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STABILITY OF A STAGE-STRUCTURED PLANT-POLLINATOR MUTUALISM MODEL WITH THE BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE

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Abstract. A stage-structured plant-pollinator mutualism model with the Beddington-DeAngelis functional response is introduced and studied. Our results show that the mutual interference concerning pollinator k_2 does not have an impact on the permanence and the extinction of the stage-structured plant-pollinator mutualism. However, when the positive equilibrium is unstable, a sufficient large k_2 can drive the system into a globally stable. Furthermore, some sufficient cases which guarantee the permanence of the system are obtained. The local stability of the system is discussed by the sign of eigenvalue. Finally, we analyze the global stability of the system via iterative methods.

Keywords. Plant-pollinator; Stage-structure; Beddington-DeAngelis functional response; Global stability.

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1. Introduction

The Beddington-DeAngelis functional response represents a positive effect of the pollinators on the plants. Beddington [1] and DeAngelis and Goldstein [2] recently proposed the Beddington-DeAngelis functional response, which has been widely studied, see [1]-[5] and the references therein. Recently, Chen, Chen and Huang [6] proposed and studied a two species non-autonomous competitive phytoplankton system with the Beddington-DeAngelis functional response and the effect of toxic substances. Particularly, sufficient conditions which guarantee the extinction of a species and global attractivity of the other one are obtained. In [7], Han, Xie

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and Chen analyzed the permanence, extinction and globally attractive of the mutualism model in the plant-pollinator system with the Beddington-DeAngelis functional response. In [8], Chen, Chen and Shi investigated the stability of the boundary solution of a nonautonomous predator-prey model with the Beddington-DeAngelis functional response, which reflects the dynamics of interacting predators and prey in a fluctuating environment. In [9], Xiao studied the uniqueness of positive equilibriums and its global asymptotic stability. The existence of limit cycles is also obtained when inference parameter of predator is small. In [10], Chen and You studied the permanence, extinction and periodic solutions of the periodic predator-prey system with the Beddington-DeAngelis functional response and the stage structure for prey. According to the dynamics of the models with properties of the evolutionary games, Wang, Wu and Sun [11] analyzed the mechanisms by which the pollination mutualism could persist in the presence of nectar robbers. They investigated the interactions between pollinators, nectar robbers, defensive plants and non-defensive plants. They also used the Beddington-DeAngelis functional response to describe the plant-pollinator interaction and assumed that the relationship is not obligate.

$$\begin{aligned} \dot{x}(t) &= x(t)\Big(r_1 - d_1 x + \frac{a_{12}y(t)}{1+ax(t)+by(t)}\Big), \\ \dot{y}(t) &= y(t)\Big(-r_2 + \frac{a_{21}x(t)}{1+ax(t)+by(t)}\Big), \end{aligned} \tag{1.1}$$

where x and y represent non-defensive plants and pollinators densities, r_1 is the intrinsic growth rate of the plants, r_2 is the mortality rate of pollinators, d_1 is their environmental capacity without visitors, a_{12} is the efficiency of the mutual transformation between pollinators and plants, a_{21} is the corresponding value of pollinators. They analyzed the global stability of the system and proved that there is no periodic orbit in the system.

In [12], Li, Han and Chen studied the global stability of the equilibrium based on iterative methods.

$$\begin{aligned} \dot{x}_1(t) &= r_1 x_2(t) - d x_1(t) - r_1 e^{-d\tau} x_2(t-\tau), \\ \dot{x}_2(t) &= r_1 e^{-d\tau} x_2(t-\tau) - b x_m^2(t) - \frac{a_1 x_2(t) y(t)}{k_1 + x_2(t)}, \\ \dot{y}(t) &= y(t)\Big(r_2 - \frac{a_2 y(t)}{x_2(t)+k_2}\Big), \end{aligned} \tag{1.2}$$

where $x_1(t), x_2(t)$ and $y(t)$ denote functions of time concerning population densities of immature prey, mature prey and predator, r_1 can be regarded as the birth rate of the immature population, d is the death rate of the immature population, $e^{-d\tau}$ denotes the surviving rate of immaturity to reach maturity. In [12], all the conclusions of the model did not consider the state during the life histories of plants. This is not realistic. In this paper, we, based on model (1.1), follow interest with the stage structure for plants

$$\begin{aligned} \dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) + \frac{m x_m(t) y(t)}{1+k_1 x_m(t)+k_2 y(t)}, \\ \dot{y}(t) &= \frac{n m x_m(t) y(t)}{1+k_1 x_m(t)+k_2 y(t)} - d y(t), \\ x_i(\theta) &= \phi_i(t), x_m(\theta) = \phi_m(t), y(\theta) = \varphi(t), -\tau \le \theta < 0, x_i > 0, x_m(0) > 0, y(0) > 0, \end{aligned} \tag{1.3}$$

where $x_i(t), x_m(t), y(t)$ can be described as the immature and mature plants densities, the pollinators densities at time t. System (1.3) satisfies the following conditions: (1) The per capita birth rate of the immature population is $\alpha > 0$. The per capita death rate of the immature population is $\gamma > 0$. The per capita death rate of the mature plants is proportional to the current mature plants population with a proportionality constant $\beta > 0$, $\tau > 0$ is the length of time from birth to maturity $e^{-\gamma\tau}$ denotes the surviving rate of immaturity to reach maturity, m can be regarded as the plants' efficiency of mutual transformation between pollinators and plants, nm represents the corresponding value for the pollinators. For the continuity of the solutions to system (1.3), we require

$$x_i(0) = \int_{-\tau}^{0} \alpha e^{-\gamma s} \phi_m(s) ds. \tag{1.4}$$

From the first equation of system (1.3), the initial conditions (1.4) and the arguments similar to Lemma 3.1 in [13], we obtain

$$x_i(t) = \int_{t-\tau}^{t} \alpha e^{-\gamma(t-s)} x_m(s) ds. \tag{1.5}$$

Hence, we only need to analyse the last two equations in (1.3)

$$\begin{aligned} \dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) + \frac{m x_m(t) y(t)}{1+k_1 x_m(t)+k_2 y(t)}, \\ \dot{y}(t) &= \frac{n m x_m(t) y(t)}{1+k_1 x_m(t)+k_2 y(t)} - d y(t), \\ x_m(\theta) &= \phi_m(\theta) \ge 0, y(\theta) = \varphi(t), -\tau \le \theta < 0, x_m(0) > 0, y(0) > 0. \end{aligned} \tag{1.6}$$

2. Permanence of the system

To investigate the persistent property of the system, we need the following lemmas.

Lemma 2.1. [14] *Consider the following equation:*

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t), \tag{2.1}$$

where $a, c, \tau > 0, b \geq 0$, *and* $x(t) > 0$, $for - \tau \leq t \leq 0$.

(1) If $a \geq b$, *then* $\lim_{t\to\infty} x(t) = \frac{(a-b)}{c}$.

