $[0,\infty)$, and $\int_0^\infty \frac{1}{p(s)}ds \leq 1$. Resonant problems are generally more difficult to handle due the special fixed point formulation they required.

Inspired and motivated by the above works, in particular, [2], and [5], our aim in this paper is to investigate the existence of positive solutions to the third-order two-point bvp (1.1)-(1.2). More precisely, we shall establish some sufficient conditions guaranteeing the existence of a positive solution. In Section 2, we present some preliminaries and the fixed point formulation of the problem. The main results and proofs are given in Section 3. Finally, we provide an example in Section 4.

2. Preliminaries

For the convenience of the reader, we start with some standard facts about the theory of Fredholm operators (more details can be found in [7]).

Let X, Y be real Banach spaces and consider a linear operator $L : \mathrm{dom} L\subset X\to Y$ and a mapping $N : X\to Y$. Assume that L is a Fredholm operator of index zero, namely $\mathrm{Im} L$ is closed and $\dim \mathrm{Ker} L = co\dim \mathrm{Im} L < \infty$, which implies that there exist continuous projections $P : X\to X$ and $Q : Y\to Y$ such that $\mathrm{Im} P = \mathrm{Ker} L$ and $\mathrm{Ker} Q = \mathrm{Im} L$. Moreover, since $\dim \mathrm{Ker} L =$

$co\dim \operatorname{Im} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Denote by K_p the inverse of the operator $L_p = L \mid_{\operatorname{Ker} P \cap \operatorname{dom}(L)}$.

It is known that the nonlinear equation $Lx = Nx$ is equivalent to $x = [P + JON + K_p (I-Q) N] x$.

We also recall that a nonempty convex closed set $C \subset X$ is said to be a cone provided that

(1) $\mu x \in C$, for all $x \in C$, $\mu \geq 0$ and
(2) $x, -x \in C$ imply $x = 0$.

Note that every cone $C \subset X$ induces a partial order in X defined by

$$x \preceq y \text{ if and only if } y - x \in C.$$

The following property is valid for every cone in a Banach space X.

Lemma 2.1. [8] *Let C be a cone in X. Then for every $x_0 \in C \backslash \{0\}$, there exists a positive number $\sigma(x_0)$ such that $||x + x_0|| \geq \sigma(x_0) ||x||$, $\forall x \in C$.*

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma x = x$ for all $x \in C$ and put

$$\Psi := P + JQN + K_p (I-Q) N \text{ and } \Psi_\gamma := \Psi \circ \gamma.$$

The following result is the main tool for the existence result.

Theorem 2.2. [9] *Let C be a cone in X. Let Ω_1 and Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap \left(\overline{\Omega}_2 \diagdown \Omega_1\right) \neq \phi$. Assume that $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero and*

- $(C1)$: *$QN : X \to Y$ is continuous and bounded and $K_p (I-Q) N : X \to Y$ is compact on every bounded subset of X,*
- $(C2)$: *$Lx \neq \lambda Nx$, $\forall x \in C \cap \operatorname{dom}(L) \cap \partial \Omega_2$ and $\lambda \in (0,1)$,*
- $(C3)$: *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*
- $(C4)$: *$d_B \left((I - (P + JQN) \gamma)_{|\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0 \right) \neq 0$, where d_B stands for the Brouwer degree,*
- $(C5)$: *there exists $x_0 \in C \diagdown \{0\}$ such that $\|x\| \leq \sigma(x_0) \|\Psi x\|$, $\forall x \in C(x_0) \cap \partial \Omega_1$, where $C(x_0) = \{x \in C : \mu x_0 \preceq x \text{for some } \mu > 0\}$ and $\sigma(x_0)$ is such that $||x + x_0|| \geq \sigma(x_0) ||x||$ for all $x \in C$,*
- $(C6)$: *$(P + JQN) \gamma(\partial \Omega_2) \subset C$,*
- $(C7)$: *$\Psi_\gamma \left(\overline{\Omega}_2 \diagdown \Omega_1\right) \subset C$.*

Then equation $Lx = Nx$ has a solution in $C \cap \left(\overline{\Omega}_2 \diagdown \Omega_1\right)$.

3. Main results

We start by setting the functional framework for bvp (1.1)-(1.2).

3.1. Auxiliary Lemmas

Here and hereafter $C[0,\infty)$ and $L^1[0,\infty)$ denote the spaces of continuous and Lebesgue integrable functions on interval $[0,\infty)$, respectively. Define the space

$$X = \left\{ x \in C[0,\infty) : \lim_{t\to\infty} \frac{x(t)}{1+t^2} \text{ exists} \right\}$$

equipped with the norm $\|x\|_X = \sup\limits_{t\geq 0} \frac{|x(t)|}{1+t^2}$ and the space

$$Y = \left\{ y \in C[0,\infty) \cap L^1[0,\infty) : \lim_{t\to\infty} y(t) \text{ exists} \right\}$$

with the norm $\|y\|_Y = \max\left(\|y\|_\infty, \|y\|_1\right)$, where $\|y\|_\infty = \sup_{t\geq 0} |y(t)|$ and $\|y\|_1 = \int_0^\infty |y(s)|\,ds$. Obviously $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are real Banach spaces. Define the operator of differentiation $L : \text{dom}\,(L) \subset X \to Y$ by

$$(Lx)(t) = -x'''(t), \quad t \geq 0, \tag{3.1}$$

where

$$\text{dom}\,(L) = \left\{ x \in X : x''' \in Y \text{ and } x''(0) = x''(\infty) = 2x(0), x'(0) = 0 \right\}$$

and the Nemytskii operator $N : X \to Y$:

$$(Nx)(t) = f(t, x(t)), \quad t \geq 0. \tag{3.2}$$

Thus, bvp (1.1)-(1.2) is equivalent to $Lx = Nx$, $x \in \text{dom}\,(L)$. Regarding operator L, we prove the following.

Lemma 3.1.

(1) L is a Fredholm operator of index 0.

(2) The associated linear continuous projectors $P : X \to X$ and $Q : Y \to Y$ are defined by $(Px)(t) = \frac{1}{3}\left(1+t^2\right) \int_0^\infty e^{-s} x(s)\,ds$ and $(Qy)(t) = e^{-t} \int_0^\infty y(s)\,ds$.

