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EXISTENCE OF POSITIVE SOLUTIONS FOR HIGHER ORDER BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS ON TIME SCALES

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Abstract. In this paper, we establish the existence of even number of positive solutions for higher order integral boundary value problems on time scales

$$(-1)^n u^{\Delta^{2n}}(t) = f(t,u), \quad t \in (0,1)_{\mathbb{T}},$$

$$u^{\Delta^{2i}}(0) = u^{\Delta^{2i}}(1) = \int_0^1 a_{i+1}(x) u^{\Delta^{2i}}(x) \Delta x, \text{ for } 0 \le i \le n-1,$$

where $n \ge 1$, by applying the Avery–Henderson fixed point theorem.

Keywords. Green's function; Integral boundary condition; Cone; Positive solution; Fixed point theorem.

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1. Introduction

The theory of time scales was introduced by Higler [13] not only to unify continuous and discrete theory, but also to provide an accurate information of phenomena that manifest themselves partly in continuous and partly in discrete time. This theory [1, 6, 7] can be applied to various real life situations like epidemic models, stock market and mathematical modeling of

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physical and biological systems. The existence of positive solutions [2] of the boundary value problems (BVPs) have created a great deal of interest due to wide applicability in both theory and applications. Davis and Henderson [9], Wong and Agarwal [23], Davis, Henderson and Wong [10], Ehme and Henderson [11], Bai and Ge [4] and Zhang and Liu [26] considered Lidstone type BVPs associated with ordinary differential equations and established the existence of positive solutions to the boundary value problems by using various methods. In recent years, the existence of positive solutions of nonlinear BVPs with integral boundary conditions have been studied extensively by the researchers. For back ground, results and recent contributions, we refer [5, 8, 12, 14, 15, 17, 18, 20, 22, 24, 25, 27, 28, 29]. However, the corresponding results for BVPs with integral boundary conditions on time scales are still very few, see [16, 19, 21].

Motivated by the papers mentioned above, we wish to extend the results to higher order boundary value problems with integral boundary conditions on time scales. We consider a $2n^{\text{th}}$ order nonlinear dynamic equations on time scales

$$(-1)^n u^{\Delta^{2n}}(t) = f(t,u),\ t \in (0,1)_{\mathbb{T}}, \tag{1.1}$$

satisfying the integral boundary conditions of the form

$$u^{\Delta^{2i}}(0) = u^{\Delta^{2i}}(1) = \int_0^1 a_{i+1}(x) u^{\Delta^{2i}}(x)\Delta x, \text{ for } 0 \le i \le n-1, \tag{1.2}$$

where $n \geq 1$, and establish the existence of even number of positive solutions by applying Avery–Henderson fixed point theorem.

We assume the following conditions hold throughout this paper:

$(A1)$ $f : [0,1]_{\mathbb{T}} \times [0,\infty) \longrightarrow [0,\infty)$ is continuous,

$(A2)$ a_j is nonnegative and rd-continuous on $[0,1]_{\mathbb{T}}$ for $1 \le j \le n$,

$(A3)$ $d_j = \int_0^1 a_j(x)\Delta x \in (0,1)$ for $1 \le j \le n$.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate the bounds for the Green's function. In Section 3, we develop criteria for the existence of even number of positive solutions of the BVP (1.1)-(1.2) by using the Avery–Henderson fixed point theorem. Finally, we give an example to illustrate our results as an application.

2. Preliminaries

In this section, we construct the Green's function for the homogeneous BVP corresponding to (1.1)-(1.2) and estimate bounds for the Green's function. We prove certain lemmas which are useful in establishing further results of this paper.

Let $G_j(t,s)(1 \leq j \leq n)$, be the Green's function for the second order homogeneous BVP,

$$-u^{\Delta\Delta}(t) = 0,\ t \in (0,1)_{\mathbb{T}}, \tag{2.1}$$

$$u(0) = u(1) = \int_0^1 a_j(x)u(x)\Delta x, \text{ for } 0 \leq j \leq n, \tag{2.2}$$

and then obtain bounds for this Green's function. Using this Green's function, the Green's function for the homogeneous BVP corresponding to (1.1)-(1.2) is constructed and bounds for the Green's function are estimated.