(2) If $a \leq b$, *then* $\lim_{t\to\infty} x(t) = 0$.

Lemma 2.2. [15] *Consider the following equation:*

$$\dot{x}(t) = ax(t-\tau) + bx(t) - cx^2(t), \tag{2.2}$$

where $a, b, c, \tau > 0$ *and* $x(t) > 0$, $for - \tau \leq t \leq 0$. *We obtain* $\lim_{t\to\infty} x(t) = \frac{a+b}{c}$.

Putting $\tau = 0$ in Theorem 4.9.1 of ([16], pp.159), we directly get the following.

Lemma 2.3.

$$\dot{y}(t) = \frac{nmMy(t)}{1+k_1M+k_2y(t)} - dy(t). \tag{2.3}$$

The coefficients of the above system are positive constants.

(1) If $(nm-k_1d)M > d$, *then* $\lim_{t\to\infty} y(t) = \frac{(nm-k_1d)M-d}{k_2d} = N$.

(2) If $(nm-k_1d)M \leq d$, *then* $\lim_{t\to\infty} y(t) = 0$.

Lemma 2.4. *Assume* $\phi(\theta) \geq 0$ *is continuous on* $\theta \in [-\tau, 0]$, $x_m(0), y(0) > 0$. *Then the solutions of system (1.6) satisfies all* $x_m(t), y(t) > 0$ *for* $t > 0$.

Theorem 2.5. *If* $mn > k_1d$, *then all solutions of system (1.6) are bounded on* Ω.

$$\Omega = \{(x_m(t), y(t)) : x_m(t) \leq M_1, y(t) \leq M_2\}. \tag{2.4}$$

Proof. Using the first equation of system (1.6), we get

$$\dot{x}_m(t) < \alpha e^{(-\gamma\tau)} x_m(t-\tau) - \beta x_m^2(t) + \frac{mx_m(t)y(t)}{k_2y(t)}. \tag{2.5}$$

Consider the following equations

$$\begin{aligned} \dot{u}(t) &= \alpha e^{(-\gamma\tau)} x_m(t-\tau) - \beta x_m^2(t) + \frac{mx_m(t)y(t)}{k_2y(t)}, \ t \geq 0, \\ u(t) &= \varphi(t), \ -\tau \leq t \leq 0. \end{aligned}$$

It follows the Lemma 2.2 that $u^* = \lim_{t\to\infty} u(t) = \dfrac{\alpha e^{-\gamma\tau} + \dfrac{m}{k_2}}{\beta}$. We also can obtain $0 < x_m(t) < u(t)$, $t > 0$. Hence,

$$\limsup_{t\to\infty} x_m(t) \quad \le \quad \frac{\alpha e^{-\gamma\tau} + \dfrac{m}{k_2}}{\beta}, \tag{2.6}$$

that is, for any $\varepsilon > 0$ sufficiently small, there exists $T_1^* > 0$ such that

$$x_m(t) \quad < \quad \frac{\alpha e^{-\gamma\tau} + \dfrac{m}{k_2}}{\beta} + \varepsilon \triangleq M_1,\ t > T_1^*.$$

Substituting M_1 into the second equation of system (1.6), we obtain

$$\dot{y}(t) \quad < \quad \frac{nmM_1 y(t)}{k_1(M_1) + k_2 y(t)} - dy(t). \tag{2.7}$$

Consider the following equations

$$\dot{u}(t) \quad = \quad \frac{nmM_1 y(t)}{k_1 M_1 + k_2 y(t)} - dy(t).$$

From the comparison principle, we obtain $u^* = \lim_{t\to\infty} u(t) = \dfrac{(nm - k_1 d)M_1}{k_2 d}$. We also can get $0 < y(t) < u(t), t > T_1^*$. Hence,

$$\limsup_{t\to\infty} y(t) \quad \le \quad \frac{(nm - k_1 d)M_1}{k_2 d}, \tag{2.8}$$

that is, for any $\varepsilon > 0$ sufficiently small, there exists $T_2^* > T_1^* > 0$ such that

$$y(t) \quad < \quad \frac{(nm - k_1 d)M_1}{k_2 d} + \varepsilon \triangleq M_2,\ t > T_2^*.$$

Theorem 2.6. *Suppose that the coefficients of the system (1.6) satisfies $(nm - k_1 d)m_1 > d$. Then system (1.6) is permanent, where $m_1 = \dfrac{\alpha e^{-\gamma\tau}}{\beta}$.*

Proof. Using the first equation of system (1.6), we get

$$\dot{x}_m(t) \quad > \quad \alpha e^{(-\gamma\tau)} x_m(t - \tau) - \beta x_m^2(t). \tag{2.9}$$

Consider the following equations

$$\begin{aligned} \dot{u}(t) &= \alpha e^{(-\gamma\tau)} x_m(t - \tau) - \beta x_m^2(t),\ t \ge 0, \\ u(t) &= \varphi(t),\ -\tau \le t \le 0. \end{aligned}$$

It follows the Lemma 2.1 that $u^* = \lim_{t\to\infty} u(t) = \dfrac{\alpha e^{-\gamma\tau}}{\beta}$. We also can get $0 < u(t) < x_m(t),\ t > 0$. Hence,

$$\liminf_{t\to\infty} x_m(t) \quad \geq \quad \frac{\alpha e^{-\gamma\tau}}{\beta} > 0, \tag{2.10}$$

that is, for any $\varepsilon > 0$ sufficiently small, there exists $\overline{T}_1 > 0$ such that

$$x_m(t) \quad > \quad \frac{\alpha e^{-\gamma\tau}}{\beta} - \varepsilon \triangleq m_1,\ t > \overline{T}_1.$$

Substituting m_1 into the second equation of system (1.6), we obtain

$$\dot{y}(t) \quad > \quad \frac{nmm_1 y(t)}{1 + k_1 m_1 + k_2 y(t)} - dy(t),\ t > \overline{T}_1. \tag{2.11}$$

Consider the following equations

$$\begin{aligned} \dot{u}(t) \quad &= \quad \frac{nmm_1 y(t)}{1 + k_1 m_1 + k_2 y(t)} - dy(t),\ t > \overline{T}_1, \\ u(t) \quad &= \quad \varphi(t),\ -\tau \leq t \leq 0. \end{aligned}$$

It follows the Lemma 2.3 that $u^* = \lim_{t\to\infty} u(t) = \dfrac{(nm - k_1 d)m_1 - d}{k_2 d}$. We can also get $0 < u(t) < y(t),\ t > \overline{T}_1$. Hence, for any $\varepsilon > 0$ sufficiently small, there exists $\overline{T}_2 > \overline{T}_1 > 0$ such that

$$\liminf_{t\to\infty} y(t) \quad \geq \quad \frac{(nm - k_1 d)m_1 - d}{k_2 d} \triangleq m_2 > 0. \tag{2.12}$$

In view of (2.6), (2.8), (2.10) and (2.12), we obtain that $m_1 \leq \liminf_{t\to\infty} x_m(t) \leq \limsup_{t\to\infty} x_m(t) \leq M_1$ and $m_2 \leq \liminf_{t\to\infty} y(t) \leq \limsup_{t\to\infty} y(t) \leq M_2$. This completes the proof.