Proof. We can easily see that $\operatorname{Ker} L=\left\{a\left(t^{2}+1\right): a \in \mathbb{R}\right\}$. It follows that $\operatorname{dim} \operatorname{Ker} L=1$ and $\operatorname{Im} L=\left\{y \in Y: \int_{0}^{\infty} y(s) d s=0\right\}$ due to $\int_{0}^{\infty} y(s) d s=0$, for all $y \in Y$. There exists $x \in \operatorname{dom}(L)$ such that $y(t)=-x^{\prime \prime \prime}(t)$, where

$$x(t)=a\left(t^{2}+1\right)-\frac{1}{2} \int_{0}^{t}(t-s)^{2} y(s) d s,$$

which imply that $y \in \operatorname{Im} L$. In addition, for each $y \in \operatorname{Im} L$, there exists $x \in \operatorname{dom}(L)$ such that $y(t)=-x^{\prime \prime \prime}(t)$. It follows that $x(t)=a t^{2}+b t+c-\frac{1}{2} \int_{0}^{t}(t-s)^{2} y(s) d s$. This combined with boundary conditions (1.2) implies $\int_{0}^{\infty} y(s) d s=0$. Now, for $y \in Y$, we have, for all $t \geq 0$,

$$\begin{aligned}
Q(Q y)(t) & =e^{-t} \int_{0}^{\infty} Q y(s) d s=e^{-t} \int_{0}^{\infty} e^{-r}\left(\int_{0}^{\infty} y(s) d s\right) d r \\
& =e^{-t} \int_{0}^{\infty} y(s) d s \int_{0}^{\infty} e^{-r} d r=(Q y)(t)
\end{aligned}$$

and $|(Q y)(t)| \leq\|y\|_{1} \leq\|y\|_{Y}$ and $\int_{0}^{\infty}|Q y(s)| d s \leq\|y\|_{1} \leq\|y\|_{Y}$. Then

$$\|Q y\|_{Y} \leq\|y\|_{Y}, \tag{3.3}$$

which shows that Q is a linear continuous projector with $\operatorname{Ker} Q=\operatorname{Im} L$. Moreover, for all $y \in Y$, $y=(I-Q) y+Q y \in \operatorname{Ker} Q+\operatorname{Im} Q$, that is, $Y=\operatorname{Im} L+\operatorname{Im} Q$. If $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, then, for all $t \geq 0$, $y(t)=a . e^{-t}$ and $\int_{0}^{\infty} a . e^{-t} d t=a=0$. As a result $y=0$. It follows that $Y=\operatorname{Im} L \oplus \operatorname{Im} Q$. Note that $\operatorname{Im} L$ is closed and $\operatorname{dim} \operatorname{Ker} L=\operatorname{Im} Q=co \operatorname{dim} \operatorname{Im} L=1$. Therefore L is a Fredholm operator of index 0. Finally, for $x \in X$ and $t \geq 0$, we have

$$\begin{aligned}
P(P x)(t) & =\frac{1}{3}\left(1+t^{2}\right) \int_{0}^{\infty} e^{-s}(P x)(s) d s \\
& =\frac{1}{3}\left(1+t^{2}\right) \int_{0}^{\infty} e^{-s}\left(\frac{1}{3}\left(1+s^{2}\right) \int_{0}^{\infty} e^{-r} x(r) d r\right) d s \\
& =\left(\frac{1}{3}\left(1+t^{2}\right) \int_{0}^{\infty} e^{-s} x(s) d s\right) \int_{0}^{\infty} e^{-s} \frac{1}{3}\left(1+s^{2}\right) d s \\
& =\frac{1}{3}\left(1+t^{2}\right) \int_{0}^{\infty} e^{-s} x(s) d s=(P x)(t)
\end{aligned}$$

and

$$\begin{aligned}
\frac{|(P x)(t)|}{1+t^{2}} & =\frac{1}{3}\left|\int_{0}^{\infty} e^{-s} x(s) d s\right| \\
& \leq \frac{1}{3} \int_{0}^{\infty} e^{-s}|x(s)| d s \\
& \leq \frac{1}{3} \int_{0}^{\infty}\left(1+s^{2}\right) e^{-s} \frac{|x(s)|}{1+s^{2}} d s \\
& \leq\|x\|_{X} .
\end{aligned}$$

So $\|P x\|_{X} \leq\|x\|_{X}$. This proves that P is a linear continuous projector such that $\operatorname{Im} P=\operatorname{Ker} L$.

Lemma 3.2. *The generalized inverse* $K_P : \mathrm{Im}\, L \to \mathrm{dom}\, L \cap \mathrm{Ker}\, P$ *of* L *can be represented by*

$$(K_P y)(t) = \frac{1}{6}\int_0^\infty k(t,s)\, y(s)\, ds,$$

where

$$k(t,s) = \begin{cases} 2\left(t^2+1\right)e^{-s} - 3(t-s)^2, & 0 \le s \le t < \infty, \\ 2\left(t^2+1\right)e^{-s}, & 0 \le t \le s < \infty, \end{cases}$$

and

$$\|K_P y\|_X \le \frac{5}{6}\|y\|_Y, \quad \forall y \in \mathrm{Im}\, L. \tag{3.4}$$

Proof. In fact, for all $y \in \mathrm{Im}\, L$, we have

$$\begin{aligned} (LK_P y)(t) &= -(K_P y)'''(t) \\ &= -\left(\frac{t^2+1}{3}\int_0^\infty e^{-s} y(s)\, ds - \frac{1}{2}\int_0^t (t-s)^2 y(s)\, ds\right)''' \\ &= y(t), \ t \ge 0 \end{aligned}$$

and for $x \in \mathrm{dom}\,(L) \cap \mathrm{Ker}\, P$, we have

$$\begin{aligned} (K_P L x)(t) &= \left(K_P\left(-x'''\right)\right)(t) \\ &= \frac{t^2+1}{3}\int_0^\infty e^{-s}\left(-x'''\right)(s)\, ds \\ &\quad -\frac{1}{2}\int_0^t (t-s)^2\left(-x'''\right)(s)\, ds \\ &= \frac{t^2+1}{3}\left(x''(0) + x(0) - \int_0^\infty e^{-s} x(s)\, ds\right) \\ &\quad + \left(-\frac{t^2}{2}x''(0) - x(0) + x(t)\right), \ t \ge 0. \end{aligned}$$

