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STRONG KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS AND DUALITY FOR NONSMOOTH MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING VIA MICHEL-PENOT SUBDIFFERENTIAL

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Abstract. The aim of this paper is to study strong Karush-Kuhn-Tucker optimality conditions and duality for nonsmooth multiobjective semi-infinite programming. By using the Michel-Penot subdifferential and suitable generalized regularity conditions, we establish the strong necessary and sufficient optimality conditions for some kind of efficient solutions of nonsmooth multiobjective semi-infinite programming. We also propose Wolfe and Mond-Weir duality schemes for multiobjective semi-infinite programming and explore weak and strong duality relations under the generalized convexity.

Keywords. Multiobjective semi-infinite programming; KKT optimality condition; Wolfe duality; Mond-Weir duality; Michel-Penot subdifferential.

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1. Introduction

A simultaneous minimization with a finitely many objective functions and an infinite set of constraints is called multiobjective semi-infinite programming (MSIP). Many theoretical aspects and various fields of applications of the semi-infinite programming have been considered by many researchers, see [1, 2] and references therein. Recently, weak and strong Karush-Kuhn-Tucker (KKT) optimality conditions and duality for MSIP have been investigated by many authors. In [3, 4], optimality conditions for some types of efficient solutions of MSIP

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and duality relations were investigated in terms of the Mordukhovich subdifferential. Weak and strong KKT optimality conditions for weakly efficient solutions and Pareto efficient solutions were obtained in [5, 6, 7] based on some regularity conditions in sense of Clarke subdifferential. Constraint qualifications in convex vector semi-infinite optimization were investigated in [8]. Recently, Caristi and Ferrara [9] considered the necessary optimality conditions for weakly efficient solution of MSIP via Michel-Penot subdifferential; see [9] and the references therein.

Strong KKT optimality conditions give more information than the weak KKT optimality conditions since all the multipliers corresponding to the objective functions are positive. In [10], many regularity conditions for differentiable functions were investigated to establish the strong KKT optimality conditions for Pareto efficient solutions of multiobjective optimization problem. Regularity conditions in sense of Clarke subdifferential were considered in [11] and [12]. In [13], strong KKT type sufficient optimality conditions for nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints were given in terms of the Clarke subdifferential. We observe that the strong KKT necessary conditions for weakly efficient solution and the Pareto efficient solution of MSIP via the Michel-Penot subdifferential were not investigated in [9]. Moreover, there is no work dealing with duality relations for nonsmooth multiobjective semi-infinite programming via the Michel-Penot subdifferential. Some models in real life are nonsmooth MSIP such as FIR filter design in [14] or moment robust optimizations and their applications in [15]. The Wolfe and Mond-Weir duality schemes can be used to give the optimality conditions for solving these problems or reformulate the nonsmooth MSIP in [16].

Motivated by the above observations, by using the Michel-Penot subdifferential as generalized derivatives, we establish necessary and sufficient conditions for the Pareto efficient solutions and weakly efiicient solution and duality theorems of the Wolfe and Mond-Weir types for MSIP. The paper is organized as follows. Section 2 recalls basic concepts and some preliminaries. In Section 3, the strong necessary and sufficient KKT optimality conditions for weakly efficient solution and the Pareto efficient solution of MSIP are established. Section 4 is devoted to investigating the Wolfe and Mond-Weir dual type problems of MSIP in term of the Michel-Penot subdifferential. Some examples are provided to illustrate our results.

2. Preliminaries

The following notations and definitions will be used throughout the paper. $\mathbb{R}^n$ stands for a finite-dimensional normed space. We write $\mathbb{R}^n$ also for the dual space $(\mathbb{R}^n)^*$ and $\langle x^*, x\rangle$ for the value of $x^* \in (\mathbb{R}^n)^*$ at $x \in \mathbb{R}^n$. For a given $\bar{x}$, $\mathscr{U}(\bar{x})$ stands for the system of the neighborhoods of $\bar{x}$. For $\rho > 0$, denote $B(\bar{x},\rho) := \{x \in \mathbb{R}^n \mid \|x-\bar{x}\| < \rho\}$ ball of radius ρ centered at $\bar{x}$. For $S \subseteq \mathbb{R}^n$, int$S$, cl$S$, bd$S$, span$S$, co$S$ and $C(S)$ denote its interior, closure, boundary, linear hull, convex hull and the cone $\{\lambda x \mid x \in S, \lambda \geq 0\}$, respectively (resp). Denote $|S|$ the cardinality of S, i.e., the number of elements of S. The convex cone containing the origin generated by S, denoted by cone$S$, is defined as

$$\text{cone}S := \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{k} \lambda_i x_i, x_i \in S, \lambda_i \geq 0, i = 1, ..., k \right\}.$$

The affine hull of S, denoted by aff$S$, defined by

$$\text{aff}S := \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{k} \lambda_i x_i, x_i \in S, \lambda_i \in \mathbb{R}, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

The relative interior of a convex set $S \subset \mathbb{R}^n$, denoted by ri$S$, defined by

$$\text{ri}S := \{x \in \text{aff}S \mid \exists \varepsilon > 0, B(x,\varepsilon) \cap (\text{aff}S) \subset S\}.$$

Note that ri$S$ of a convex set S is also a convex set and ri$S \subset S \subset$ cl$S \subset$ aff$S$. The negative polar cone and strictly negative polar cone of S are defined resp by

$$S^- := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle \leq 0 \; \forall x \in S\},$$

$$S^s = \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle < 0 \; \forall x \in S\}.$$

The contingent cone of S at $\bar{x} \in$ cl$S$ is

$$T(S,\bar{x}) := \{x \in \mathbb{R}^n \,|\, \exists \tau_k \downarrow 0, \exists x_k \to x, \forall k \in \mathbb{N}, \bar{x} + \tau_k x_k \in S\}.$$

The right-sided directional derivative of a function $\phi : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is denoted by $\phi'(\bar{x},d)$ and defined by

$$\phi^+(\bar{x},d) := \limsup_{\tau \downarrow 0} \frac{\phi(\bar{x}+\tau d) - \phi(\bar{x})}{\tau}.$$

Definition 2.1. [17] Let $\bar{x} \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of $\phi : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$ in direction u is defined by

$$\phi^o(\bar{x}, u) := \limsup_{\tau \downarrow 0, x \to \bar{x}} \frac{\phi(x + \tau u) - \phi(x)}{\tau}.$$

The Clarke subdifferential of ϕ at $\bar{x}$ is

$$\partial^C \phi(\bar{x}) := \{x^* \in \mathbb{R}^n | \langle x^*, d \rangle \le \phi^o(\bar{x}, d), \ \forall d \in \mathbb{R}^n\}.$$

We say that ϕ is Clarke regular at $\bar{x}$ if $\phi'(\bar{x}, d)$ exists and $\phi^o(\bar{x}, d) = \phi'(\bar{x}, d)$ for all $d \in \mathbb{R}^n$.