Lemma 2.1. *Suppose that $d_j = \int_0^1 a_j(x)\Delta x \in (0,1)$, for $1 \leq j \leq n$. If $h(t) \in (C[0,1]_{\mathbb{T}}, \mathbb{R}^+)$, then the BVP*

$$u^{\Delta\Delta} + h(t) = 0,\ t \in (0,1)_{\mathbb{T}}, \tag{2.3}$$

satisfying (2.2) *has a unique solution*

$$u(t) = \int_0^1 G_j(t,s)h(s)\Delta s, \text{ for } 1 \leq j \leq n,$$

where

$$G_j(t,s) = G(t,s) + \frac{1}{1-d_j}\int_0^1 G(x,s)a_j(x)\Delta x, \text{ for } 1 \leq j \leq n, \tag{2.4}$$

and

$$G(t,s) = \begin{cases} t(1-\sigma(s)), & 0 \leq t \leq s \leq 1, \\ \sigma(s)(1-t), & 0 \leq \sigma(s) \leq t \leq 1. \end{cases} \tag{2.5}$$

Proof. Integrating both sides of (2.3) from 0 to t, we have

$$u^{\Delta}(t) = -\int_0^t h(s)\Delta s + B, \tag{2.6}$$

where $B = u^{\Delta}(0)$. Again integrating (2.6) from 0 to t, we get

$$u(t) = -\int_0^t \left(\int_0^x h(s)\Delta s\right)\Delta x + Bt + A.$$

where $A = u(0)$. Which means that

$$u(t) = -\int_0^t (t-\sigma(s))h(s)\Delta s + Bt + A. \tag{2.7}$$

In particular, $u(1) = -\int_0^1 (1-\sigma(s))h(s)\Delta s + B + A$. Using the boundary conditions (2.2), we get

$$B = \int_0^1 (1-\sigma(s))h(s)\Delta s \tag{2.8}$$

and

$$\begin{aligned}
A &= \int_0^1 a_j(x)u(x)\Delta x \\
&= \int_0^1 a_j(x)\left[-\int_0^x (x-\sigma(s))h(s)\Delta s + Bx + A\right]\Delta x \\
&= \int_0^1 a_j(x)\left[-\int_0^x (x-\sigma(s))h(s)\Delta s + x\int_0^1 (1-\sigma(s))h(s)\Delta s\right]\Delta x + Ad_j \\
&= \int_0^1 a_j(x)\left[-\int_0^x (x-\sigma(s))h(s)\Delta s + x\left(\int_0^x (1-\sigma(s))h(s)\Delta s + \int_x^1 (1-\sigma(s))h(s)\Delta s\right)\right]\Delta x + Ad_j \\
&= \int_0^1 a_j(x)\left[\int_0^x \sigma(s)(1-x)h(s)\Delta s + \int_x^1 x(1-\sigma(s))h(s)\Delta s\right]\Delta x + Ad_j \\
&= \int_0^1 a_j(x)\left[\int_0^1 G(x,s)h(s)\Delta s\right]\Delta x + Ad_j \\
&= \int_0^1 \left[\int_0^1 G(x,s)a_j(x)\Delta x\right]h(s)\Delta s + Ad_j,
\end{aligned}$$

which implies that

$$A = \frac{1}{(1-d_j)}\int_0^1 \left[\int_0^1 G(x,s)a_j(x)\Delta x\right]h(s)\Delta s. \tag{2.9}$$

From (2.7), (2.8) and (2.9), the solution of boundary value problem (2.3), (2.2) is