Also $x''(0) = 2x(0)$ and $\int_0^\infty e^{-s} x(s)\, ds = 0$. Thus $(K_P L x)(t) = x(t)$, for all $t \ge 0$ and $K_p = \left(L\,|_{\mathrm{dom}\,(L)\cap \mathrm{Ker}\, P}\right)^{-1}$. Finally, for $y \in \mathrm{Im}\, L$ and $t \ge 0$, we have

$$\begin{aligned} \frac{|(K_P y)(t)|}{1+t^2} &\le \frac{1}{3}\int_0^\infty e^{-s}|y(s)|\, ds + \frac{1}{2}\int_0^t \frac{(t-s)^2}{1+t^2}|y(s)|\, ds \\ &\le \frac{1}{3}\int_0^\infty |y(s)|\, ds + \frac{1}{2}\int_0^\infty |y(s)|\, ds \\ &= \frac{5}{6}\|y\|_1 \le \frac{5}{6}\|y\|_Y \end{aligned}$$

which implies that $\|K_P y\|_X \le \frac{5}{6}\|y\|_Y$. This completes the proof.

We always assume that the following condition holds throughout this paper.

(H_0)**:** The function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and, for every $\rho > 0$, there exists $\varphi_\rho \in C[0,\infty) \cap L^1[0,\infty)$ satisfying $\varphi_\rho(t) > 0$, for all $t \geq 0$ and $\lim_{t\to\infty} \varphi_\rho(t) < \infty$ such that $\left|f(t,\left(1+t^2\right)x)\right| \leq \varphi_\rho(t)$ for all $x \in \mathbb{R}$ such that $|x| < \rho$ and all $t \geq 0$.

Also, the condition $f(t,0) \not\equiv 0$, on $(0,+\infty)$ allows to avoid the trivial solution. The second technical lemma we need is as follows.

Lemma 3.3. *The operator N defined as in (3.2) is L-completely continuous on X. (i.e., continuous and L-compact on every bounded subset of X).*

Since the Arzelà-Ascoli theorem fails in the noncompact interval case, we will make use of the following criterion to prove this lemma.

Theorem 3.4. ([10]) *A subset M is relatively compact in X if and only if the following conditions are satisfied:*
(1) M is uniformly bounded, that is, there exists a constant $m > 0$, such that $\|x\| \leq m$, $\forall x \in M$,
(2) functions from M are quasi-equicontinuous, i.e., equicontinuous on every compact subinterval of $[0,+\infty)$, that is, given a compact subinterval $J \subset [0,+\infty)$, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for t_1,t_2 from J with $|t_2 - t_1| < \delta$,

$$\left|\frac{x(t_2)}{1+t_2^2} - \frac{x(t_1)}{1+t_1^2}\right| < \varepsilon, \quad \forall x \in M,$$

(3) functions from M are equiconvergent at infinity, that is, given $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that, for all $t \geq T$,

$$\left|\frac{x(t)}{1+t^2} - \lim_{t\to\infty} \frac{x(t)}{1+t^2}\right| < \varepsilon, \quad \forall x \in M.$$

Proof of Lemma 3.3. Letting B be a nonempty bounded subset of X, one sees that there exists $r > 0$ such that, for all $x \in \overline{B}$, $\|x\|_X = \sup_{t\geq 0} \frac{|x(t)|}{1+t^2} < r$. By condition (H_0), we get

$$\begin{aligned} |Nx(s)| = |f(s,x(s))| &= \left|f\left(s,\left(1+s^2\right)\tfrac{x(s)}{1+s^2}\right)\right| \\ &\leq \|\varphi_r\|_\infty \leq \|\varphi_r\|_Y . \end{aligned}$$

This implies that

$$\begin{aligned} \|Nx\|_1 &= \textstyle\int_0^\infty |f(s,x(s))|\,ds \\ &\leq \textstyle\int_0^\infty \varphi_r(s)\,ds = \|\varphi_r\|_1 \leq \|\varphi_r\|_Y . \end{aligned}$$

It follows that

$$\|Nx\|_Y \leq \|\varphi_r\|_Y < \infty. \tag{3.5}$$

From (3.3), one sees that

$$\|QNx\|_Y \leq \|Nx\|_Y, \tag{3.6}$$

which shows that QN is bounded. Next, we show that $K_p(I-Q)N$ is compact on B. For all $t \geq 0$, we have

$$\begin{aligned} K_p(I-Q)Nx(t) &= \frac{1}{6}\left(\int_0^\infty k(t,s)f(s,x(s))\,ds \right.\\ &\quad \left. -\int_0^\infty k(t,s)e^{-s}ds\int_0^\infty f(s,x(s))\,ds\right)\\ &= \frac{1}{6}\left(\int_0^\infty k(t,s)f(s,x(s))\,ds \right.\\ &\quad \left. +\left(2t^2-6t-6e^{-t}+5\right)\int_0^\infty f(s,x(s))\,ds\right). \end{aligned}$$

In view of (3.4), (3.5), and (3.6), one obtains

$$\begin{aligned} \left\|K_p(I-Q)Nx\right\|_X &\leq \left\|K_pNx\right\|_X+\left\|K_pQNx\right\|_X\\ &\leq \frac{5}{6}\left(\|Nx\|_Y+\|QNx\|_Y\right)\\ &\leq \frac{5}{3}\|Nx\|_Y\\ &\leq \frac{5}{3}\|\varphi_r\|_Y<\infty, \end{aligned}$$

which shows that $K_p(I-Q)N(B)$ is uniformly bounded. Using the Lebesgue dominated convergence theorem and condition (H_0), we can easily find that that QN and $K_p(I-Q)N$ are continuous. It remains to study the equicontinuity. For $x \in B$ and $0 \leq t_1 \leq t_2 \leq T < \infty$, we have the estimates