3. Stability of equilibria

First, according to the equations of system (1.6), we can obtain three nonnegative equilibrium points $P_0(0,0)$, $P_1(\dfrac{\alpha e^{-\gamma\tau}}{\beta}, 0)$ and $P_2(x_m^*, y^*)$. Here x_m^*, y^* are the unique positive equilibrium point of system (1.6) and satisfy the following equations

$$\alpha e^{-\gamma\tau} x_m - \beta (x_m)^2 + \frac{m x_m y}{1 + k_1 x_m + k_2 y} = 0, \tag{3.1}$$

$$\frac{m x_m y(t)}{1 + k_1 x_m + k_2 y} - dy = 0. \tag{3.2}$$

From system (3.2), we obtain that

$$x_m^* = \frac{d(1 + k_2 y^*)}{mn - k_1 d}. \tag{3.3}$$

Substituting x_m^* into system (3.1), we obtain

$$A\big(y^*\big)^2 - By^* - C = 0, \tag{3.4}$$

where

$$\begin{aligned} A &= \frac{nd\beta k_2^2}{(mn-k_1d)^2}, B = \frac{\alpha e^{-\gamma\tau}k_2 n}{mn-k_1 d} - \frac{2\beta d k_2 n}{(mn-k_1d)^2} + 1, \\ C &= \frac{\alpha e^{-\gamma\tau} n(mn-k_1d) - \beta dn}{(mn-k_1d)^2}. \end{aligned}$$

Therefore, the unique positive equilibrium point P_2 exists if $C > 0$, which is equivalent to the following equation

$$\alpha e^{-\gamma\tau}(mn-k_1d) - \beta d > 0. \tag{3.5}$$

By solving inequality (3.5), we get

$$d < \frac{nm\alpha e^{-\gamma\tau}}{\beta + k_1\alpha e^{-\gamma\tau}}. \tag{3.6}$$

Hence, the unique positive equilibrium point P_2 exists if $d < \dfrac{nm\alpha e^{-\gamma\tau}}{\beta + k_1\alpha e^{-\gamma\tau}}$.

Theorem 3.1. *The equilibrium point P_0 is unstable.*

Proof. The varitional matrix of system (1.6) at the equilibrium point P_0 is

$$V(P_0) = \begin{bmatrix} \alpha e^{-(\gamma+\lambda)\tau} & 0 \\ 0 & -d \end{bmatrix}.$$

We can obtain the characteristic equation at equilibrium point P_0

$$F(\lambda) = (\lambda - \alpha e^{-(\gamma+\lambda)\tau})(\lambda + d) = 0. \tag{3.7}$$

Since $F(0) = -d\alpha e^{-\gamma\tau} < 0$ and $F(+\infty) = +\infty$, one finds that $F(\lambda) = 0$ has at least one positive solution and P_0 is unstable.

Clearly, one positive eigenvalue is $\lambda = -d$. Hence equilibrium point P_0 is a saddle point.

Theorem 3.2. *The equilibrium point P_1 is*

(1) unstable assume that $d < \dfrac{nm\alpha e^{-\gamma\tau}}{\beta + k_1\alpha e^{-\gamma\tau}}$,

(2) linearly neutrally stable assume $d = \dfrac{nm\alpha e^{-\gamma\tau}}{\beta + k_1\alpha e^{-\gamma\tau}}$,

(3) locally asymptotically stable assume $d > \dfrac{nm\alpha e^{-\gamma\tau}}{\beta + k_1\alpha e^{-\gamma\tau}}$.

Proof. The variational matrix of system (1.6) at the equilibrium point P_1 is

$$V(P_1) = \begin{bmatrix} \alpha e^{-(\gamma+\lambda)\tau} - 2\alpha e^{-\gamma\tau} & \dfrac{m\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}} \\ 0 & \dfrac{nm\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}} - d \end{bmatrix}.$$

We can obtain the characteristic equation at the equilibrium point P_1

$$F(\lambda) = (\lambda - \alpha e^{-(\gamma+\lambda)\tau} + 2\alpha e^{-\gamma\tau})(\lambda + d - \frac{nm\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}}) = 0. \tag{3.8}$$

The solutions of characteristic equation at point P_1 are given by

$$\lambda - \alpha e^{-(\gamma+\lambda)\tau} + 2\alpha e^{-\gamma\tau} = 0, \tag{3.9}$$

$$G(\lambda) = \lambda - \frac{nm\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}} + d = 0. \tag{3.10}$$

The equation (3.9) implies $Re\lambda < 0$. We prove it by contradiction. Letting $Re\lambda \geq 0$, we obtain

$$\begin{aligned} Re\lambda &= \alpha e^{-\gamma\tau} e^{-Re(\lambda)\tau} \cos(Im\lambda)\tau - 2\alpha e^{-\gamma\tau} \\ &\leq \alpha e^{-\gamma\tau} - 2\alpha e^{-\gamma\tau} < 0. \end{aligned} \tag{3.11}$$

It is a contradiction. Hence, Re$\lambda < 0$.

Next, we consider equation (3.10).

(1) Assumed $d < \frac{nm\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}}$. Then $G(0) = d - \frac{nm\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}} < 0, G(+\infty) = \infty$. Therefore, $G(\lambda) = 0$ has at least a positive root and P_1 is unstable.

(2) As $d = \frac{nm\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}}$, $G(0) = 0$ implies that $\lambda = 0$ is a root of $G(\lambda) = 0$. Furthermore, since $\dot{G}(\lambda) = 1$, we obtain $\dot{G}(0) > 0$. So $\lambda = 0$ is a simple root. Hence equilibrium point P_1 is locally asymptotically stable.

(3) Supposed that $d > \frac{nm\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}}$. Then $\lambda = \frac{nm\alpha e^{-\gamma\tau}}{\beta + k_1 \alpha e^{-\gamma\tau}} - d < 0$.

To prove equilibrium point P_2 is locally asymptotically stable, we need to use the following lemma.

Lemma 3.3. (see [17]) *The necessary and sufficient conditions for equilibrium point P_2 to be asymptotically stable for $\tau \geq 0$ as the following*

(1) The real parts of all the eigenvalue are negative in $F(\lambda, 0) = 0$.

(2) For all real b and $\tau \geq 0$, $F(ib, \tau) \neq 0$, where $i = \sqrt[2]{-1}$.