$$
\begin{aligned}
u(t) &= -\int_0^t (t-\sigma(s))h(s)\Delta s + t\int_0^1 (1-\sigma(s))h(s)\Delta s + \frac{1}{(1-d_j)}\int_0^1 \left[\int_0^1 G(x,s)a_j(x)\Delta x\right] h(s)\Delta s \\
&= -\int_0^t (t-\sigma(s))h(s)\Delta s + t\left[\int_0^t \left[(1-\sigma(s))h(s)\Delta s + \int_t^1 (1-\sigma(s))h(s)\Delta s\right]\right. \\
&\quad + \frac{1}{(1-d_j)}\int_0^1 \left[\int_0^1 G(x,s)a_j(x)\Delta x\right] h(s)\Delta s \\
&= \int_0^t \sigma(s)(1-t)h(s)\Delta s + \int_t^1 t(1-\sigma(s))h(s)\Delta s + \frac{1}{(1-d_j)}\int_0^1 \left[\int_0^1 G(x,s)a_j(x)\Delta x\right] h(s)\Delta s \\
&= \int_0^1 G(t,s)h(s)\Delta s + \frac{1}{(1-d_j)}\int_0^1 \left[\int_0^1 G(x,s)a_j(x)\Delta x\right] h(s)\Delta s \\
&= \int_0^1 G_j(t,s)h(s)\Delta s.
\end{aligned}
$$

Lemma 2.2. *Assume that the conditions $(A2)$-$(A3)$ are satisfied. Then for $1 \le j \le n$, $G_j(t,s)$ satisfies the following inequalities:*

(i) $G(t,s) > 0$ *and* $G_j(t,s) > 0$, *for all* $t,s \in (0,1)_{\mathbb{T}}$,

(ii) $G(\sigma(s),s)G(t,t) \le G(t,s) \le G(\sigma(s),s)$, *for all* $t,s \in [0,1]_{\mathbb{T}}$,

(iii) $\gamma_j G_j(\sigma(s),s) \le G_j(t,s) \le G_j(\sigma(s),s)$, *for all* $t,s \in [0,1]_{\mathbb{T}}$,

where

$$\gamma_j = \frac{\eta_j}{(1-d_j+\eta_j)} \in (0,1), \tag{2.10}$$

and

$$\eta_j = \int_0^1 G(x,x)a_j(x)\Delta x.$$

Proof. We can easily establish the inequalities (i) and (ii). For the inequality (iii), let

$$E_j(s) = \frac{1}{(1-d_j)}\int_0^1 G(x,s)a_j(x)\Delta x, \text{ for } 1 \le j \le n.$$

From (ii), the second inequality of (iii) is obvious, we prove the first inequality of (iii). Using the inequality $G(\sigma(s),s)G(t,t) \le G(t,s)$, then for $t,s \in [0,1]_{\mathbb{T}}$, we have

$$
\begin{aligned}
E_j(s) &\ge \frac{1}{(1-d_j)}\int_0^1 G(\sigma(s),s)G(x,x)a_j(x)\Delta x \\
&= \frac{\eta_j}{(1-d_j)}G(\sigma(s),s),
\end{aligned}
$$

which implies that $(1-d_j)E_j(s) \geq \eta_j G(\sigma(s),s)$. So, we arrive

$$(1-d_j+\eta_j)E_j(s) \geq \eta_j[G(\sigma(s),s)+E_j(s)]$$
$$= \eta_j G_j(\sigma(s),s).$$

Subsequently, we have

$$E_j(s) \geq \frac{\eta_j}{(1-d_j+\eta_j)} G_j(\sigma(s),s)$$
$$= \gamma_j G_j(\sigma(s),s).$$

It follows that $G_j(t,s) = G(t,s)+E_j(s) \geq E_j(s) \geq \gamma_j G_j(\sigma(s),s)$. This completes the proof.

Lemma 2.3. *Assume that the condition* $(A2)$-$(A3)$ *are satisfied . Let* $G_1(t,s) = H_1(t,s)$ *and recursively define*

$$H_j(t,s) = \int_0^1 H_{j-1}(t,r)G_j(r,s)\Delta r, \text{ for } 2 \leq j \leq n. \tag{2.11}$$

Then the Green's function for the homogeneous boundary value problem corresponding to (1.1)-(1.2) is $H_n(t,s)$, *where* $G_j(t,s)$ $(1 \leq j \leq n)$ *is given in* (2.4).