$$\begin{aligned} &\frac{K_p(I-Q)Nx(t_2)}{1+t_2^2}-\frac{K_p(I-Q)Nx(t_1)}{1+t_1^2}\\ =\;& \frac{1}{6}\int_0^\infty\left(\frac{k(t_2,s)}{1+t_2^2}-\frac{k(t_1,s)}{1+t_1^2}\right)f(s,x(s))\,ds\\ &+\left(\frac{1}{6}\left(\frac{2t_2^2}{1+t_2^2}-\frac{2t_1^2}{1+t_1^2}\right)-\left(\frac{t_2}{1+t_2^2}-\frac{t_1}{1+t_1^2}\right)\right.\\ &\left.-\left(\frac{e^{-t_2}}{1+t_2^2}-\frac{e^{-t_1}}{1+t_1^2}\right)+\frac{5}{6}\left(\frac{1}{1+t_2^2}-\frac{1}{1+t_1^2}\right)\right)\int_0^\infty f(s,x(s))\,ds. \end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\frac{K_p(I-Q)Nx(t_2)}{1+t_2^2}-\frac{K_p(I-Q)Nx(t_1)}{1+t_1^2}\right|\\
\leq\ &\frac{1}{6}\int_0^\infty\left|\frac{k(t_2,s)}{1+t_2^2}-\frac{k(t_1,s)}{1+t_1^2}\right|\left|f(s,x(s))\right|ds\\
&+\left(\frac{1}{6}\left|\frac{2t_2^2}{1+t_2^2}-\frac{2t_1^2}{1+t_1^2}\right|+\left|\frac{t_2}{1+t_2^2}-\frac{t_1}{1+t_1^2}\right|+\left|\frac{e^{-t_2}}{1+t_2^2}-\frac{e^{-t_1}}{1+t_1^2}\right|\right.\\
&\left.+\frac{5}{6}\left|\frac{1}{1+t_2^2}-\frac{1}{1+t_1^2}\right|\right)\int_0^\infty\left|f(s,x(s))\right|ds.
\end{aligned}$$

By simple calculation, we find that

$$\Delta(s):=\left|\frac{k(t_2,s)}{1+t_2^2}-\frac{k(t_1,s)}{1+t_1^2}\right|=\begin{cases}0, & \text{if } 0\leq t_1\leq t_2\leq s,\\ 3\frac{(t_2-s)^2}{1+t_2^2}, & \text{if } 0\leq t_1\leq s\leq t_2\leq T,\\ 3\left|\frac{(t_2-s)^2}{1+t_2^2}-\frac{(t_1-s)^2}{1+t_1^2}\right|, & \text{if } 0\leq s\leq t_1\leq t_2\leq T.\end{cases}$$

In view of (3.5), (3.6), we have

$$\begin{aligned}
\left|\frac{K_p(I-Q)Nx(t_2)}{1+t_2^2}-\frac{K_p(I-Q)Nx(t_1)}{1+t_1^2}\right| \leq\ &\|\varphi_r\|_Y\left(\frac{1}{6}\int_0^\infty\Delta(s)\,ds+\frac{1}{3}\left|\frac{t_2^2}{1+t_2^2}-\frac{t_1^2}{1+t_1^2}\right|\right.\\
&\left.+\left|\frac{t_2}{1+t_2^2}-\frac{t_1}{1+t_1^2}\right|+\left|\frac{e^{-t_2}}{1+t_2^2}-\frac{e^{-t_1}}{1+t_1^2}\right|+\frac{5}{6}\left|\frac{1}{1+t_2^2}-\frac{1}{1+t_1^2}\right|\right).
\end{aligned}$$

By the uniform continuity of functions $\frac{t^2}{1+t^2},\frac{t}{1+t^2},\frac{e^{-t}}{1+t^2},\frac{1}{1+t^2}$ in $[0,T]$ and the fact that $\Delta(s)\to 0$ uniformly as $|t_2-t_1|\to 0$ in all possible cases above, we deduce that $K_p(I-Q)N$ is equicontinuous on $[0,T]$ for all $T>0$. Since

$$\begin{aligned}
K_p(I-Q)Nx(\infty) &= \lim_{t\to\infty}\frac{K_p(I-Q)Nx(t)}{1+t^2}\\
&= \frac{1}{6}\int_0^\infty\left(2e^{-s}-1\right)f(s,x(s))\,ds,
\end{aligned}$$

one has

$$\begin{aligned}
&\left|\frac{K_p(I-Q)Nx(t)}{1+t^2}-K_p(I-Q)Nx(\infty)\right|\\
=\ &\frac{1}{6}\left|\int_0^\infty\frac{k(t,s)}{1+t^2}f(s,x(s))\,ds+\left(\frac{2t^2-6t-6e^{-t}+5}{1+t^2}\right)\int_0^\infty f(s,x(s))\,ds\right.\\
&\left.-\int_0^\infty\left(2e^{-s}-1\right)f(s,x(s))\,ds\right|\\
=\ &\frac{1}{6}\left|\left(\int_0^\infty\left(\frac{k(t,s)}{1+t^2}-\left(2e^{-s}-3\right)\right)f(s,x(s))\,ds\right)\right.\\
&\left.+\left(\frac{2t^2-6t-6e^{-t}+5}{1+t^2}-2\right)\int_0^\infty f(s,x(s))\,ds\right|\\
\leq\ &\frac{1}{6}\int_0^\infty\left|\frac{k(t,s)}{1+t^2}-\left(2e^{-s}-3\right)\right|\left|f(s,x(s))\right|ds\\
&+\left|\frac{2t^2-6t-6e^{-t}+5}{1+t^2}-2\right|\int_0^\infty\left|f(s,x(s))\right|ds\\
\leq\ &\frac{\|\varphi_r\|_Y}{6}\left(\int_0^\infty\left|\frac{k(t,s)}{1+t^2}-\left(2e^{-s}-3\right)\right|ds+\left|\frac{-6t-6e^{-t}+3}{1+t^2}\right|\right),
\end{aligned}$$

where the last term tends to 0 uniformly, as $t\to\infty$ for $\lim_{t\to\infty}\frac{k(t,s)}{1+t^2}=2e^{-s}-3$. So $K_p(I-Q)N$ is equiconvergent at infinity. Therefore $Kp(I-Q)N(B)$ is relatively compact, which completes the proof.