Theorem 3.4. *Assume that $k_2 \alpha e^{-\gamma\tau} > m$. Then equilibrium point P_2 is locally asymptotically stable.*

Proof. The variational matrix of system (1.6) at the equilibrium point P_2 is

$$V(P_2) = \begin{bmatrix} \alpha e^{-(\gamma+\lambda)\tau} - 2\beta x_m^* + A_1 & A_2 \\ nA_1 & nA_2 - d \end{bmatrix},$$

where $A_1 = \dfrac{my^*(1+k_2y^*)}{(1+k_1x_m^*+k_2y^*)^2}$ and $A_2 = \dfrac{mx_m^*(1+k_1x_m^*)}{(1+k_1x_m^*+k_2y^*)^2}$. We can have characteristic equation at the equilibrium point P_2

$$F(\lambda,\tau) = \lambda^2 + \lambda P_1(\tau) + P_0(\tau) + (\lambda Q_1(\tau) + Q_0(\tau))e^{-\lambda\tau} = 0. \tag{3.12}$$

$$\begin{aligned} P_1(\tau) &= 2\beta x_m^* - A_1 - nA_2 + d, \\ P_0(\tau) &= (2\beta x_m^* - A_1)(d - nA_2) - nA_1A_2, \\ Q_1(\tau) &= -\alpha e^{-\gamma\tau}, \\ Q_0(\tau) &= -\alpha e^{-\gamma\tau}(d - nA_2). \end{aligned} \tag{3.13}$$

First, we prove

$$P_0(\tau) + Q_0(\tau) \neq 0, \tag{3.14}$$

which implies that $\lambda = 0$ cannot be a root of equation (3.12) for any $\tau \in I = [0,\tau^*)$, where $\tau^* = \dfrac{1}{\gamma}\ln\dfrac{\alpha(mn-k_1d)}{d\beta}$.

$$\begin{aligned} P_0(\tau) + Q_0(\tau) &= (2\beta x_m^* - A_1)(d - nA_2) - nA_1A_2 - \alpha e^{-\gamma\tau}(d - nA_2) \\ &= (2\beta x_m^* - \alpha e^{-\gamma\tau})(d - nA_2) - dA_1 \\ &= (\alpha e^{-\gamma\tau} + \frac{2my^*}{1+k_1x_m^*+k_2y^*})(1 - \frac{1+k_1x_m^*}{1+k_1x_m^*+k_2y^*})d - A_1d \\ &= \Big(\frac{\alpha e^{-\gamma\tau}k_2y^*}{1+k_1x_m^*+k_2y^*} + \frac{k_2my^{*2}}{(1+k_1x_m^*+k_2y^*)^2} - \frac{my^*}{(1+k_1x_m^*+k_2y^*)^2}\Big)d \\ &= \Big[\alpha e^{-\gamma\tau}k_2(1+k_1x_m^*+k_2y^*) + k_2y^* - m\Big]\frac{dy^*}{(1+k_1x_m^*+k_2y^*)^2}. \end{aligned}$$

Since $k_2\alpha e^{-\gamma\tau} > m$, we have $\alpha e^{-\gamma\tau}k_2(1+k_1x_m^*+k_2y^*) + k_2y^* - m > 0$. It follows that

$$P_0(\tau) + Q_0(\tau) > 0,\ \tau \in I = [0,\tau^*), \tag{3.15}$$

that is, $\lambda = 0$ is not a root of (3.12). Next, we consider the following steps. (1) From equation (3.12), we have

$$F(\lambda,0) = \lambda^2 + \lambda(P_1(0) + Q_1(0)) + \big(P_0(0) + Q_0(0)\big) = 0. \tag{3.16}$$

We consider the sign of $P_1(0)+Q_1(0)$ and $P_0(0)+Q_0(0)$. Since $P_0(\tau)+Q_0(\tau)>0$ as $\tau\in[0,\tau^*)$, one has $P_0(0)+Q_0(0)>0$,

$$\begin{aligned}
P_1(0)+Q_1(0) &= \left(\frac{2my^*}{1+k_1x_m^*+k_2y^*}+2\alpha\right)-A_1-nA_2+d-\alpha \\
&= \frac{2my^*}{1+k_1x_m^*+k_2y^*}-\frac{my^*(1+k_2y^*)}{(1+k_1x_m^*+k_2y^*)^2} \\
&\quad -\frac{nmx_m^*(1+k_1x_m^*)}{(1+k_1x_m^*+k_2y^*)^2}+d+2\alpha-\alpha \\
&> 0.
\end{aligned}$$

Since $P_1(0)+Q_1(0)>0$, $P_0(0)+Q_0(0)>0$, one sees that the all roots of equation (3.16) are negative real parts. Hence, equilibrium point P_2 is locally asymptotically stable at $\tau=0$. (2) We consider $F(ib,\tau)\neq 0$, for real b. If $b=0$, then $F(0,\tau)=P_0(\tau)+Q_0(\tau)>0$. If $b\neq 0$, then system (3.12) does not have the positive real roots. Note that

$$\begin{aligned}
F(ib,\tau) &= -b^2+ibP_1(\tau)+P_0(\tau)+\big(ibQ_1(\tau)+Q_0(\tau)\big)(\cos b\tau-i\sin b\tau)=0, \\
F(ib,\tau) &= F_R(ib,\tau)+iF_I(ib,\tau), \\
F_R(ib,\tau) &= -b^2+P_0(\tau)+bQ_1(\tau)\sin b\tau+Q_0(\tau)\cos b\tau=0, \\
F_I(ib,\tau) &= bP_1(\tau)+bQ_1(\tau)\cos b\tau-Q_0(\tau)\sin b\tau=0.
\end{aligned}$$

It follows that $F(b,\tau)=b^4+A(\tau)b^2+B(\tau)=0$, $A(\tau)=-2P_0(\tau)+P_1^2(\tau)-Q_1^2(\tau)$ $B(\tau)=P_0^2(\tau)-Q_0^2(\tau)$.

$$\begin{aligned}
P_0(\tau)-Q_0(\tau) &= (2\beta x_m^*-A_1)(d-nA_2)-nA_1A_2+\alpha e^{-\gamma\tau}(d-nA_2) \\
&= (2\beta x_m^*+\alpha e^{-\gamma\tau})(d-nA_2)-dA_1 \\
&= \left[(3\alpha e^{-\gamma\tau}+\frac{2my^*}{1+k_1x_m^*+k_2y^*})(1-\frac{1+k_1x_m^*}{1+k_1x_m^*+k_2y^*})-A_1\right]d \\
&= (3\alpha e^{-\gamma\tau}+\frac{2my^*}{1+k_1x_m^*+k_2y^*})(1-\frac{1+k_1x_m^*}{1+k_1x_m^*+k_2y^*})d \\
&\quad -\frac{my^*(1+k_2y^*)}{(1+k_1x_m^*+k_2y^*)^2}d \\
&= \Big(\frac{3\alpha e^{-\gamma\tau}k_2y^*}{1+k_1x_m^*+k_2y^*}+\frac{k_2my_2^{*2}}{(1+k_1x_m^*+k_2y^*)^2}\Big)d \\
&\quad -\Big(\frac{my^*}{1+k_1x_m^*+k_2y^*)^2}\Big)d \\
&= \Big[3\alpha e^{-\gamma\tau}k_2(1+k_1x_m^*+k_2y^*)+k_2y^*-m\Big]\frac{dy^*}{(1+k_1x_m^*+k_2y^*)^2}.
\end{aligned}$$