Lemma 2.4. *Assume that the condition* $(A2)$-$(A3)$ *are satisfied. If we define* $K = \prod_{j=1}^{n-1} K_j$ *and* $L = \prod_{j=1}^{n-1} \gamma_j K_j$, *then the Green's function* $H_n(t,s)$ *in* (2.11) *satisfies the following inequalities:*

(*i*) $0 \leq H_n(t,s) \leq KG_n(\sigma(s),s)$, *for all* $t,s \in [0,1]_{\mathbb{T}}$, *and*

(*ii*) $H_n(t,s) \geq \gamma_n L G_n(\sigma(s),s)$, *for all* $t,s \in [0,1]_{\mathbb{T}}$,

where γ_n *is given in* (2.10) *and* $K_j = \int_0^1 G_j(\sigma(s),s)\Delta s$, *for* $1 \leq j \leq n$.

3. Even number of positive solutions

In this section, we establish the existence of at least two positive solutions for the boundary value problem (1.1)-(1.2) by Avery–Henderson fixed point theorem . And then, we establish the existence of at least $2m$ positive solutions to the boundary value problem (1.1)-(1.2) for an arbitrary positive integer m.

Let B be a real Banach space. A nonempty closed convex set $P \subset B$ is called a cone, if it satisfies the following conditions:

(*i*) $y \in P$, $\lambda \geq 0$ implies $\lambda y \in P$, and

(*ii*) $y \in P$ and $-y \in P$ implies $y = 0$.

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for nonnegative real numbers a' and b', we define the sets

$$P(\psi, a') = \{y \in P : \psi(y) < a'\}$$

and

$$P_{b'} = \{y \in P : \|y\| < b'\}.$$

In obtaining multiple positive solutions of the boundary value problem (1.1)-(1.2), the following Avery–Henderson functional fixed point theorem [3] will be the fundamental tool.

Theorem 3.1. [3] *Let P be a cone in a real Banach space B. Suppose α and γ are increasing, nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive numbers c' and k, $\gamma(y) \le \theta(y) \le \alpha(y)$ and $\|y\| \le k\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose that there exist positive numbers a' and b' with $a' < b' < c'$ such that $\theta(\lambda y) \le \lambda \theta(y)$, for all $0 \le \lambda \le 1$ and $y \in \partial P(\theta, b')$. Further, let $T : \overline{P(\gamma, c')} \to P$ be a completely continuous operator such that*

- $(B1)$ $\gamma(Ty) > c'$, *for all* $y \in \partial P(\gamma, c')$,
- $(B2)$ $\theta(Ty) < b'$, *for all* $y \in \partial P(\theta, b')$,
- $(B3)$ $P(\alpha, a') \neq \emptyset$ *and* $\alpha(Ty) > a'$, *for all* $y \in \partial P(\alpha, a')$.

Then T has at least two fixed points $y_1, y_2 \in \overline{P(\gamma, c')}$ such that $a' < \alpha(y_1)$ with $\theta(y_1) < b'$ and $b' < \theta(y_2)$ with $\gamma(y_2) < c'$.

Let

$$M = \prod_{j=1}^{n} \gamma_j. \tag{3.1}$$

Let $B = \{u : u \in C[0,1]_{\mathbb{T}}\}$ be the Banach space equipped with the norm $\|u\| = \max_{t \in [0,1]_{\mathbb{T}}} |u(t)|$. Define the cone $P \subset B$ by

$$P = \big\{u \in B : u(t) \ge 0 \text{ on } [0,1]_{\mathbb{T}} \text{ and } \min_{t \in [0,1]_{\mathbb{T}}} u(t) \ge M\|u\|\big\},$$

where M is given in (3.1).

Define the nonnegative increasing continuous functionals β, θ and α on the cone P by

$$\beta(u) = \min_{t \in [0,1]_{\mathbb{T}}} u(t),\ \theta(u) = \max_{t \in [0,1]_{\mathbb{T}}} u(t) \text{ and } \alpha(u) = \max_{t \in [0,1]_{\mathbb{T}}} u(t).$$

We observe that for any P,

$$\beta(u) \le \theta(u) \le \alpha(u) \tag{3.2}$$

and

$$\|u\| \le \frac{1}{M} \min_{t\in[0,1]_{\mathbb{T}}} u(t) = \frac{1}{M}\beta(u) \le \frac{1}{M}\alpha(u). \tag{3.3}$$