For simplicity of notations, we set

$$U(t,s)=\frac{1}{6}\begin{cases} 8+\frac{1+6e^{-t}}{1+t^2}+2e^{-s}-\frac{6t+8+3(t-s)^2}{1+t^2}, & \text{if } 0\le s\le t<\infty,\\ 8+\frac{1+6e^{-t}}{1+t^2}+2e^{-s}-\frac{6t+8}{1+t^2}, & \text{if } 0\le t\le s<\infty.\end{cases}$$

We can verify that $0.17<\frac{5}{6}+\frac{e^{-t}-t-\frac{2}{3}}{1+t^2}<U(t,s)\le\frac{17}{6}$.

3.2. Existence theorems

We are now in position to prove our main result.

Theorem 3.5. *Further to condition* (H_0), *we assume that*

(H_1)**:** *there exist three positive real numbers* $\alpha,\beta,\ \kappa$ *and three non negative functions* q,v,r *such that* $\alpha\neq 0,\ 0<\kappa<\frac{2}{17},\ r(t),v(t)\in L^1[0,\infty),\ \ q(t)>0,$ *for all* $t\ge 0$ *and* $S_0=\sup_{t\ge 0}\frac{t^2+1}{q(t)}<\infty$ *satisfying*

$$-\kappa e^{-t}x\le f(t,x)\le -q(t)\,e^{-t}\frac{x}{1+t^2}+v(t), \tag{3.7}$$

$$f(t,x)\le -\alpha\,|f(t,x)|+\beta e^{-t}x+r(t), \tag{3.8}$$

for all $t\ge 0$ *and* $x\in\mathbb{R}$ *with* $0\le\frac{x}{1+t^2}\le M_1$, *where*

$$M_1\ge M_0:=S_0\left(\frac{1}{3}+\frac{5}{6}\frac{\beta}{\alpha}\right)\|v\|_1+\frac{5}{6\alpha}\|r\|_1,$$

(H_2)**:** *there exists a real* R *such that* $M_0<R\le M_1$ *and* $f\left(t,R\left(1+t^2\right)\right)<0,$ *for all* $t\ge 0,$

(H_3)**:** *there exists* $r\in(0,M_0),t_0\in[0,\infty),\eta\in(0,1),$ *and* $b\in(0,1)$, *such that for each* $t\in[0,\infty)$, *the function* $x\mapsto\frac{f\left(t,\left(1+t^2\right)x\right)}{x^\eta}$ *is nonincreasing on* $(0,r]$, *with*

$$\int_0^\infty U(t_0,s)\,\frac{f\left(s,\left(1+s^2\right)r\right)}{r}ds\ge\frac{1-b}{b^\eta}. \tag{3.9}$$

Then bvp (1.1)-(1.2) has at least one positive solution in $\operatorname{dom}(L)$.

Proof. In view of Lemma 3.1, one sees that L is a Fredholm operator of index 0. From Lemma 3.3, the condition (C_1) of Theorem 2.2 is fulfilled.

Define the cone C of nonnegative functions and the subsets Ω_1,Ω_2 of X by

$$C=\{x\in X: x(t)\ge 0,\ t\ge 0\},$$

$$\Omega_1=\left\{x\in X: b\,\|x\|_X<\frac{|x(t)|}{1+t^2}<r,\ t\ge 0\right\}\ \text{and}\ \Omega_2=\{x\in X:\|x\|_X<R\},$$

where R, r and b are defined in conditions $(H_2),(H_3)$ of Theorem 3.5. Clearly Ω_1,Ω_2 are bounded open sets in X. Furthermore, one has

$$\overline{\Omega_1}=\left\{x\in X: b\|x\|_X\le \frac{|x(t)|}{1+t^2}\le r<R,\ t\ge 0\right\}\subset\Omega_2$$

and

$$C\cap\left(\overline{\Omega_2}\diagdown\Omega_1\right)\neq\phi$$

because $x(t)=\frac{1}{2}(r+R)t^2$ satisfies

$$x(t)\ge 0,\ t\ge 0 \text{ and } r<\|x\|_X=\frac{1}{2}(r+R)\le R.$$

Define a mapping $\gamma: X\to C$ by $(\gamma x)(t)=|x(t)|$. Then γ is a retraction and maps subsets of $\overline{\Omega_2}$ into bounded subsets of C because we have $(\gamma x)(t)\ge 0$, $t\ge 0$ and $\frac{|(\gamma x)(t)|}{1+t^2}=\frac{|x(t)|}{1+t^2}\le\|x\|_X\le R$ for all $x\in\overline{\Omega_2}$. This means that condition (C_3) of Theorem 2.2 holds.

To show that (C_2) holds, suppose, by contradiction, that there exists $x_0\in C\cap\operatorname{dom}(L)\cap\partial\Omega_2$ and $\lambda_0\in(0,1)$ such that $Lx_0=\lambda_0 Nx_0$. Then $x_0'''(t)+\lambda_0 f(t,x_0(t))=0$, $t\ge 0$, $x_0(t)\ge 0$, $t\ge 0$, $\|x_0\|_X=R$, and x_0 satisfies the boundary conditions (1.2). For all $t\ge 0$, one has

$$\begin{aligned}\frac{x_0(t)}{1+t^2} &= \frac{(Px_0)(t)+(K_PLx_0)(t)}{1+t^2}\\ &= \frac{1}{3}\int_0^\infty e^{-s}x_0(s)\,ds+\frac{1}{6}\int_0^\infty\frac{k(t,s)}{1+t^2}\left(-x_0'''(s)\right)ds\\ &= \frac{1}{3}\int_0^\infty e^{-s}x_0(s)\,ds+\frac{1}{6}\int_0^\infty\frac{k(t,s)}{1+t^2}\lambda_0 f(s,x_0(s))\,ds\\ &\le \frac{1}{3}\int_0^\infty e^{-s}x_0(s)\,ds+\frac{5}{6}\int_0^\infty|f(s,x_0(s))|\,ds.\end{aligned}$$

Using (H_1) and (3.8), we obtain that

$$-x_0'''(s)=\lambda_0 f(s,x_0(s))\le\lambda_0\left(-\alpha|f(s,x_0(s))|+\beta e^{-s}x_0(s)+r(s)\right).$$