Definition 2.2. [18, 19] Let $\bar{x} \in \mathbb{R}^n$ and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Michel-Penot (MP) directional derivative of $\phi : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$ in direction u is defined by

$$\phi^\diamond(\bar{x}, u) := \sup_{v \in \mathbb{R}^n} \limsup_{\tau \downarrow 0} \frac{\phi(\bar{x} + \tau(u + v)) - \phi(\bar{x} + \tau v)}{\tau}.$$

The MP subdifferential of ϕ at $\bar{x}$ is

$$\partial^{MP} \phi(\bar{x}) := \{x^* \in \mathbb{R}^n | \langle x^*, d \rangle \le \phi^\diamond(\bar{x}, d), \ \forall d \in \mathbb{R}^n\}.$$

We say that ϕ is MP regular at $\bar{x}$ if $\phi'(\bar{x}, d)$ exists and $\phi^\diamond(\bar{x}, d) = \phi'(\bar{x}, d)$ for all $d \in \mathbb{R}^n$.

The following properties of MP directional derivative and MP subdifferential are useful in the sequel; see [18, 19] and the references therin.

Lemma 2.3. *Let ϕ be function from $\mathbb{R}^n$ to $\mathbb{R}$, which is Lipschitz near $\bar{x}$. Then, the following assertions hold.*

(i) *The function $v \to \phi^\diamond(\bar{x}, v)$ is finite, positively homogenous, subadditive on $\mathbb{R}^n$, $\phi^\diamond(\bar{x}, 0) = 0$ and $\partial(\phi^\diamond(\bar{x}, .))(0) = \partial^{MP} \phi(\bar{x})$, where ∂ denotes the subdifferential in sense of convex analysis.*

(ii) *$\partial^{MP} \phi(\bar{x})$ is nonempty, convex and compact subset of $\mathbb{R}^n$.*

(iii) *$\phi^\diamond(\bar{x}, v) = \max_{\xi \in \partial^{MP} \phi(\bar{x})} \langle \xi, v \rangle$.*

(iv) *If ϕ is Gâteaux differentiable at $\bar{x}$, then $\partial^{MP} \phi(\bar{x}) = \{\nabla \phi(\bar{x})\}$. If ϕ is convex, then $\partial^{MP} \phi(\bar{x}) = \partial \phi(\bar{x})$.*

(v) *If ϕ is Clarke regular at $\bar{x}$, then ϕ is MP regular at $\bar{x}$.*

(vi) *$\partial^{MP} \phi(\bar{x}) \subseteq \partial^C \phi(\bar{x})$.*

(vii) *For $x, y \in \mathbb{R}^n$, there exist a point c in the open line segment (x, y) and $x^* \in \partial^{MP} \phi(c)$ such that $\phi(y) - \phi(x) = \langle x^*, y - x \rangle$.*

By Lemma 2.3 (vi), we can see that the necessary optimality conditions in terms of the MP subdifferential are sharper than the necessary optimality conditions via the Clarke subdifferential; see, e.g., [20, 21, 22, 23] and the references therein. The following example shows that the inclusion in Lemma 2.3 (vi) may be strict.

Example 2.4. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$\phi(x) = \begin{cases} x^2 \sin \frac{2}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then, for $\bar{x} = 0$, one has

$$\partial^{MP}\phi(\bar{x}) = \{0\},$$

$$\partial^{C}\phi(\bar{x}) = [-2, 2].$$

Hence,

$$\partial^{MP}\phi(\bar{x}) \subsetneqq \partial^{C}\phi(\bar{x}).$$

Lemma 2.5. [24] *Let* $\{C_i | i = 1, ..., m\}$ *be a collection of nonempty convex sets in* $\mathbb{R}^n$ *and* $K = \text{co}\left(\bigcup\limits_{i=1}^{m} C_i\right)$. *Then,* $\text{ri}K = \bigcup\left\{\sum\limits_{i=1}^{m} \lambda_i \text{ri}C_i \mid \sum\limits_{i=1}^{m} \lambda_i = 1, \lambda_i > 0, i = 1, ..., m\right\}$.

Lemma 2.6. [24] *Let* C_1 *and* C_2 *be non-empty convex sets in* $\mathbb{R}^n$. *In order that there exist a hyperplane separating* C_1 *and* C_2 *properly, it is necessary and sufficient that* $\text{ri}C_1$ *and* $\text{ri}C_2$ *have no point in common.*

Lemma 2.7. [24] *Let* $\{C_t | t \in \Gamma\}$ *be an arbitrary collection of nonempty convex sets in* $\mathbb{R}^n$ *and* $K = \text{co}\left(\bigcup\limits_{t\in\Gamma} C_t\right)$. *Then, every nonzero vector of* K *can be expressed as a non-negative linear combination of* n *or fewer linear independent vectors, each belonging to a different* C_t.

3. Optimality conditions

In this section, we consider the following multiobjective semi-infinite programming

$$\text{(P)} \qquad \begin{aligned} &\min\nolimits_{\mathbb{R}^m_+} f(x) := (f_1(x), ..., f_m(x)) \\ &\text{s.t.} \quad g_t(x) \leq 0, \quad t \in T, \\ &\qquad\;\; x \in \mathbb{R}^n, \end{aligned}$$

where f_i, $i = 1,...,m$, $g_t, t \in T$ are Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$. The index set T is arbitrary nonempty set, not necessary finite. Set $I := \{1,...,m\}$, $f := (f_1,...,f_m)$ and $g_T := (g_t)_{t\in T}$. Denote the feasible solution set of (P) $\Omega := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0,\ t \in T\}$.

Definition 3.1. For problem (P), let $\bar{x} \in \Omega$.

(i) A point $\bar{x}$ is a locally weakly efficient solution of (P), denoted by $\bar{x} \in$ LWE(P), if there exists $U \in \mathscr{U}(\bar{x})$ such that $f(\bar{x}) - f(x) \notin \text{int}\mathbb{R}^m_+, \forall x \in \Omega \cap U$.

(ii) A point $\bar{x}$ is a locally (Pareto) efficient solution of (P), denoted by $\bar{x} \in$ LE(P), if there exists $U \in \mathscr{U}(\bar{x})$ such that $f(\bar{x}) - f(x) \notin \mathbb{R}^m_+ \setminus \{0\}, \forall x \in \Omega \cap U$.