Theorem 3.2. *Assume that the conditions* $(A1)$-$(A3)$ *are satisfied. Suppose that there exist real numbers* a', b' *and* c' *with* $0 < a' < b' < c'$ *such that f satisfies the following conditions:*

$(D1)$ $f(t,u) > \frac{c'}{\prod_{j=1}^{n}\gamma_j K_j}$, *for* $t \in [0,1]_{\mathbb{T}}$ *and* $u \in [c', \frac{c'}{M}]$,

$(D2)$ $f(t,u) < \frac{b'}{\prod_{j=1}^{n} K_j}$, *for* $t \in [0,1]_{\mathbb{T}}$ *and* $u \in [0, \frac{b'}{M}]$,

$(D3)$ $f(t,u) > \frac{a'}{\prod_{j=1}^{n}\gamma_j K_j}$, *for* $t \in [0,1]_{\mathbb{T}}$ *and* $u \in [a', \frac{a'}{M}]$.

Then the boundary value problem (1.1)-(1.2) *has at least two positive solutions.*

Proof. Define the operator $T : P \to B$ by

$$Tu(t) = \int_0^1 H_n(t,s) f(s,u(s))\Delta s. \tag{3.4}$$

It is obvious that a fixed point of T is the solution of the boundary value problem (1.1)-(1.2). We seek two fixed points $u_1, u_2 \in P$ of T. First we show that $T : P \to P$. Let $u \in P$. Clearly, $Tu(t) \ge 0$ on $[0,1]_{\mathbb{T}}$. From Lemma 2.4, we have

$$\begin{aligned} Tu(t) &= \int_0^1 H_n(t,s) f(s,u(s))\Delta s \\ &\le K \int_0^1 G_n(\sigma(s),s) f(s,u(s))\Delta s \end{aligned}$$

so that

$$\|Tu(t)\| \le K \int_0^1 G_n(\sigma(s),s) f(s,u(s))\Delta s. \tag{3.5}$$

Next, if $u \in P$, then from Lemma 2.4 and (3.5), we have

$$\begin{aligned} \min_{t\in[0,1]_{\mathbb{T}}} Tu(t) &= \min_{t\in[0,1]_{\mathbb{T}}} \int_0^1 H_n(t,s) f(s,u(s))\Delta s \\ &\ge \gamma_n L \int_0^1 G_n(\sigma(s),s) f(s,u(s))\Delta s \\ &\ge \gamma_n \left(\prod_{j=1}^{n-1}\gamma_j\right)\|Tu\| \\ &= M\|Tu\|. \end{aligned}$$

Hence $Tu \in P$ and so $T : P \to P$. Moreover, T is completely continuous. From (3.2) and (3.3), for each $u \in P$, we have $\beta(u) \le \theta(u) \le \alpha(u)$ and $\|u\| \le \frac{1}{M}\beta(u)$. Also, for any $0 \le \lambda \le 1$ and $u \in P$, we have $\theta(\lambda u) = \max_{t\in[0,1]_{\mathbb{T}}} (\lambda u)(t) = \lambda \max_{t\in[0,1]_{\mathbb{T}}} u(t) = \lambda\theta(u)$. It is clear that $\theta(0) = 0$. We now show that the remaining conditions of the Theorem 3.1 are satisfied.

Firstly, we shall verify that the condition $(B1)$ of Theorem 3.1 is satisfied. Since $u \in \partial P(\beta, c')$, from (3.3) we have that $c' = \min_{t\in[0,1]_{\mathbb{T}}} u(t) \le \|u\| \le \frac{c'}{M}$, for $t \in [0,1]_{\mathbb{T}}$. Then,

$$\begin{aligned}\beta(Tu) &= \min_{t\in[0,1]_{\mathbb{T}}} \int_0^1 H_n(t,s) f(s,u(s))\Delta s \\ &\ge \gamma_n L \int_0^1 G_n(\sigma(s),s) f(s,u(s))\Delta s \\ &> \frac{c'}{\prod_{j=1}^n \gamma_j K_j} \gamma_n L \int_0^1 G_n(\sigma(s),s)\Delta s \\ &= c',\end{aligned}$$

using hypothesis $(D1)$.