By integrating both sides of inequality, we get

$$\begin{aligned}\textstyle\int_0^\infty -x_0'''(s)\,ds &= \lambda_0\textstyle\int_0^\infty f(s,x_0(s))\,ds\\ &\le \lambda_0\left(-\alpha\textstyle\int_0^\infty|f(s,x_0(s))|\,ds+\beta\int_0^\infty e^{-s}x_0(s)\,ds\right.\\ &\quad\left.+\textstyle\int_0^\infty r(s)\,ds\right).\end{aligned}$$

Since

$$\int_0^\infty -x_0'''(s)\,ds=x_0''(0)-x_0''(\infty)=0,$$

one has

$$\int_0^\infty |f(s,x_0(s))|\,ds \le \frac{\beta}{\alpha}\int_0^\infty e^{-s}x_0(s)\,ds + \frac{1}{\alpha}\int_0^\infty r(s)\,ds. \tag{3.10}$$

Using (3.7) in (H_1), we find

$$-x_0'''(s) = \lambda_0 f(s,x_0(s)) \le \lambda_0\left(-q(s)e^{-s}\frac{x_0(s)}{1+s^2}+v(s)\right).$$

Then

$$\begin{aligned}\int_0^\infty -x_0'''(s)\,ds &= \lambda_0\int_0^\infty f(s,x_0(s))\,ds\\ &\le \lambda_0\left(-\int_0^\infty q(s)e^{-s}\frac{x_0(s)}{1+s^2}ds+\int_0^\infty v(s)\,ds\right).\end{aligned}$$

This implies that

$$\int_0^\infty q(s)e^{-s}\frac{x_0(s)}{1+s^2}ds \le \int_0^\infty v(s)\,ds. \tag{3.11}$$

From (3.10), (3.11), we deduce that

$$\begin{aligned}\frac{x_0(t)}{1+t^2} &\le \frac{1}{3}\int_0^\infty e^{-s}x_0(s)\,ds+\frac{5}{6}\int_0^\infty |f(s,x_0(s))|\,ds\\ &\le \frac{1}{3}\int_0^\infty e^{-s}x_0(s)\,ds+\frac{5}{6}\frac{\beta}{\alpha}\int_0^\infty e^{-s}x_0(s)\,ds+\frac{5}{6\alpha}\int_0^\infty r(s)\,ds\\ &= \left(\frac{1}{3}+\frac{5}{6}\frac{\beta}{\alpha}\right)\int_0^\infty \frac{(1+s^2)}{q(s)}q(s)e^{-s}\frac{x_0(s)}{1+s^2}ds+\frac{5}{6\alpha}\int_0^\infty r(s)\,ds\\ &\le S_0\left(\frac{1}{3}+\frac{5}{6}\frac{\beta}{\alpha}\right)\int_0^\infty q(s)e^{-s}\frac{x_0(s)}{1+s^2}ds+\frac{5}{6\alpha}\int_0^\infty r(s)\,ds\\ &\le S_0\left(\frac{1}{3}+\frac{5}{6}\frac{\beta}{\alpha}\right)\int_0^\infty v(s)\,ds+\frac{5}{6\alpha}\int_0^\infty r(s)\,ds.\end{aligned}$$

Therefore

$$\|x_0\|_X = R \le S_0\left(\frac{1}{3}+\frac{5}{6}\frac{\beta}{\alpha}\right)\|v\|_1+\frac{5}{6\alpha}\|r\|_1 = M_0,$$

which contradicts $R > M_0$. Let $J : \operatorname{Im} Q \to \operatorname{Ker} L$ be the isomorphism defined by $J(ae^{-t}) = a(t^2+1)$ and consider the mapping $G(x,\lambda) = [I-\lambda(P+JQN)\gamma]x$, $\lambda\in[0,1]$. For $x \in \operatorname{Ker} L \cap \Omega_2$ and $t \ge 0$, we have $x(t) = a(t^2+1)$, where $|a| < R$. Hence

$$\begin{aligned}G(x,\lambda)(t) &= a(t^2+1)-\lambda\Big(\tfrac{1}{3}(1+t^2)\int_0^\infty e^{-s}|a|(s^2+1)\,ds\\ &\quad +(1+t^2)\int_0^\infty f(s,|a|(s^2+1))\,ds\Big).\end{aligned}$$

By simple calculation, we find that

$$G(x,\lambda)(t) = (1+t^2)\left(a-\lambda\left(|a|+\int_0^\infty f(s,|a|(s^2+1))\,ds\right)\right).$$

Suppose $G(x,\lambda)=0$. By (H_1) (3.7), we get

$$\begin{aligned}a &= \lambda\left(|a|+\int_0^\infty f(s,|a|(s^2+1))\,ds\right)\\ &\ge \lambda\left(|a|+\int_0^\infty -\kappa e^{-s}|a|(s^2+1)\,ds\right) = \lambda|a|(1-3\kappa)\ge 0.\end{aligned}$$

Hence $G(x,\lambda)=0$, which implies that $a\geq 0$. For $x\in \operatorname{Ker}L\cap\partial\Omega_2$, if $G(x,\lambda)=0$, then $R=|a|=a$. Therefore $R=\lambda\left(R+\int_0^\infty f\left(s,R\left(s^2+1\right)\right)ds\right)$. By (H_2), we get

$$(1-\lambda)R=\lambda\int_0^\infty f\left(s,R\left(1+s^2\right)\right)ds<0,$$

which contradicts $(1-\lambda)R\geq 0$. Then $G(x,\lambda)\neq 0$ for all $x\in \operatorname{Ker}L\cap\partial\Omega_2$. Thus, the degree

$$d_B\left([I-\lambda(P+JQN)\gamma]_{|\operatorname{Ker}L},\operatorname{Ker}L\cap\Omega_2,0\right)$$

is well defined. By the homotopy property of the topological degree, we have

$$\begin{aligned} d_B\left([I-(P+JQN)\gamma]_{|\operatorname{Ker}L},\operatorname{Ker}L\cap\Omega_2,0\right) &= d_B\left(G(x,1),\operatorname{Ker}L\cap\Omega_2,0\right)\\ &= d_B\left(G(x,0),\operatorname{Ker}L\cap\Omega_2,0\right)\\ &= 1\neq 0, \end{aligned}$$