If $U = \mathbb{R}^n$, the word "locally" is omitted. In this case, the weakly efficient solution sets/the weakly efficient solution sets are denoted by WE(P)/E(P). It is easy to see that LE(P) $\subset$ LWE(P); see, e.g., [25] for more details.

Denote $\mathbb{R}^{|T|}_+$ the collection of all the functions $\lambda : T \to \mathbb{R}$ taking values λ_t's positive only at finitely many points of T, and equal to zero at the other points, i.e., there exists a finite index set $J := \{1,2,...,k\} \subset T$ such that $\lambda_t > 0$ for all $t \in J$ and $\lambda_t = 0$ for all $t \in T \setminus J$. For a given $\bar{x} \in \Omega$, denote $T(\bar{x}) := \{t \in T | g_t(\bar{x}) = 0\}$ the index set of all active constraints at $\bar{x}$. The set of active constraint multipliers at $\bar{x} \in \Omega$ is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}^{|T|}_+ | \lambda_t g_t(\bar{x}) = 0, \forall t \in T\}.$$

Note that $\lambda \in \Lambda(\bar{x})$ if there exists a finite index set $I := \{1,2,...,m\} \subset T(\bar{x})$ such that $\lambda_t > 0$ for all $t \in I$ and $\lambda_t = 0$ for all $t \in T(\bar{x}) \setminus I$. For a given $\bar{x} \in \Omega$, define the extension of constraints system of (P)

$$Q^i := \left\{x \in \mathbb{R}^n \mid f_k(x) \leq f_jk(\bar{x}), g_t(x) \leq 0 j \in I \setminus \{i\}, t \in T\right\}, i \in I,$$

$$Q := \bigcap_{i\in I} Q^i.$$

Then, we have $Q = \{x \in \mathbb{R}^n \mid f_k(x) \leq f_k(\bar{x}), g_t(x) \leq 0, j \in I, t \in T\}$. Recall the following conditions in [9] (with the convention $\cup_{\alpha\in\emptyset} X_\alpha = \emptyset$)

$$\text{(RC1)} : \left(\bigcup_{i=1}^{m} \partial^{MP} f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} g_t(\bar{x})\right)^- \subseteq T(Q,\bar{x}),$$

$$(\text{CQ1}):\left(\bigcup_{t\in T(\bar{x})} g_t(\bar{x})\right)^s \neq \emptyset,$$

(C1): T is a compat set and the set-valued map $j \rightrightarrows g_j(\bar{x})$ is upper semicontinuous on $T(\bar{x})$.

Theorem 3.2. [9] *Suppose that (C1) is satisfied at $\bar{x}\in\Omega$. Then, (CQ1) implies (RC1) at $\bar{x}$.*

Theorem 3.3. [9] *Let $\bar{x}\in\Omega$. If one of the following assertions holds:*

(I) (RC1) holds at $\bar{x}$ and closedness of cone$\left(\bigcup_{t\in T(\bar{x})} g_t(\bar{x})\right)$,

(II) (CQ1) and (C1) hold at $\bar{x}$,

then there exist $\alpha_i \geq 0$ (for $i\in I$) with $\sum_{i=1}^m \alpha_i = 1$ and $\lambda\in\Lambda(\bar{x})$ such that

$$0\in\sum_{i=1}^m \alpha_i\partial^{MP} f_i(\bar{x})+\sum_{t\in T}\lambda_t\partial^{MP} g_t(\bar{x}).$$

Now, in line of [5, 6, 10], we propose the following conditions:

$$(\text{RC2}):\left(\bigcup_{i=1}^m \partial^{MP} f_i(\bar{x})\right)^- \cap\left(\bigcup_{t\in T(\bar{x})} g_t(\bar{x})\right)^- \subseteq\bigcap_{i=1}^m T(Q^i,\bar{x}),$$

$$(\text{RC3}):\left(\bigcup_{i=1}^m \partial^{MP} f_i(\bar{x})\right)^s \cap\left(\bigcup_{t\in T(\bar{x})} g_t(\bar{x})\right)^s \neq\emptyset,$$

$$(\text{C2}):\left(\bigcup_{i=1}^m \partial^{MP} f_i(\bar{x})\right)^- \setminus\{0\}\subseteq\bigcup_{i=1}^m (\partial^{MP} f_i(\bar{x}))^s,$$

(C3): T is a compat set, the function $(x,t)\to g_t(x)$ is upper semicontinuous on $\mathbb{R}^n\times T$ and the set-valued map $j\rightrightarrows g_j(x)$ is an upper semicontinuous mapping in t on each x.

Remark 3.4. Let $\bar{x}\in\Omega$. The following relations of the above conditions are easy to see.

(i) (RC3) implies (CQ1).

(ii) (C3) implies (C1).

Now, we establish the strong necessary KKT condition for locally weakly efficient solutions of (P).

Proposition 3.5. *Suppose that $\bar{x}\in LWE(P)$ and (RC3), (C3) are satisfied at $\bar{x}$. Then, there exist $\alpha\in\text{int}\mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$ and $\lambda\in\Lambda(\bar{x})$ such that*

$$0\in\sum_{i=1}^m \alpha_i\partial^{MP} f_i(\bar{x})+\sum_{t\in T}\lambda_t\partial^{MP} g_t(\bar{x}).$$

Proof. It follows from Theorem 3.2 and Theorem 3.3 that there exist $\alpha \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$ and $\lambda \in \Lambda(\bar{x})$ such that

$$0 \in \sum_{i=1}^{m} \alpha_i \partial^{MP} f_i(\bar{x}) + \sum_{t\in T} \lambda_t \partial^{MP} g_t(\bar{x}).$$

This implies that there exist $x_i^* \in \partial^{MP} f_i(\bar{x})$ and $y_j \in \partial^{MP} g_k(\bar{x})$ with $(i,k) \in I \times K$, where K is a finite subset of $T(\bar{x})$ such that

$$\sum_{i=1}^{m} \alpha_i x_i^* + \sum_{j\in J} \lambda_t \partial^{MP} y_j^* = 0. \tag{3.1}$$

Suppose to contrary that $\alpha_i = 0$ for some $i \in I$. Since (RC3) holds, there exists $u \in \mathbb{R}^n$ such that

$$\begin{cases} \langle x_j^*, u\rangle < 0, & j \in I \setminus \{i\}, \\ \langle y_t^*, u\rangle < 0, & t \in T(\bar{x}). \end{cases}$$

The above inequalities together with (3.1) deduce that

$$0 = \sum_{i=1}^{m} \alpha_i \langle x_i^*, u\rangle + \sum_{k\in K} \lambda_k \partial^{MP} \langle y_k^*, u\rangle < 0,$$

which is a contradiction. Hence, $\alpha_i > 0, \forall i \in I$ and the conclusion is obtained.