This implies that $P_0(\tau)-Q_0(\tau)>0$. Note that $B(\tau)=P_0^2(\tau)-Q_0^2(\tau)=\big(P_0(\tau)-Q_0(\tau)\big)\big(P_0(\tau)+Q_0(\tau)\big)$. It follows from (3.15) that $P_0(\tau)+Q_0(\tau)>0, P_0(\tau)+Q_0(\tau)>0$. Therefore, we can obtain $B(\tau)>0$. Next, we judge the sign of $A(\tau)$

$$\begin{aligned}
A(\tau) &= -2(2\beta dx_m^*-2\beta nA_2x_m^*-dA_1)+(2\beta dx_m^*-A_1-nA_2+d)^2-\alpha^2e^{-2\gamma\tau}\\
&= -4\beta dx_m^*+(2\beta x_m^*+d)^2+4\beta nA_2x_m^*+2A_1d\\
&\quad +(nA_2+A_1)^2-2(2\beta x_m^*+d)(A_1+nA_2)-\alpha^2e^{-2\gamma\tau}\\
&= (2\beta x_m^*)^2-\alpha^2e^{-2\gamma\tau}+d^2+(nA_2+A_1)^2-2dnA_2-4\beta A_1x_m^*\\
&= 2\beta x_m^*\Big(\frac{2my^*}{1+k_1x_m^*+k_2y^*}-\frac{2my^*(1+k_2y^*)}{(1+k_1x_m^*+k_2y^*)^2}+\alpha e^{\gamma\tau}\Big)\\
&\quad +\alpha e^{\gamma\tau}(2\beta x_m^*-\alpha e^{\gamma\tau})+(nA_2-d)^2+A_1^2+2nA_1A_2\\
&> 0.
\end{aligned}$$

Therefore, $F(b,\tau)\neq 0$ for any real b. Using Lemma 3.3, we obtain equilibrium point P_2 is locally asymptotically stable. This completes the proof.

4. Global attractivity

In this Section, we investigate the global Stability of equilibrium point P_1 and P_2. To prove the results, we need to the following lemma.

Lemma 4.1. *Consider the following equation:*

$$\dot{v}(t)=\alpha e^{-\gamma\tau}v(t-\tau)-\beta v^2(t)+\frac{my_0v(t)}{1+k_1v(t)+k_2y_0}, \tag{4.1}$$

where $\alpha,\beta,m,k_1,k_2>0$ and $v(\theta)>0$, $for -\tau\leq\theta\leq 0$. System (4.1) has a single positive root

$$\lim_{t\to\infty}v(t)=\frac{k_1\alpha e^{-\gamma\tau}-\beta-k_2y_0\beta+\sqrt{(k_1\alpha e^{-\gamma\tau}+\beta+k_2y_0\beta)^2+4\beta k_1my_0}}{2k_1\beta}.$$

Proof. From system (4.1), we obtain that

$$\alpha e^{-\gamma\tau}v(t-\tau)-\beta v^2(t)<\dot{v}(t)<\alpha e^{-\gamma\tau}v(t-\tau)-\beta v^2(t)+\frac{mv(t)}{k_2}.$$

It follows from Lemma 2.1, Lemma 2.2 and the comparison principle, for $\varepsilon>0$ sufficiently small that there exists $T_1>0$ such that

$$\underline{v}_1=\frac{\alpha e^{-\gamma\tau}}{\beta}-\varepsilon<v(t)<\frac{\alpha e^{-\gamma\tau}+\dfrac{m}{k_2}}{\beta}+\varepsilon=\overline{v}_1.\ \ t\geq T_1.$$

Substituting $\underline{v}_1$ and $\overline{v}_1$ into system (4.1), we have

$$\dot{v}(t) > \alpha e^{-\gamma\tau} v(t-\tau) - \beta v^2(t) + \frac{my_0 v(t)}{1+k_1\overline{v}_1+k_2y_0}.$$
$$\dot{v}(t) < \alpha e^{-\gamma\tau} v(t-\tau) - \beta v^2(t) + \frac{my_0 v(t)}{1+k_1\underline{v}_1+k_2y_0}.$$

It follows from Lemma 2.2 and the comparison principle for the above $\varepsilon > 0$ sufficiently small that there exists $T_2 > T_1 > 0$ such that

$$v(t) > \frac{\alpha e^{-\gamma\tau} + \dfrac{my_0}{1+k_1\overline{v}_1+k_2y_0}}{\beta} - \varepsilon = \underline{v}_2.$$
$$v(t) < \frac{\alpha e^{-\gamma\tau} + \dfrac{my_0}{1+k_1\underline{v}_1+k_2y_0}}{\beta} + \varepsilon = \overline{v}_2 \ t > T_2.$$

Hence, we obtain that $\underline{v}_1 < \underline{v}_2 < v(t) < \overline{v}_1 < \overline{v}_2 \ t > T_2$. By repeating the above steps, we obtain that $0 < \underline{v}_1 < \underline{v}_2 < \cdots < \underline{v}_n < v(t) < \overline{v}_n < \cdots < \overline{v}_2 < \overline{v}_1$. In view of the monotone bounded theorem, we obtain that the limit of $\{\underline{v}_n\}_{n=1}^{\infty}$, $\{\overline{v}_n\}_{n=1}^{\infty}$ exist. Put $\overline{v} = \lim\limits_{t\to\infty} \overline{v}_n$ and $\underline{v} = \lim\limits_{t\to\infty} \underline{v}_n$, where

$$\underline{v}_n = \frac{\alpha e^{-\gamma\tau} + \dfrac{my_0}{1+k_1\overline{v}_{n-1}+k_2y_0}}{\beta} - \varepsilon.$$
$$\overline{v}_n = \frac{\alpha e^{-\gamma\tau} + \dfrac{my_0}{1+k_1\underline{v}_{n-1}+k_2y_0}}{\beta} + \varepsilon.$$

Then we have $\underline{v}_n - \varepsilon < v(t) < \overline{v}_n + \varepsilon$ as $t \to \infty$. It follows that $\overline{v} = \underline{v} = v^*$

$$v^* = \lim_{t\to\infty} v(t) = \frac{k_1\alpha e^{-\gamma\tau} - \beta - k_2y_0\beta + \sqrt{(k_1\alpha e^{-\gamma\tau} + \beta + k_2y_0\beta)^2 + 4\beta k_1 m y_0}}{2k_1\beta}.$$

This completes the proof.