which shows that condition (C_4) of Theorem 2.2 holds. For $x\in\partial\Omega_2$, by condition (H_1), we have

$$\begin{aligned} (P+JQN)\gamma x(t) &= \left(1+t^2\right)\left(\frac{1}{3}\int_0^\infty e^{-s}|x(s)|\,ds+\int_0^\infty f(s,|x(s)|)\,ds\right)\\ &\geq \left(1+t^2\right)\left(\frac{1}{3}\int_0^\infty e^{-s}|x(s)|\,ds-\kappa\int_0^\infty e^{-s}|x(s)|\,ds\right)\\ &\geq \left(1+t^2\right)\left(\frac{1}{3}-\kappa\right)\left(\int_0^\infty e^{-s}|x(s)|\,ds\right)\\ &\geq 0,\quad \forall t\geq 0. \end{aligned}$$

Hence $(P+JQN)\gamma(\partial\Omega_2)\subset C$. Notice that

$$\begin{aligned} 0\leq \int_0^\infty e^{-s}|x(s)|\,ds &= \int_0^\infty\left(1+s^2\right)e^{-s}\tfrac{|x(s)|}{1+s^2}ds\\ &\leq R\int_0^\infty\left(1+s^2\right)e^{-s}ds=3R. \end{aligned}$$

Taking $x_0(t)=1+t^2$ on $[0,\infty)$ yields $x_0\in C\setminus\{0\}$ and

$$C(x_0)=\left\{x\in C:\inf_{t\geq 0}\frac{x(t)}{x_0(t)}=\mu>0\right\}=\left\{x\in C:\frac{x(t)}{1+t^2}>0\right\}.$$

So we can choose $\sigma(x_0)=1$. For $x\in C(x_0)\cap\partial\Omega_1$, we have $\frac{x(t)}{1+t^2}>0$, $\frac{x(t)}{1+t^2}\geq b\|x\|_X$ for all $t\geq 0$, and $0<\|x\|_X\leq r$. Therefore, in view of (H_3) ,we have

$$(\Psi x)(t_0)=\left(1+t_0^2\right)\left(\frac{1}{3}\int_0^\infty e^{-s}x(s)\,ds+\int_0^\infty U(t_0,s)\,f(s,x(s))\,ds\right),\quad \forall x\in C(x_0)\cap\partial\Omega_1.$$

Putting $X(s)=\frac{x(s)}{1+s^2}$, we obtain the estimates

$$\begin{aligned}
\frac{(\Psi x)(t_0)}{1+t_0^2} &= \frac{1}{3}\int_0^\infty \left(1+s^2\right)e^{-s}X(s)\,ds \\
&\quad +\int_0^\infty U(t_0,s)\frac{f\left(s,\left(1+s^2\right)X(s)\right)(X(s))^\eta}{(X(s))^\eta}ds \\
&\geq b\|x\|_X\frac{1}{3}\int_0^\infty \left(1+s^2\right)e^{-s}ds \\
&\quad +(b\|x\|_X)^\eta\int_0^\infty U(t_0,s)\frac{f\left(s,\left(1+s^2\right)X(s)\right)}{(X(s))^\eta}ds \\
&\geq b\|x\|_X+(b\|x\|_X)^\eta\int_0^\infty U(t_0,s)\frac{f\left(s,\left(1+s^2\right)r\right)}{r^\eta}ds \\
&\geq b\|x\|_X+\frac{(b\|x\|_X)^\eta}{r^{\eta-1}}\int_0^\infty U(t_0,s)\frac{f\left(s,\left(1+s^2\right)r\right)}{r}ds \\
&\geq b\|x\|_X+\frac{(b\|x\|_X)^\eta}{r^{\eta-1}}\frac{1-b}{b^\eta} \\
&= b\|x\|_X+(1-b)\|x\|_X\left(\frac{r}{\|x\|_X}\right)^{1-\eta} \\
&\geq b\|x\|_X+(1-b)\|x\|_X=\|x\|_X.
\end{aligned}$$

Consequently, $\|\Psi x\|_X\geq\frac{(\Psi x)(t_0)}{1+t_0^2}\geq\|x\|_X$ and hence condition (C_5) is also fulfilled. Now, let $x\in\overline{\Omega_2}\diagdown\Omega_1$. By (H_1) (3.7), we have, for all $t\geq 0$,

$$\begin{aligned}
\Psi_\gamma x\ (t) &= \left(1+t^2\right)\left(\frac{1}{3}\int_0^\infty e^{-s}|x(s)|\,ds+\int_0^\infty U(t,s)f(s,|x(s)|)\,ds\right) \\
&\geq \left(1+t^2\right)\left(\frac{1}{3}\int_0^\infty e^{-s}|x(s)|\,ds-\kappa\int_0^\infty U(t,s)e^{-s}|x(s)|\,ds\right) \\
&\geq \left(1+t^2\right)\left(\frac{1}{3}\int_0^\infty e^{-s}|x(s)|\,ds-\frac{17}{6}\kappa\int_0^\infty e^{-s}|x(s)|\,ds\right) \\
&= \left(1+t^2\right)\left(\frac{1}{3}-\frac{17}{6}\kappa\right)\int_0^\infty e^{-s}|x(s)|\,ds\geq 0.
\end{aligned}$$

Thus $\Psi_\gamma\left(\overline{\Omega}_2\diagdown\Omega_1\right)\subset C$, which completes the proof of the theorem.

4. Example

Consider the two-point boundary value problem

$$\begin{cases} x'''(t) - \frac{1}{18}\frac{(t+1)^2}{t^2+1}e^{-t}\left(x - \frac{e^{\frac{x}{1+t^2}}}{e^{\frac{x}{1+t^2}}+1}\right) + 6e^{-t} = 0,\ t \geq 0, \\ x'(0) = 0,\ x''(\infty) = x''(0) = 2x(0). \end{cases} \tag{4.1}$$

Remark 4.1. We have, for all $t \geq 0$ and $x \in \mathbb{R}$, $x - 1 \leq x - \frac{e^{\frac{x}{1+t^2}}}{e^{\frac{x}{1+t^2}}+1} \leq x$; $1 \leq \frac{(t+1)^2}{t^2+1} \leq 2$.