Similarly, the strong necessary KKT condition for locally efficient solutions of (P) is established as follows.

Proposition 3.6. *Suppose that $\bar{x} \in LE(P)$ and (RC2), (C2) are satisfied at $\bar{x}$. Then, there exist $\alpha \in \mathrm{int}\mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$ and $\lambda \in \Lambda(\bar{x})$ such that*

$$0 \in \sum_{i=1}^{m} \alpha_i \partial^{MP} f_i(\bar{x}) + \sum_{t\in T} \lambda_t \partial^{MP} g_t(\bar{x}).$$

Proof. First, we prove that, for every $i \in I$,

$$(\partial^{MP} f_i(\bar{x}))^s \cap T(Q^i, \bar{x}) = \emptyset. \tag{3.2}$$

Suppose to the contrary that there exist $i_0 \in I$ and a vector d such that

$$d \in (\partial^{MP} f_i(\bar{x}))^s \cap T(Q^{i_0}, \bar{x}). \tag{3.3}$$

Since $d \in T(Q^{i_0}, \bar{x})$, there exist $\tau_k \downarrow 0, d_k \to d$ such that $\bar{x} + \tau_k d_k \in Q^{i_0}$ for all k, i.e.,

$$\begin{cases} f_i(\bar{x} + \tau_k d_k) \le f_i(\bar{x}) & \forall i \in I \setminus \{i_0\}, \forall k, \\ \bar{x} + \tau_k d_k \in \Omega, & \forall k. \end{cases} \tag{3.4}$$

By Lemma 2.3 (vii), for each k, there exist c_k in the open line segment $(\bar{x}, \bar{x}+\tau_k d_k)$ and $x_k^* \in \partial^{MP} f_{i_0}(c_k)$ such that

$$f_{i_0}(\bar{x}+\tau_k d_k) - f_{i_0}(\bar{x}) = \tau_k \langle x_k^*, d_k \rangle. \tag{3.5}$$

It follows from the upper semicontinuity of $\partial^{MP} f_{i_0}(.)$ and $c_k \to \bar{x}$ that we can assume, taking subsequence if necessary, $x_k^* \to \bar{x}^* \in \partial^{MP} f_{i_0}(\bar{x})$. Thus, we deduce from (3.3) and (3.5) that

$$\lim_{k\to\infty} \frac{f_{i_0}(\bar{x}+\tau_k d_k) - f_{i_0}(\bar{x})}{\tau_k} = \lim_{k\to\infty} \langle x_k^*, d_k \rangle = \langle \bar{x}^*, d \rangle < 0.$$

Hence, for k large enough, one has

$$f_{i_0}(\bar{x}+\tau_k d_k) < f_{i_0}(\bar{x}),$$

which contradicts with $\bar{x} \in LE(P)$. Therefore, (3.2) holds. It follows that

$$\left(\bigcup_{i=1}^{m} (\partial^{MP} f_i(\bar{x}))^s \right) \cap \left(\bigcap_{i=1}^{m} T(Q^i, \bar{x}) \right) = \emptyset. \tag{3.6}$$

Now, we prove that

$$0 \in \operatorname{ri}\left(\operatorname{co} \bigcup_{i=1}^{m} \partial^{MP} f_i(\bar{x}) \right) + \operatorname{cone} \bigcup_{t \in T(\bar{x})} \partial^{MP} g_t(\bar{x}). \tag{3.7}$$

Suppose to contrary that (3.7) does not hold. Then,

$$\operatorname{ri}\left(\operatorname{co} \bigcup_{i=1}^{m} \partial^{MP} f_i(\bar{x}) \right) \cap \left(-\operatorname{cone} \bigcup_{t \in T(\bar{x})} \partial^{MP} g_t(\bar{x}) \right) = \emptyset.$$

Using Lemma 2.5, one sees that there exists $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$d \in \left(\operatorname{co} \bigcup_{i=1}^{m} \partial^{MP} f_i(\bar{x}) \right)^- \cap \left(\operatorname{cone} \bigcup_{t \in T(\bar{x})} \partial^{MP} g_t(\bar{x}) \right)^- = \left(\bigcup_{i=1}^{m} \partial^{MP} f_i(\bar{x}) \right)^- \cap \left(\bigcup_{t \in T(\bar{x})} \partial^{MP} g_t(\bar{x}) \right)^-,$$

which together with (C2) contradicts with (3.6). Hence, (3.7) holds. Then, it follows from (3.7) and Lemma 2.6 that there exist $\alpha \in \operatorname{int}\mathbb{R}_+^m$ with $\sum_{i=1}^m \alpha_i = 1$ and $\lambda \in \Lambda(\bar{x})$ such that

$$0 \in \sum_{i=1}^{m} \alpha_i \partial^{MP} f_i(\bar{x}) + \sum_{t \in T} \lambda_t \partial^{MP} g_t(\bar{x}).$$

The following example shows that condition (C2) is essential.

Example 3.7. Let $n=2, T=[0,1], D:=\{a\in\mathbb{R}^2 \mid -1\le a_1\le -a_2^2, -1\le a_2\le 1\}$ and let $f:\mathbb{R}^2\to\mathbb{R}$, $g_t:\mathbb{R}^2\to\mathbb{R}$ be defined as follows

$$f_1(x)=-2x_2,\ f_2(x)=\sup_{a\in D}\langle a,x\rangle,$$

$$g_t(x)=-x_2-t, t\in T.$$

Then $\Omega:=\{x\in\mathbb{R}^2 \mid x_2\ge 0\}$. For $\bar{x}=(0,0)\in\Omega$, one has

$$Q^1=\mathbb{R}_+\times\{0\}, Q^2=\mathbb{R}\times\mathbb{R}_+,$$

$$\partial^{MP}f_1(\bar{x})=\{(0,-2)\}, \partial^{MP}f_2(\bar{x})=D,$$

$$T(\bar{x})=\{0\},\ \bigcup_{t\in T(\bar{x})}\partial^{MP}g_t(\bar{x})=\{(0,-1)\}.$$

Hence, by some calculations, we have

$$T(Q^1,\bar{x})=Q^1, T(Q^2,\bar{x})=Q^2,$$

$$\left(\bigcup_{i=1}^{2}\partial^{MP}f_i(\bar{x})\right)^{-}=\mathbb{R}_+\times\{0\},$$

$$(\partial^{MP}f_1(\bar{x}))^s=\mathbb{R}\times\operatorname{int}\mathbb{R}_+, (\partial^{MP}f_2(\bar{x}))^s=\emptyset,$$

$$\left(\bigcup_{t\in T(\bar{x})}\partial^{MP}g_t(\bar{x})\right)^{-}=\mathbb{R}\times\mathbb{R}_+.$$