Theorem 4.2. *Assume that $mn < k_1 d$. Then equilibrium point P_1 is globally asymptotically stable.*

Proof. Note that P_1 is locally asymptotically stable. Next, we only need to show the globally asymptotically stable. From system (1.6), we obtain

$$\dot{y} \ \le \ \Big(\frac{mnx_m(t)}{k_1x_m(t)} - d\Big)y(t) = \Big(\frac{mn}{k_1} - d\Big)y(t).$$

In view of $mn < k_1 d$, we can get $y(t) \to 0$ as $t \to \infty$. Therefore, for any sufficiently small positive number ε, there exists a t_ε such that $y(t) \le \varepsilon$ as $t \ge t_\varepsilon$. From the first equation of system (1.6), we get $\dot{x}_m < \alpha^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) + m x_m(t)\varepsilon$. It follows the Lemma 2.2 that

$u^* = \lim_{t\to\infty} u(t) = \dfrac{\alpha e^{-\gamma\tau} + m\varepsilon}{\beta}$. We also get $0 < x_m < u(t)$, $t > 0$. Hence, for any $\varepsilon > 0$ sufficiently small, there exists $T_1^* > 0$ such that

$$x_m \quad < \quad \frac{\alpha e^{-\gamma\tau} + m\varepsilon}{\beta} + \varepsilon,\ t > T_1^*.$$

On the other hand, one has $\dot{x}_m(t) \geq \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t)$. It follows the Lemma 2.1 that $u^* = \lim_{t\to\infty} u(t) = \dfrac{\alpha e^{-\gamma\tau}}{\beta}$. We also get $0 < u(t) < x_m(t)$, $t > 0$. Hence, for any $\varepsilon > 0$ sufficiently small, there exists $T_2^* > T_1^* > 0$ such that $x_m > \dfrac{\alpha e^{-\gamma\tau}}{\beta} - \varepsilon$. Hence, one has $\lim_{t\to\infty} x_m(t) = \dfrac{\alpha e^{-\gamma\tau}}{\beta}$ and $\lim_{t\to\infty}(x_m(t), y(t)) = (\dfrac{\alpha e^{-\gamma\tau}}{\beta}, 0)$. This proof is completed.

Theorem 4.3. *If* $1 - \dfrac{m(nm - k_1 d)}{k_2 d\beta} > 0$*, then equilibrium point* P_2 *is globally attractive.*

Proof. Using the first equation of system (1.6), we obtain $\dot{x}_m(t) \leq \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) + \dfrac{m x_m(t)}{k_2}$. Using Lemma 2.2 and the comparison theorem, we obtain that there exists a $T_1 > 0$ such that

$$x_m(t) < \frac{\alpha e^{-\gamma\tau} + \dfrac{m}{k_2}}{\beta} + \varepsilon = \overline{u}_1.$$

It follows from system (1.6) that

$$\dot{y}(t) \quad \leq \quad \frac{nm\overline{u}_1 y(t)}{1 + k_1\overline{u}_1 + k_2 y(t)} - dy(t),\ t \geq T_1.$$

By comparison theorem and Lemma 2.3, we obtain that there exists a $T_2 > T_1 > 0$ such that $y(t) < \dfrac{(nm - k_1 d)\overline{u}_1 - d}{k_2 d} + \varepsilon = \overline{v}_1$. Using the first equation of system (1.6), we get that $\dot{x}_m \geq \alpha^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t)$. Using the comparison theorem and Lemma 2.1, we conclude there exists a $T_3 > T_2 + \tau > 0$ such that $x_m(t) > \dfrac{\alpha e^{-\gamma\tau}}{\beta} - \varepsilon = \underline{u}_1$. Using the second equation of system (1.6), we have

$$\dot{y}(t) \quad \geq \quad \frac{nm\underline{u}_1 y(t)}{1 + k_1\underline{u}_1 + k_2 y(t)} - dy,\ t \geq T_3.$$

From the comparison theorem and Lemma 2.3, one sees that there exists a $T_4 > T_3 > 0$ such that

$$y(t) \quad > \quad \frac{(nm - k_1 d)\underline{u}_1 - d}{k_2 d} - \varepsilon = \underline{v}_1,\ t \geq T_4.$$

$$\underline{v}_1 \quad = \quad \frac{(nm - k_1 d)\underline{u}_1 - d}{k_2 d} - \varepsilon = \frac{(nm - k_1 d)(\dfrac{\alpha e^{-\gamma\tau}}{\beta} - \varepsilon) - d}{k_2 d} - \varepsilon.$$

According to the hypothesis condition, we have $\underline{v}_1 > 0$, for sufficiently small $\varepsilon > 0$. Using the first equation of system (1.6) again, we get

$$\dot{x}_m(t) \leq \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) + \frac{m\overline{v}_1 x_m(t)}{1+k_1 x_m(t)+k_2\overline{v}_1}.$$

Applying Lemma 4.1 and the comparison theorem, we obtain that there exists a $T_5 > T_4 + \tau > 0$ such that $x_m(t) < z^* + \varepsilon = \overline{u}_2$, where

$$z^* = \frac{k_1\alpha e^{-\gamma\tau} - \beta - k_2\overline{v}_1\beta + \sqrt{(k_1\alpha e^{-\gamma\tau} + \beta + k_2\overline{v}_1\beta)^2 + 4\beta k_1 m\overline{v}_1}}{2k_1\beta}.$$

By the simple computation, we get $\overline{u}_1 < \overline{u}_2$. Using the second equation of system (1.6) again, we get the following equation $\dot{y}(t) \leq \frac{nm\overline{u}_2 y(t)}{1+k_1\overline{u}_2+k_2 y(t)} - dy$ $t \geq T_5$. By using the comparison theorem and Lemma 2.3, there exists a $T_6 > T_5 > 0$ such that

$$y(t) < \frac{(nm-k_1 d)\overline{u}_2 - d}{k_2 d} + \varepsilon = \overline{v}_2,\ t \geq T_6.$$

For $\overline{u}_2 < \overline{u}_1$, we obtain $\overline{v}_2 < \overline{v}_1$. Taking $\underline{v}_1$ into the first equation of system (1.6), we find that

$$\dot{x}_m(t) \geq \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) + \frac{m\underline{v}_1 x_m(t)}{1+k_1 x(t)+k_2\underline{v}_1}.$$

Applying the comparison theorem and Lemma 4.1, we conclude there exists a $T_7 > T_6 + \tau > 0$ such that $x_m(t) > z_1^* - \varepsilon = \underline{u}_2$, where

$$z_1^* = \frac{k_1\alpha e^{-\gamma\tau} - \beta - k_2\underline{v}_1\beta + \sqrt{(k_1\alpha e^{-\gamma\tau} + \beta + k_2\underline{v}_1\beta)^2 + 4\beta k_1 m\underline{v}_1}}{2k_1\beta}.$$