Now, we shall show that condition $(B2)$ of Theorem 3.1 is satisfied. Since $u \in \partial P(\theta, b')$, from (3.3) we have that $0 \le u(t) \le \|u\| \le \frac{b'}{M}$, for $t \in [0,1]_{\mathbb{T}}$. Thus,

$$\begin{aligned}\theta(Tu) &= \max_{t\in[0,1]_{\mathbb{T}}} \int_0^1 H_n(t,s) f(s,u(s))\Delta s \\ &\le K \int_0^1 G_n(\sigma(s),s) f(s,u(s))\Delta s \\ &< \frac{b'}{\prod_{j=1}^n K_j} K \int_0^1 G_n(\sigma(s),s)\Delta s \\ &= b',\end{aligned}$$

by hypothesis $(D2)$.

Finally, using hypothesis $(D3)$, we shall show that condition $(B3)$ of Theorem 3.1 is satisfied. Since $0 \in P$ and $a' > 0, P(\alpha, a') \neq \emptyset$. Since $u \in \partial P(\alpha, a')$, $a' = \max_{t\in[0,1]_{\mathbb{T}}} u(t) \le \|u\| \le \frac{a'}{M}$, for

$t \in [0,1]_{\mathbb{T}}$. Therefore,

$$\begin{aligned}\alpha(Tu) &= \max_{t\in[0,1]_{\mathbb{T}}} \int_0^1 H_n(t,s)f(s,u(s))\Delta s \\ &\geq \int_0^1 H_n(t,s)f(s,u(s))\Delta s \\ &\geq \gamma_n L \int_0^1 G_n(\sigma(s),s)f(s,u(s))\Delta s \\ &> \frac{a'}{\prod_{j=1}^n \gamma_j K_j}\gamma_n L \int_0^1 G_n(\sigma(s),s)\Delta s \\ &= a'.\end{aligned}$$

Thus, all the conditions of Theorem 3.1 are satisfied. So there exist at least two positive solutions $u_1, u_2 \in \overline{P(\beta,c')}$ for the boundary value problem (1.1)-(1.2). This completes the proof of the theorem.

Theorem 3.3. *Let m be an arbitrary positive integer. Suppose there exist real numbers $a_r(r = 1,2,\ldots,m+1)$ and $b_s(s = 1,2,\ldots,m)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < a_{m+1}$ such that f satisfies following conditions:*

$$f(t,u) > \frac{a_r}{\prod_{j=1}^n \gamma_j K_j}, \text{ for } t \in [0,1]_{\mathbb{T}} \text{ and } u \in \left[a_r, \frac{a_r}{M}\right], \quad r = 1,2,\ldots,m+1, \tag{3.6}$$

$$f(t,u) < \frac{b_s}{\prod_{j=1}^n K_j}, \text{ for } t \in [0,1]_{\mathbb{T}} \text{ and } u \in \left[0, \frac{b_s}{M}\right], \quad s = 1,2,\ldots,m. \tag{3.7}$$

Then the boundary value problem (1.1)-(1.2) has at least $2m$ positive solutions in $\bar{P}_{a_{m+1}}$.

Proof. We use induction on m. For $m = 1$, from (3.6) and (3.7), it is clear that $T : \bar{P}_{a_2} \to P_{a_2}$, then it follows from Avery–Henderson fixed point theorem that the boundary value problem (1.1)-(1.2) has at least two positive solutions in $\bar{P}_{a_2}$. Let us assume that this conclusion holds for $m = l$. In order to prove this conclusion holds for $m = l+1$, we suppose that there exist real numbers $a_r(r = 1,2,\ldots,l+2)$ and $b_s(s = 1,2,\ldots,l+1)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{l+1} < b_{l+1} < a_{l+2}$ such that

$$f(t,u) > \frac{a_r}{\prod_{j=1}^n \gamma_j K_j}, \text{ for } t \in [0,1]_{\mathbb{T}} \text{ and } u \in \left[a_r, \frac{a_r}{M}\right], \quad r = 1,2,\ldots,l+2, \tag{3.8}$$

$$f(t,u) < \frac{b_s}{\prod_{j=1}^n K_j}, \text{ for } t \in [0,1]_{\mathbb{T}} \text{ and } u \in \left[0, \frac{b_s}{M}\right], \quad s = 1,2,\ldots,l+1. \tag{3.9}$$