It follows that

$$\left(\bigcup_{i=1}^{2}\partial^{MP}f_i(\bar{x})\right)^{-}\cap\left(\bigcup_{t\in T(\bar{x})}g_t(\bar{x})\right)^{-}\subseteq\bigcap_{i=1}^{2}T(Q^i,\bar{x}),$$

i.e., (RC2) holds at x; but (C2) does not hold at $\bar{x}$ since

$$(1,0)\in\left(\bigcup_{i=1}^{m}\partial^{MP}f_i(\bar{x})\right)^{-}\setminus\{0\},\ (1,0)\notin\bigcup_{i=1}^{m}(\partial^{MP}f_i(\bar{x}))^s.$$

It is easy to check that there are no $\alpha\in\operatorname{int}\mathbb{R}_+^2$ with $\alpha_1+\alpha_2=1$ and $\lambda\in\Lambda(\bar{x})$ such that

$$(0,0)\in\alpha_1(0,-2)+\alpha_2 D+\sum_{t\in T}\lambda_t\partial^{MP}g_t(\bar{x})=\alpha_1(0,-2)+\alpha_2 D+\lambda_0(0,-1).$$

Definition 3.8. [23] Let $\phi:\mathbb{R}^n\to\mathbb{R}$ be a locally Lipschitz function.

(i) ϕ is said to be ∂^{MP}-convex at $\bar{x}$ if for each $x\in\mathbb{R}^n$ and any $x^*\in\partial^{MP}\phi(\bar{x})$,

$$\phi(x)-\phi(\bar{x})\ge\langle x^*,x-\bar{x}\rangle.$$

(ii) ϕ is said to be strictly ∂^{MP}-convex at $\bar{x}$ if for each $x \in \mathbb{R}^n \setminus \{\bar{x}\}$ and any $x^* \in \partial^{MP}\phi(\bar{x})$,

$$\phi(x) - \phi(\bar{x}) > \langle x^*, x - \bar{x}\rangle.$$

(iii) ϕ is said to be ∂^{MP}-pseudoconvex at $\bar{x}$ if for each $x \in \mathbb{R}^n$,

$$\phi(x) - \phi(\bar{x}) < 0 \Rightarrow \langle x^*, x - \bar{x}\rangle < 0, \forall x^* \in \partial^{MP}\phi(\bar{x}).$$

(iv) ϕ is said to be strictly ∂^{MP}-pseudoconvex at $\bar{x}$ if for each $x \in \mathbb{R}^n \setminus \{\bar{x}\}$,

$$\phi(x) - \phi(\bar{x}) \leq 0 \Rightarrow \langle x^*, x - \bar{x}\rangle < 0, \forall x^* \in \partial^{MP}\phi(\bar{x}).$$

(v) ϕ is said to be ∂^{MP}-quasiconvex at $\bar{x}$ if for each $x \in \mathbb{R}^n$ and any $x^* \in \partial^{MP}\phi(\bar{x})$,

$$\phi(x) - \phi(\bar{x}) \leq 0 \Rightarrow \langle x^*, x - \bar{x}\rangle \leq 0.$$

Remark 3.9. Let f be differentiable at $\bar{x}$. Then if f is convex/pseudoconvex/quasiconvex at x then f is ∂^{MP}-convex/ ∂^{MP}-pseudoconvex/∂^{MP}-quasiconvex at $\bar{x}$.

Proposition 3.10. *Let $\bar{x} \in \Omega$. Suppose that there exist $\alpha \in \operatorname{int}\mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$ and $\lambda \in \Lambda(\bar{x})$ such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^{MP} f_i(\bar{x}) + \sum_{t\in T} \lambda_t \partial^{MP} g_t(\bar{x}). \tag{3.8}$$

(i) If $f_i, i \in I$, is ∂^{MP}-pseudoconvex at $\bar{x}$ and $g_t, t \in T$, is ∂^{MP}-quasiconvex $\bar{x}$, then $\bar{x}$ is a weakly efficient solution of (P).

(ii) If $f_i, i \in I$, is strictly ∂^{MP}-pseudoconvex at $\bar{x}$ and $g_t, t \in T$, is ∂^{MP}-quasiconvex $\bar{x}$, then $\bar{x}$ is an efficient solution of (P).

Proof. (i) Suppose on the contrary that $\bar{x}$ is not a weakly efficient solution. Then there exists a feasible point x such that $f_i(x) < f_i(\bar{x})$, $\forall i = 1, ..., m$. Since $\bar{x} \in \Omega$ satisfies (3.8), there exist $x_i^* \in \partial^{MP} f_i(\bar{x}), i \in I$ and $y_t^* \in \partial^{MP} g_t(\bar{x}), t \in J$, where J is a finite subset of $T(\bar{x})$, such that

$$-\sum_{t\in J} \lambda_t y_t^* = \sum_{i=1}^m \alpha_i x_i^*. \tag{3.9}$$

Since each f_i is ∂^{MP}-pseudoconvex, we have

$$\langle x_i^*, x - \bar{x}\rangle < 0, \; \forall x_i^* \in \partial^{MP} f_i(\bar{x}).$$

Hence, we deduce from $\alpha_i > 0$ and (3.9) that

$$-\left\langle \sum_{t\in J} \lambda_t y_t^*, x-\bar{x} \right\rangle = \left\langle \sum_{i=1}^{m} \alpha_i x_i^*, x-\bar{x} \right\rangle < 0. \tag{3.10}$$

For each $t \in J, g_t(x) \leq 0 = g_t(\bar{x})$. Thus, by the ∂^{MP}-pseudoconvexity of $g_t, t \in T$, at $\bar{x}$, one has

$$\langle y_t^*, x-\bar{x}\rangle \leq 0, \forall y_t^* \in \partial^{MP} g_t(\bar{x}), t \in J.$$

Consequently,

$$\left\langle \sum_{t\in J} \lambda_t y_t^*, x-\bar{x} \right\rangle \leq 0,$$

which contradicts with (3.10).