Using the second equation of system (1.6), we obtain

$$\dot{y}(t) \geq \frac{nm\underline{u}_2 y(t)}{1+k_1\underline{u}_2+k_2 y(t)} - dy,\ t \geq T_7.$$

Using the comparison theorem and Lemma 2.3, there exists a $T_8 > T_7 > 0$ such that

$$y(t) > \frac{(nm-k_1 d)\underline{u}_2 - d}{k_2 d} - \varepsilon = \underline{v}_2,\ t \geq T_8.$$

It follows that

$$0 < \underline{u}_1 < \underline{u}_2 < \cdots < \underline{u}_n < x_m(t) < \overline{u}_n < \cdots < \overline{u}_2 < \overline{u}_1,$$

$$0 < \underline{v}_1 < \underline{v}_2 < \cdots < \underline{v}_n < y(t) < \overline{v}_n < \cdots < \overline{v}_2 < \overline{v}_1.$$

Hence, the limits of $\{\overline{u}_n\}_{n=1}^{\infty}$, $\{\underline{u}_n\}_{n=1}^{\infty}$, $\{\underline{v}_n\}_{n=1}^{\infty}$ and $\{\overline{v}_n\}_{n=1}^{\infty}$ exist. Putting $\overline{u} = \lim_{t\to\infty}\overline{u}_n$ $\overline{v} = \lim_{t\to\infty}\overline{v}_n$, $\underline{u} = \lim_{t\to\infty}\underline{u}_n$ and $\underline{v} = \lim_{t\to\infty}\underline{v}_n$, one see that $\overline{u} \geq \underline{u}$ and $\overline{v} \geq \underline{v}$. Next, we only need to prov $\overline{u} = \underline{u}$ and $\overline{v} = \underline{v}$. According to the relationship between u_n and v_n, we see that

$$\overline{v}_n - \underline{v}_n = \frac{(nm-k_1 d)(\overline{u}_n - \underline{u}_n)}{k_2 d} + 2\varepsilon.$$

From the definitions of $\overline{u}_n$ and $\underline{u}_n$, we find that

$$\begin{aligned}
\overline{u}_{n+1}-\underline{u}_{n+1} &= \frac{k_1\alpha e^{-\gamma\tau}-\beta-k_2\overline{v}_n\beta+\sqrt{(k_1\alpha e^{-\gamma\tau}+\beta+k_2\overline{v}_n\beta)^2+4\beta k_1 m\overline{v}_n}}{2k_1\beta} \\
&\quad -\frac{k_1\alpha e^{-\gamma\tau}-\beta-k_2\underline{v}_n\beta+\sqrt{(k_1\alpha e^{-\gamma\tau}+\beta+k_2\underline{v}_n\beta)^2+4\beta k_1 m\underline{v}_n}}{2k_1\beta}+2\varepsilon \\
&= \frac{k_2(\underline{v}_n-\overline{v}_n)}{2k_1}+\frac{\sqrt{(k_1\alpha e^{-\gamma\tau}+\beta+k_2\overline{v}_n\beta)^2+4\beta k_1 m\overline{v}_n}}{2k_1\beta} \\
&\quad -\frac{\sqrt{(k_1\alpha e^{-\gamma\tau}+\beta+k_2\underline{v}_n\beta)^2+4\beta k_1 m\underline{v}_n}}{2k_1\beta}+2\varepsilon \\
&< \frac{k_2(\underline{v}_n-\overline{v}_n)}{2k_1}+\frac{4\beta k_1 m(\overline{v}_n-\underline{v}_n)}{2\beta k_1(2k_1\alpha e^{-\gamma\tau}+2\beta+k_2(\overline{v}_n+\underline{v}_n)\beta)} \\
&\quad +\frac{[2k_1\alpha e^{-\gamma\tau}+2\beta+k_2(\overline{v}_n+\underline{v}_n)\beta](\overline{v}_n-\underline{v}_n)\beta k_2}{2\beta k_1(2k_1\alpha e^{-\gamma\tau}+2\beta+k_2(\overline{v}_n+\underline{v}_n)\beta)}+2\varepsilon \\
&< \frac{2m(\overline{v}_n-\underline{v}_n)}{(2k_1\alpha e^{-\gamma\tau}+2\beta+k_2(\overline{v}_n+\underline{v}_n)\beta)} \\
&< \frac{m(\overline{v}_n-\underline{v}_n)}{\beta}+2\varepsilon.
\end{aligned}$$

Taking the limit on both sides of the inequality and letting $n\to\infty$, we obtain that

$$\overline{u}-\underline{u} \leq \frac{m(nm-k_1d)(\overline{u}-\underline{u})}{k_2d\beta}+\frac{2m\varepsilon}{\beta}+2\varepsilon,$$

$$\left(1-\frac{m(nm-k_1d)}{k_2d\beta}\right)(\overline{u}-\underline{u}) \leq \frac{2m\varepsilon}{\beta}+2\varepsilon.$$

In view of $1-\frac{m(nm-k_1d)}{k_2d\beta}>0$, we have $\overline{u}=\underline{u}$, and $\overline{v}=\underline{v}$. If $1-\frac{m(nm-k_1d)}{k_2d\beta}>0$, then equilibrium point P_2 is globally attractive. This completes the proof.

5. Numerical simulations

The following examples demonstrate the feasibility of main results.

Example 5.1. Consider the following system

$$\begin{aligned}
\dot{x}_m(t) &= 6e^{(-2.5)}x_m(t-5)-x_m^2(t)+\frac{x_m(t)y(t)}{1+x_m(t)+6y(t)}, \\
\dot{y}(t) &= \frac{0.8x_m(t)y(t)}{1+x_m(t)+6y(t)}-y(t),
\end{aligned}$$

where $\alpha=6$, $\gamma=0.5$, $\beta=1$, $k_1=1$, $d=1$, $k_2=6$, $\tau=5$, $m=1$, $n=0.8$. By computation, one sees that $nm-k_1d=0.8*1-1*1=-0.2<0$, $\alpha e^{-\gamma\tau}(mn-k_1d)-\beta d=6*e^{(-2.5)}(-0.2)-1*1<0$, which satisfies the conditions of Theorem 4.2.

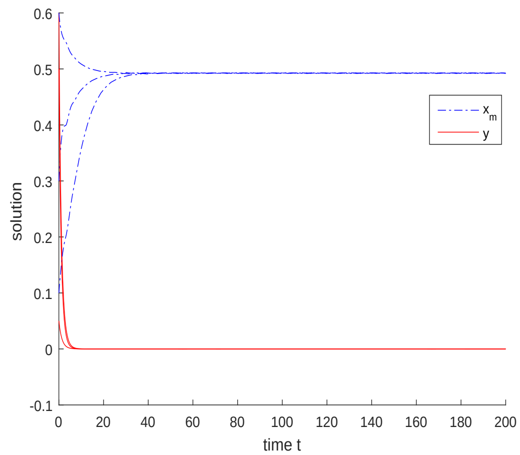

FIGURE 1. The boundary equilibrium $P_1(\dfrac{\alpha e^{-\gamma\tau}}{\beta},0)$ of system (1.6) with the initial conditions $(x_m(\theta),y(0))$=$(0.1,0.5),(0.6,0.05)$ and $(0.3,0.6)$ for $-5<\theta<0$ is globally asymptotically stable.