(1) By taking $\varphi_\rho(t) = \frac{1}{18}(t+1)^2 e^{-t}\rho + \frac{55}{9}e^{-t}$, we can see that $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, where $f(t,x) = -\frac{1}{18}\frac{(t+1)^2}{t^2+1}e^{-t}\left(x - \frac{e^{\frac{x}{1+t^2}}}{e^{\frac{x}{1+t^2}}+1}\right) + 6e^{-t}$ is a continuous function that satisfies condition (H_0).

(2) It can be checked that

$$-\frac{1}{9}e^{-t}x \leq f(t,x) \leq -\frac{1}{18}(t+1)^2 e^{-t}\frac{x}{t^2+1} + \frac{55}{9}e^{-t},$$

i.e. $\kappa = \frac{1}{9}, q(t) = \frac{1}{18}(t+1)^2, v(t) = \frac{55}{9}e^{-t}$, $S_0 = \sup_{t\geq 0}\frac{t^2+1}{q(t)} = 18$ and $\|v\|_1 = \frac{55}{9}$. In addition

$$\begin{aligned} |f(t,x)| &\leq \frac{1}{18}\frac{(t+1)^2}{t^2+1}e^{-t}x + \frac{1}{18}\frac{(t+1)^2}{t^2+1}e^{-t}\frac{e^{\frac{x}{1+t^2}}}{e^{\frac{x}{1+t^2}}+1} + 6e^{-t} \\ &\leq \frac{2}{18}e^{-t}x + \frac{2}{18}e^{-t} + 6e^{-t} = \frac{2}{18}e^{-t}x + \frac{55}{9}e^{-t}, \end{aligned}$$

one has

$$\begin{aligned} |f(t,x)| + f(t,x) &\leq \frac{1}{18}\left(2 - \frac{(t+1)^2}{t^2+1}\right)e^{-t}x + \frac{110}{9}e^{-t} \\ &= \frac{1}{18}\left(1 - \frac{2t}{t^2+1}\right)e^{-t}x + \frac{110}{9}e^{-t} \\ &\leq \frac{1}{18}e^{-t}x + \frac{110}{9}e^{-t}. \end{aligned}$$

Therefore

$$f(t,x) \leq -|f(t,x)| + \frac{1}{18}e^{-t}x + \frac{110}{9}e^{-t},$$

i.e., $\alpha = 1, \beta = \frac{1}{18}, r(t) = \frac{110}{9}e^{-t}, \|r\|_1 = \frac{110}{9}$, and

$$M_0 = S_0\left(\frac{1}{3} + \frac{5}{6}\frac{\beta}{\alpha}\right)\|v\|_1 + \frac{5}{6\alpha}\|r\|_1 = 18(\frac{1}{3} + \frac{5}{6}\frac{1}{18})\frac{55}{9} + \frac{5}{6}\frac{110}{9} \simeq 52.$$

(3) Taking $R=111$ implies that, for $t\geq 0$,

$$\begin{aligned} f\left(t,R\left(1+t^2\right)\right) &= -\frac{1}{18}(t+1)^2e^{-t}R+\frac{1}{18}\frac{(t+1)^2}{t^2+1}e^{-t}\frac{e^R}{e^R+1}+6e^{-t} \\ &\leq -\frac{1}{18}(t+1)^2e^{-t}R+\frac{1}{9}e^{-t}+6e^{-t} \\ &\leq -\frac{1}{18}(t+1)^2e^{-t}R+\frac{55}{9}e^{-t} \\ &= \left(110-(t+1)^2R\right)\frac{e^{-t}}{18} \\ &\leq (110-R)\frac{e^{-t}}{18}=-\frac{e^{-t}}{18}<0. \end{aligned}$$

(4) For $t_0=0$, $\eta=\frac{1}{2}, r=8$ and $b=0.64$, let

$$h_t(x)=\frac{f\left(t,\left(1+t^2\right)x\right)}{\sqrt{x}}=-\frac{1}{18}(t+1)^2e^{-t}\sqrt{x}+\frac{1}{18}\frac{(t+1)^2}{t^2+1}e^{-t}\frac{e^x}{e^x+1}\frac{1}{\sqrt{x}}+\frac{6e^{-t}}{\sqrt{x}}.$$

Since $h_t'(x)=-\left(\frac{1}{18}\frac{(t+1)^2e^{-t}}{2\sqrt{x}}+\frac{1}{18}\frac{(t+1)^2}{t^2+1}e^{-t}\left(e^x+1-2x\right)\frac{e^x}{2x\sqrt{x}(e^x+1)^2}+\frac{6e^{-t}}{2x\sqrt{x}}\right)<0$, for all $x\in(0,r]$ (because, we have $e^x+1-2x>3-2\ln 2$,for all $x\geq 0$) , one sees that h_t is non increasing on $(0,r]$, for all $t\geq 0$ and

$$\begin{aligned} &\int_0^\infty U(0,s)\frac{f\left(s,\left(1+s^2\right)r\right)}{r}ds \\ =\ & \frac{1}{6}\int_0^\infty\left(2e^{-s}+7\right)\left(-\frac{1}{18}(1+s)^2e^{-s}+\frac{1}{18}\frac{(s+1)^2}{s^2+1}e^{-s}\frac{e^r}{e^r+1}\frac{1}{r}+\frac{6}{r}e^{-s}\right)ds \\ \geq\ & \frac{1}{6}\int_0^\infty\left(2e^{-s}+7\right)\left(-\frac{1}{18}(1+s)^2e^{-s}+\frac{6}{r}e^{-s}\right)ds \\ =\ & -\frac{1}{108}\int_0^\infty\left(2e^{-2s}+7e^{-s}\right)\left((1+s)^2-\frac{27}{2}\right)ds \\ =\ & \frac{47}{72}\geq\frac{1-0.64}{0.8}. \end{aligned}$$

Hence, all the conditions of the existence theorem holds. This guarantees that problem (4.1) has at least one positive solution.

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