By assumption, the boundary value problem (1.1)-(1.2) has at least $2l$ positive solutions $u_i (i = 1,2,\dots,2l)$ in $\bar{P}_{a_{l+1}}$. At the same time, it follows from Theorem 3.2, (3.8) and (3.9) that the boundary value problem (1.1)-(1.2) has at least two positive solutions u_1, u_2 in $\bar{P}_{a_{l+2}}$ such that $a_{l+1} < \alpha(u_1)$ with $\theta(u_1) < b_{l+1}$ and $b_{l+1} < \theta(u_2)$ with $\beta(u_2) < a_{l+2}$. Obviously u_1 and u_2 are different from $u_i (i = 1,2,\dots,2l)$. Therefore, the boundary value problem (1.1)-(1.2) has at least $2l+2$ positive solutions in $\bar{P}_{a_{l+2}}$, which shows that conclusion holds for $m = l+1$.

4. Example

Let us consider an example to illustrate our established results. Let $n = 2$ and $\mathbb{T} = \{0, \frac{1}{8}, \frac{1}{5}, \frac{1}{3}, 1\}$. Now, we consider the boundary value problem

$$u^{\Delta^4}(t) = f(t,u(t)), \quad t \in (0,1)_{\mathbb{T}}, \tag{4.1}$$

with

$$\begin{aligned} u(0) = u(1) &= \int_0^1 a_1(x)u(x)\Delta x, \\ u^{\Delta^2}(0) = u^{\Delta^2}(1) &= \int_0^1 a_2(x)u^{\Delta^2}(x)\Delta x, \end{aligned} \tag{4.2}$$

where

$$f(t,u(t)) = \frac{400(u+1)^6}{81(u^2+999)}, \; a_1(x) = \frac{38}{41} \text{ and } a_2(x) = \frac{29}{35}.$$

By direct calculations, we get

$$\eta_1 = 0.023428184, \; \eta_2 = 0.020944444, \; \gamma_1 = 0.242530509, \; \gamma_1 = 0.108874128$$

$$M = 0.026405297, \; K_1 = 0.900171164 \text{ and } K_2 = 0.377685980$$

Clearly f is continuous on $[0,\infty)$. Choosing $a' = 0.00001$, $b' = 0.004$, $c' = 10$ then $0 < a' < b' < c'$ and f satisfies

(i) $f(t,u(t)) > 1113.917351 = \frac{c'}{\prod_{j=1}^{2} \gamma_j K_j}$, for $t \in [0,1]_{\mathbb{T}}$ and $u \in [10, 378.712]$,
(ii) $f(t,u(t)) < 0.01176532 = \frac{b'}{\prod_{j=1}^{2} K_j}$, for $t \in [0,1]_{\mathbb{T}}$ and $u \in [0, 0.15148475]$,
(iii) $f(t,u(t)) > 0.001113917 = \frac{a'}{\prod_{j=1}^{2} \gamma_j K_j}$, for $t \in [0,1]_{\mathbb{T}}$ and $u \in [0.00001, 0.00037811]$.

Thus, all the conditions of Theorem 3.2 are satisfied and hence, the boundary value problem (4.1)-(4.2) has at least two positive solutions.

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REFERENCES

[1] R. P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications, Results Math. 35 (1999), 3-22.

[2] R. P. Agarwal, D. O'Regan, P. J. Y. Wong, Positive Solutions of Differential, Difference and Integral Equations, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[3] R. I. Avery, J. Henderson, Two positive fixed points of nonlinear operators on ordered Banach spaces, Comm. Appl. Nonlinear Anal. 8 (2001), 27-36.

[4] Z. Bai, W. Ge, Solutions of $2n$th Lidstone boundary value problems and dependence on higher order derivatives, J. Math. Anal. Appl. 279 (2003), 442-450.

[5] A. Belarbi, M. Benchohra, Existence results for nonlinear boundary value problems with integral boundary conditions, Electron. J. Differ. Equ. 2005 (2005), Article ID 6.

[6] M. Bohner, A. C. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhauser, Boston, MA, 2001.