(ii) Suppose on the contrary that $\bar{x}$ is not an efficient solution. Then there exists a feasible point x and at least $i_0 \in I$ such that

$$\begin{cases} f_i(x) \leq f_i(\bar{x}), & \forall i \in I \setminus \{i_0\}, \\ f_{i_0}(x) < f_{i_0}(\bar{x}). \end{cases}$$

Since $\bar{x} \in \Omega$ satisfies (3.8), there exist $x_i^* \in \partial^{MP} f_i(\bar{x}), i \in I$ and $y_t^* \in \partial^{MP} g_t(\bar{x}), t \in J$, where J is a finite subset of $T(\bar{x})$, such that

$$-\sum_{t\in J} \lambda_t y_t^* = \sum_{i=1}^{m} \alpha_i x_i^*. \tag{3.11}$$

Since each f_i is strictly ∂^{MP}-pseudoconvex, we have

$$\langle x_i^*, x-\bar{x}\rangle < 0,\ \forall x_i^* \in \partial^{MP} f_i(\bar{x}).$$

Hence, we deduce from $\alpha_i > 0$ that

$$-\left\langle \sum_{t\in J} \lambda_t y_t^*, x-\bar{x} \right\rangle = \left\langle \sum_{i=1}^{m} \alpha_i x_i^*, x-\bar{x} \right\rangle < 0. \tag{3.12}$$

For each $t \in J, g_t(x) \leq 0 = g_t(\bar{x})$. Thus, by the ∂^{MP}-pseudoconvexity of $g_t, t \in T$, at $\bar{x}$, one has

$$\langle y_t^*, x-\bar{x}\rangle \leq 0, \forall y_t^* \in \partial^{MP} g_t(\bar{x}), t \in J.$$

Consequently,

$$\left\langle \sum_{t\in J} \lambda_t y_t^*, x-\bar{x} \right\rangle \leq 0,$$

which contradicts with (3.12).

Corollary 3.11. *Let $\bar{x} \in \Omega$. Suppose that there exist $\alpha \in \text{int}\mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$ and $\lambda \in \Lambda(\bar{x})$ such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^{MP} f_i(\bar{x}) + \sum_{t \in T} \lambda_t \partial^{MP} g_t(\bar{x}).$$

(i) If $f_i, i \in I$, is ∂^{MP}-convex at $\bar{x}$ and $g_t, t \in T$, is ∂^{MP}-convex $\bar{x}$, then $\bar{x}$ is a weakly efficient solution of (P).

(ii) If $f_i, i \in I$, is strictly ∂^{MP}-convex at $\bar{x}$ and $g_t, t \in T$, is ∂^{MP}-convex $\bar{x}$, then $\bar{x}$ is an efficient solution of (P).

The following example shows that the strictly ∂^{MP}-convexity of $f_i, i \in I$, and $g_t, t \in T$, imposed in the Proposition 3.10 cannot be dropped even in the smooth case.

Example 3.12. Let $n = 2, T = (0, +\infty)$ and let $f : \mathbb{R}^2 \to \mathbb{R}$, $g_t : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows

$$f_1(x) = f_2(x) = -2x^3,$$

$$g_t(x) = -tx^2, t \in T.$$

Then, $\Omega = \mathbb{R}_+$ and for $x \in \mathbb{R}_+$, one has

$$\partial^{MP} f_1(\bar{x}) = \partial^{MP} f_2(\bar{x}) = \{-6x^2\},$$

$$T(\bar{x}) = T, \partial^{MP} g_t(\bar{x}) = \{-2tx\}, \forall t \in T.$$

Let $\bar{x} = 0 \in \Omega$. Then, it is easy to see that

$$\partial^{MP} f_1(\bar{x}) = \partial^{MP} f_2(\bar{x}) = \partial^{MP} g_t(\bar{x}) = \{0\},$$

and hence, $\bar{x}$ satisfies (3.8) with any $\alpha \in \mathbb{R}^2_+, \alpha_1 + \alpha_2 = 1$ and $\lambda \in \Lambda(\bar{x})$. However, we can check that $\bar{x}$ is not a weakly efficient solution or an efficient solution of (P). The reason is that $f_i, i \in I$, is not (strictly) ∂^{MP}-convex at $\bar{x}$ and $g_t, t \in T$, is not ∂^{MP}-convex $\bar{x}$.

4. Duality

In this section, we consider the Wolfe [26] and Mond-Weir [27] duality schemes for (P) in terms of the Michel-Penot subdifferential. In what follows, we use the notations:

$$u \prec v \Leftrightarrow u - v \in -\text{int}\mathbb{R}^m_+,\ u \not\prec v \text{ is the negation of } u \preceq v.$$

$$u \preceq v \Leftrightarrow u - v \in -\mathbb{R}^m_+ \setminus \{0\},\ u \not\preceq v \text{ is the negation of } u \preceq v.$$

4.1. The Wolfe type duality

For $u \in \mathbb{R}^n$, $\alpha \in \text{int}\mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$ and $\lambda \in \mathbb{R}^{|T|}_+$, define

$$L(u,\alpha,\lambda) := f(u) + \left(\sum_{t\in T} \lambda_t g_t(u)\right) e,$$

where $e := (1,...,1) \in \mathbb{R}^m$. We define the Wolfe type dual problem as follows:

(D$_W$): $\max L(u,\alpha,\lambda) = f(u) + \left(\sum_{t\in T} \lambda_t g_t(u)\right) e$

s.t. $0 \in \sum_{i=1}^m \alpha_i \partial^{MP} f_i(u) + \sum_{t\in T} \lambda_t \partial^{MP} g_t(u)$,

$\alpha \in \text{int}\mathbb{R}^m_+$, $\lambda \in \mathbb{R}^{|T|}_+$, $u \in \mathbb{R}^n$.

The feasible set of (D$_W$) is defined by

$$\Omega_W := \left\{(u,\alpha,\lambda) \in \mathbb{R}^n \times \text{int}\mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \mid \sum_{i=1}^m \alpha_i = 1, 0 \in \sum_{i=1}^m \alpha_i \partial^* f_i(u) + \sum_{t\in T} \lambda_t \partial^* g_t(u)\right\}.$$

Definition 4.1. Let $(\bar{u},\bar{\alpha},\bar{\lambda}) \in \Omega_W$.

(i) $(\bar{u},\bar{\alpha},\bar{\lambda})$ is a weakly efficient solution of (D_W), denoted by $(\bar{u},\bar{\alpha},\bar{\lambda}) \in WE(D_W)$, if

$$L(u,\alpha,\lambda) - L(\bar{u},\bar{\alpha},\bar{\lambda}) \notin \text{int}\mathbb{R}^m_+, \ \forall (u,\alpha,\lambda) \in \Omega_W.$$

(ii) $(\bar{u},\bar{\alpha},\bar{\lambda})$ is an efficient solution of (D_W), denoted by $(\bar{u},\bar{\alpha},\bar{\lambda}) \in E(D_W)$, if

$$L(u,\alpha,\lambda) - L(\bar{u},\bar{\alpha},\bar{\lambda}) \notin \mathbb{R}^m_+ \setminus \{0\}, \ \forall (u,\alpha,\lambda) \in \Omega_W.$$

The following proposition describes weak duality relations between the primal problem (P) and the dual problem (D_W).

Proposition 4.2. *(Weak duality) Let $x \in \Omega, (u,\alpha,\lambda) \in \Omega_W$. If $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex at u, then*

(i) $f(x) \not\preceq L(u,\alpha,\lambda)$,

(i) $f(x) \not\prec L(u,\alpha,\lambda)$.

Proof. (i) For $x \in \Omega$ and $(u,\alpha,\lambda) \in \Omega_W$, we have

$$g_t(x) \leq 0, \forall t \in T, \tag{4.1}$$

and there exist $x_i^* \in \partial^{MP} f_i(\bar{x}), i \in I$ and $y_t^* \in \partial^{MP} g_t(\bar{x}), t \in T$ such that

$$\sum_{i=1}^{m} \alpha_i x_i^* + \sum_{t\in T} \lambda_t y_t^* = 0. \tag{4.2}$$

Suppose to contrary that

$$f(x) \preceq L(u,\alpha,\lambda). \tag{4.3}$$

This implies that $x \neq u$. If $x = u$, then

$$f(x) - L(u,\alpha,\lambda) = -\left(\sum_{t\in T} \lambda_t g_t(x)\right) e \in -\mathbb{R}^m_+ \setminus \{0\},$$

which is impossible since $g_t(x) \leq 0, \forall t \in T$. Moreover, we deduce from (4.3) and $\alpha \in \mathrm{int}\mathbb{R}^m_+$ and that $\langle \alpha, f(x) - L(u,\alpha,\lambda)\rangle < 0$, i.e.,

$$\sum_{i=1}^{m} \alpha_i (f_i(x) - f_i(u)) - \sum_{i=1}^{m} \alpha_i \left(\sum_{t\in T} \lambda_t g_t(x)\right) < 0. \tag{4.4}$$

Since $\sum_{i=1}^{m} \alpha_i = 1$, we have

$$\sum_{i=1}^{m} \alpha_i (f_i(x) - f_i(u)) - \sum_{t\in T} \lambda_t g_t(x) < 0. \tag{4.5}$$

Since $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex at u, we have

$$f_i(x) - f_i(u) \geq \langle x_i^*, x - u\rangle, \forall x_i^* \in \partial^{MP} f_i(u), \forall i \in I,$$

$$g_t(x) - g_t(u) \geq \langle y_t^*, x - u\rangle, \forall y_t^* \in \partial^{MP} g_t(u), \forall t \in T.$$

It follows from the above inequalities and (4.2) that

$$0 = \left\langle \sum_{i=1}^{m} \alpha_i x_i^* + \sum_{t\in T} \lambda_t y_t^*, x - u \right\rangle \leq \sum_{i=1}^{m} \alpha_i (f_i(x) - f_i(u)) + \sum_{t\in T} \lambda_t (g_t(x) - g_t(u)). \tag{4.6}$$

The above inequalities and (4.1) lead to

$$0 \leq \sum_{i=1}^{m} \alpha_i (f_i(x) - f_i(u)) + \sum_{t\in T} \lambda_t g_t(u),$$

which contradicts with (4.5).

(ii) Note that

$$(f(x) \prec L(u,\alpha,\lambda) \Rightarrow f(x) \preceq L(u,\alpha,\lambda))$$

is equivalent to

$$(f(x) \npreceq L(u,\alpha,\lambda) \Rightarrow f(x) \nprec L(u,\alpha,\lambda)).$$

Hence, the conclusion can be deduced from (i). This completes the proof.

Proposition 4.3. *(Strong duality) Suppose that $\bar{x} \in LE(P)$ and (RC2) and (C2) are satisfied at $\bar{x}$. Then there exist $\bar{\alpha} \in \text{int}\mathbb{R}^m_+$ with $\sum_{\bar{\lambda}_i} = 1$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in \Omega_W$ and $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda})$. Moreover, if $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an efficient solution of (D_W).*

Proof. According to Proposition 3.6, there exist $\bar{\alpha} \in \text{int}\mathbb{R}^m_+$ with $\sum_{i=1}^m \bar{\alpha}_i = 1$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that

$$0 \in \sum_{i=1}^m \bar{\alpha}_i \partial^{MP} f_i(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \partial^{MP} g_t(\bar{x}).$$

Since $\bar{\lambda} \in \Lambda(\bar{x})$ and $\bar{x} \in \Omega$, one has $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = 0$. Therefore,

$$f(\bar{x}) = f(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}),$$

i.e., $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in \Omega_W$ and $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda})$. Moreover, if $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex, by invoking Proposition 4.2, we obtain

$$L(\bar{x}, \bar{\alpha}, \bar{\lambda}) = f(\bar{x}) \not\leq L(x, \alpha, \lambda), \forall (x, \alpha, \lambda) \in \Omega_W.$$

This means that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an efficient solution of (DW). This completes the proof.

Now we give an example to illustrate the results in Proposition 4.3.

Example 4.4. Let $n = 1, T = [0, +\infty)$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g_t : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows

$$f_1(x) = x_1, f_2(x) = |x_1| + |x_2|,$$

$$g_t(x) = -x_1 - t, t \in T.$$

Then, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$ and for $u \in \mathbb{R}^2$,

$$\partial^{MP} f_1(u) = \{(1,0)\}, = \partial^{MP} f_2(u) = [-1,1] \times [-1,1],$$

$$\partial^{MP} g_t(u) = \{(-1,0)\}, \forall t \in T.$$

Hence, the Wolfe type dual problem of (P) is

(D_W): $\max L(u, \alpha, \lambda) = (u_1, |u_1| + |u_2|) + \left(\sum_{t \in T} \lambda_t (-u_1 + t) \right) (1,1)$

s.t. $(0,0) \in \alpha_1(1,0) + \alpha_2[-1,1] \times [-1,1] + \sum_{t \in T} \lambda_t(-1,0)$,

$\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$, $\lambda \in \mathbb{R}^{|T|}_+, u \in \mathbb{R}^2$.