[7] M. Bohner, A.C. Peterson, Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[8] A. K. Boucherif, Positive solutions of second order differential equations with integral boundary conditions, Discrete Cont. Dyn. Syst. 2007, 155–159.

[9] J. M. Davis, J. Henderson, Triple positive symmetric solutions for a Lidstone boundary value problem, Differ. Equ. Dyn. Syst. 7 (1999), 321-330.

[10] J. M. Davis, J. Henderson, P. J. Y. Wong, General Lidstone problems: Multiplicity and symmetry of solutions, J. Math. Anal. Appl. 251 (2000), 527-548.

[11] J. Ehme, J. Henderson, Existence and local uniqueness for nonlinear Lidstone boundary value problems, J. Inequal. Pure Appl. Math. 1 (2000), Article ID 8.

[12] J. Galvis, E. M. Rojas, A. V. Sinitsyn, Existence of positive solutions of a nonlinear second order boundary value problem with integral boundary conditions, Electron. J. Differ. Equ. 2015 (2015), Article ID 236.

[13] S. Hilger, Analysis on measure chains-A unified approach to continuous and discrete calculus, Results Math. 18 (1990), 18-56.

[14] Y. Ji, Y. Guo, Y. Yao, Positive solutions for higher order differential equations with integral boundary conditions, Boundary Value Probl. 2015 (2015), Article ID 214.

[15] P. Kang, Z. Wei, J. Xu, Positive solutions to fourth order singular boundary value problems with integral boundary conditions in abstract spaces, Appl. Math. Comput. 206 (2008), 245-256.

[16] Y. Li, Y. Dong, Multiple positive solutions for a fourth order integral boundary value problem on time scales, Boundary Value Probl. 2011 (2011), Article ID 59.

[17] L. Liu, X. Hao, Y. Wu, Positive solutions for singular second order differential equations with integral boundary conditions, Math. Comp. Modelling 57 (2013), 836–847.

[18] H. Ma, Symmetric positive solutions for nonlocal boundary value problems of fourth order, Nonlinear Anal. 68 (2008), 645-651.

[19] A. D. Oguz, F. S. Topal, Symmetric positive solutions for second order boundary value problems with integral boundary conditions on time scales, J. Appl. Anal. Comput. 6 (2016), 531-542.

[20] Y. P. Sun, Three Symmetric positive solutions for second order nonlocal boundary value problems, Acta Math. Appl. Sin. Engl. Ser. 27 (2011), 233-242.

[21] S. G. Topal, A. Denk, Existence of symmetric positive solutions for a semipositone problem on time scales, Hacettepe J. Math. Stat. 45 (2016), 23-31.

[22] Q. Wang, Y. Guo, Y. Ji, Positive solutions for fourth order nonlinear differential equation with integral boundary conditions, Discrete Dyn. Nature Soc. 2013 (2013), Article ID 684962.

[23] P. J. Y. Wong, R. P. Agarawal, Eigenvalues of Lidstone boundary value problems, Appl. Math. Comput. 104 (1999), 15-31.

[24] F. Xu, Three symmetric positive solutions of fourth order nonlocal boundary value problems, Electron. J. Qual. Theory Differ. Equ. 2011 (2011), Article ID 96.

[25] F. Xu, J. Liu, Symmetric positive solutions for nonlinear singular fourth order eigenvalue problems with nonlocal boundary condition, Discrete Dyn. Nature Soc. 2010 (2010), Article ID 187827.

[26] B. Zhang, X. Liu, Existence of multiple symmetric positive solutions of higher order Lidstone problems, J. Math. Anal. Appl. 284 (2003), 672-689.

[27] L. Zhang, Z. Xuan, Multiple Positive solutions for a second order boundary value problem with integral boundary conditions, Boundary Value Probl. 2016 (2016), Article ID 60.

[28] X. Zhang, M. Feng, W. Ge, Existence results for nonlinear boundary value problems with integral boundary conditions in Banach spaces, Nonlinear Anal. 69 (2008), 3310-3321.

[29] X. Zhang, W. Ge, Positive solutions for a class of boundary value problems with integral boundary conditions, Comput. Math. Appl. 58 (2009), 203-215.