Let $\bar{x} = (0,0) \in \Omega$ be a local efficient solution of (P). Then,

$$\partial^{MP} f_1(\bar{x}) = \{(1,0)\}, = \partial^{MP} f_2(\bar{x}) = [-1,1] \times [-1,1],$$

$$T(\bar{x}) = \{0\}, \bigcup_{t \in T(\bar{x})} g_t(\bar{x}) = \{(-1,0)\},$$

$$Q^1 = \{0,0\}, T(Q^1, \bar{x}) = \{(0,0)\},$$

$$Q^2 = \{x \in \mathbb{R}^2 \mid x_1 = 0\}, T(Q^2, \bar{x}) = \{u \in \mathbb{R}^2 \mid u_1 = 0\}.$$

Hence, (C2) holds and

$$\left(\bigcup_{i=1}^{2} \partial^{MP} f_i(\bar{x})\right)^{-} \cap \left(\bigcup_{t \in T(\bar{x})} g_t(\bar{x})\right)^{-} = \{(0,0)\} \subset \bigcap_{i=1}^{2} T(Q^i, \bar{x}),$$

i.e., (RC2) holds. Let $\bar{\alpha} = (1/2, 1/2)$ and $\bar{\lambda} : T \to \mathbb{R}_+$ be defined respectively by

$$\bar{\lambda}(t) = \begin{cases} 1/2, & \text{if } t = 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Then, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in \Omega_W$. Moreover, $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex at $\bar{x}$. By employing Proposition 4.3, we get that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an efficient solution of (D$_W$).

Proposition 4.5. *(Strong duality) Suppose that $\bar{x} \in W(P)$ and (RC3) and (C3) are satisfied at $\bar{x}$. Then there exist $\bar{\alpha} \in \text{int}\mathbb{R}^m_+$ with $\sum_{i=1}^{m} \bar{\alpha}_i = 1$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in \Omega_W$ and $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda})$. Moreover, if $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a weakly efficient solution of (D_W).*

4.2. The Mond-Weir type duality

Our Mond-Weir type dual problem is

(D$_{MW}$): $\max f(u)$

s.t. $0 \in \sum_{i=1}^{m} \alpha_i \partial^{MP} f_i(u) + \sum_{t \in T} \lambda_t \partial^{MP} g_t(u)$,

$\sum_{t \in T} \lambda_t g_t(u) \geq 0$,

$\alpha \in \text{int}R^m_+$, $\lambda \in \mathbb{R}^{|T|}_+$, $u \in \mathbb{R}^n$.

The feasible set of (D_{MW}) is defined by

$$\Omega_{MW} := \{(u,\alpha,\lambda) \in \mathbb{R}^n \times \text{int}\mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \mid 0 \in \sum_{i=1}^m \alpha_i \text{co}\partial^{MP} f_i(u)$$
$$+ \sum_{t\in T} \lambda_t \text{co}\partial^{MP} g_t(u), \sum_{t\in T} \lambda_t g_t(u) \geq 0\}.$$

Proposition 4.6. *(Weak duality) Let $x \in \Omega, (u,\alpha,\lambda) \in \Omega_{MW}$. If $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex at u, then*

(i) $f(x) \not\preceq L(u,\alpha,\lambda)$,

(i) $f(x) \not\prec L(u,\alpha,\lambda)$.

Proof. We only need to prove (i). Since $x \in \Omega$ and $(u,\alpha,\lambda) \in \Omega_{MW}$, we have

$$g_t(x) \leq 0, \forall t \in T. \tag{4.7}$$

There exist $x_i^* \in \partial^{MP} f_i(\bar{x}), i \in I$ and $y_t^* \in \partial^{MP} g_t(\bar{x}), t \in T$ such that

$$\sum_{i=1}^m \alpha_i x_i^* + \sum_{t\in T} \lambda_t y_t^* = 0. \tag{4.8}$$

$$\sum_{t\in T} \lambda_t g_t(u) \geq 0. \tag{4.9}$$

Assume to contrary that $f(x) \preceq f(u)$. Thus, $\langle \alpha, f(x) - L(u,\alpha,\lambda)\rangle < 0$. This is equivalent to

$$\sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)) < 0. \tag{4.10}$$

Since $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex at u, we have

$$f_i(x) - f_i(u) \geq \langle x_i^*, x-u\rangle, \forall x_i^* \in \partial^{MP} f_i(u), \forall i \in I,$$

$$g_t(x) - g_t(u) \geq \langle y_t^*, x-u\rangle, \forall y_t^* \in \partial^{MP} g_t(u), \forall t \in T.$$

By the above inequalities, we deduce from (4.8) that

$$0 \leq \sum_{i=1}^m \alpha_i \langle x_i^*, x-u\rangle + \sum_{t\in T} \lambda_t \langle y_t^* x - u\rangle \leq \sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)) + \sum_{t\in T} \lambda_t (g_t(x) - g_t(u)). \tag{4.11}$$

The above inequality together with (4.7) implies that

$$0 \leq \sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)) + \sum_{t\in T} \lambda_t g_t(u).$$

By combining (4.9) and (4.11), one has $0 \leq \sum_{i=1}^m \alpha_i (f_i(x) - f_i(u))$, which contradicts with with (4.10). This completes the proof.

Proposition 4.7. *(Strong duality) Suppose that $\bar{x} \in LE(P)$ and (RC2) and (C2) are satisfied at $\bar{x}$. Then there exist $\bar{\alpha} \in \text{int}\mathbb{R}^m_+$ with $\sum_{\bar{\lambda}_i} = 1$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in \Omega_{MW}$ and $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda})$. Moreover, if $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a efficient solution of (D_{MW}).*

Proof. According to Proposition 3.6, there exist $\bar{\alpha} \in \text{int}\mathbb{R}^m_+$ with $\sum_{i=1}^m \bar{\alpha}_i = 1$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that

$$0 \in \sum_{i=1}^m \bar{\alpha}_i \partial^{MP} f_i(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \partial^{MP} g_t(\bar{x}).$$

Since $\bar{\lambda} \in \Lambda(\bar{x}), \bar{\lambda}_t g_t(\bar{x}) = 0$ for all $t \in T$, one has $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = 0$. Hence, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in \Omega_{MW}$. urthermore, for any $(u, \alpha, \lambda) \in \Omega_{MW}$, from Proposition 4.1, we derive that

$$L(\bar{x}, \bar{\alpha}, \bar{\lambda}) = f(\bar{x}) \not\preceq L(x, \alpha, \lambda), \forall (x, \alpha, \lambda) \in \Omega_W.$$

So, $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a efficient solution of (D_{MW}). This completes the proof.

Proposition 4.8. *(Strong duality) Suppose that $\bar{x} \in LWE(P)$ and (RC3) and (C3) are satisfied at $\bar{x}$. Then there exist $\bar{\alpha} \in \text{int}\mathbb{R}^m_+$ with $\sum_{\bar{\lambda}_i} = 1$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in \Omega_{MW}$ and $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda})$. Moreover, if $f_i, i \in I$, and $g_t, t \in T$, are ∂^{MP}-convex, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an efficient solution of (D_{MW}).*

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