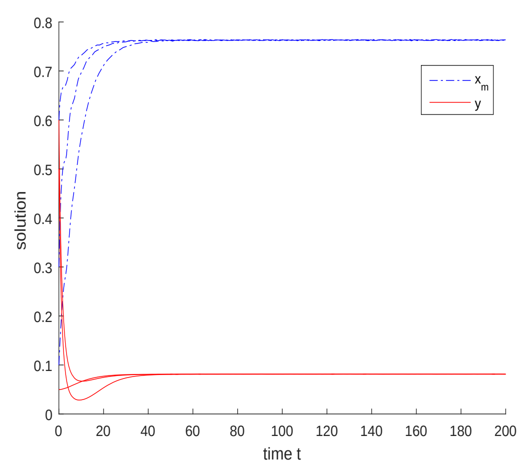

FIGURE 2. The unique positive equilibrium $P_2(x_m^*,y^*)$ of system (1.1) with the initial conditions $(x_m(\theta),y(0))$=$(1.1,0.25),(0.6,0.05)$, and $(0.5,0.01)$ for $-5<\theta<0$ is globally asymptotically stable.

Example 5.2. Consider the following system

$$\begin{aligned}\dot{x}_m(t) &= 10e^{(-2.5)}x_m(t-5) - 1.2x_m^2(t) + \frac{3x_m(t)y(t)}{1+x_m(t)+10y(t)},\\ \dot{y}(t) &= 3\frac{0.9x_m(t)y(t)}{1+x_m(t)+10y(t)} - 0.8y(t),\end{aligned}$$

where $\alpha = 10, \gamma = 0.5, \beta = 1.2, k_1 = 1, d = 0.8, k_2 = 10, \tau = 5, m = 3, n = 0.9$. By computation, one sees that $(mn - k_1 d)\frac{\alpha e^{-\gamma\tau}}{\beta} - d = \frac{(2.7-0.8)10 * e^{(-2.5)}}{1.2} - 0.8 = 0.499 > 0$ and $1 - \frac{m(nm-k_1 d)}{k_2 d\beta} = 1 - \frac{3*1.9}{10*0.8*1.2} = 0.406 > 0$, which implies the above values satisfy the conditions of the Theorem 4.3.

6. Conclusion

In this paper, we study plant-pollinator system (1.6) of the Beddington-DeAngelis type functional response with stage structure on plant, which is an extension of plant-pollination system analyzed by Wang, Wu and Sun [11]. We give the sufficient conditions for the permanence and extinction of system (1.6) and the sufficient conditions for the global stability of the coexistence equilibrium. We also proved that system (1.6) is permanent if condition $\frac{\alpha mne^{-\gamma\tau}}{k_1\alpha e^{-}\gamma\tau+\beta} > d$ holds. Assumed that system (1.6) satisfies $\frac{mn}{k_1} < d$. Then $\lim_{t\to\infty}(x_m(t), y(t)) = (\frac{\alpha e^{-\gamma\tau}}{\beta}, 0)$. Our conclusions show that the pollinator coexists with plant permanently if pollinators recruitment rate at the peak of plant abundance is larger than its mortality rate and that the pollinator goes extinct if pollinator highest recruitment rate is less than its mortality rate. Furthermore, these conditions show that the mutual interference concerning pollinator k_2 does not have an impact on the permanence and the extinction of system (1.6). But a sufficient large the mortality rate of the immature plant can destroy the permanent condition make the pollinator become extinction. Next, we will further study the impact of pollinator interference k_2 on the system. In view of the condition $\frac{m(nm-k_1 d)}{d\beta} < k_2$ of Theorem 4.3, we get interesting conclusions when the positive equilibrium P_2 is unstable. A sufficient large k_2 can drive the system into a globally asymptotically stable and the increase of structure in the plant population does not change the qualitative dynamics of system (1.6).

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REFERENCES

[1] J. R. Beddington, Mutual interference between parasites or predator and its effect on searching efficiency, J. Animal Ecology 44 (1975), 331-340.

[2] D. DeAngelis, R. Goldstein, A model for tropic interaction, Ecology 56 (1975), 881-892.

[3] S. Yu, Global stability of a modified Leslie-Gower model with Beddington-DeAngelis functional response, Adv. Difference Equ. 2014 (2014), Article ID 84.

[4] S. Liu, E. Beretta, A stage-structured predator-prey model of Beddington-DeAngelis type, SIAM J. Appl. Math. 66 (2006), 1101-1129.

[5] S. Liu, E. Beretta, Predator-prey model of Beddington-DeAngelis type with maturation and gestation delays, Nonlinear Anal. Real World Appl. 11 (2010), 4072-4091.

[6] F. Chen, X. Chen, S. Huang, Extinction of a two species non-autonomous competitive system with Beddington-DeAngelis functional response and the effect of toxic substances, Open Math. 14 (2016), 1157-1173.

[7] R. Han, X. Xie, F. Chen, Permanence and global attractivity of a discrete pollination mutualism in plant-pollinator system with feedback controls, Adv. Difference Equ. 2016 (2016), Article ID 199.

[8] F. Chen, Y. Chen, J. Shi, Stability of the boundary solution of a nonautonomous predator-prey system with the Beddington-DeAngel is functional response, J. Math. Anal. Appl. 344 (2008), 1057-1067.

[9] H. Xiao, Positive equilibrium and its stability of the Beddington-DeAngelis type predator-prey dynamical system, Appl. Math. J. Chinese Univ. 21 (2006),429-436.

[10] F. Chen, M. You, Permanence, extinction and periodic solution of the predator-prey system with Beddington-DeAngelis functional response and stage structure for prey, Nonlinear Anal. Real World Appl. 9 (2008), 207-221.

[11] Y. Wang, H. Wu, S. Sun, Persistence of pollination mutualisms in plant-pollination-robber system, Theor. Population Biolo. 81 (2012), 243-250.

[12] Z. Li, M. Han, F. Chen, Global stability of stage-structured predator-prey model with modified Leslie-Gower and Holling-type II schemes, Int. J. Biomath. 5 (2012), Article ID 1250057.

[13] S. Liu, L. Chen, G. Luo, Y. Jiang, Asymptotic behaviors of competitive Lotka-Volterra system with stage structure, J. Math. Anal. Appl. 271 (2002), 124-138.

[14] S. Liu, L. Chen, Z. Liu, Extinction and permanence in nonautonomous competitive system with stage structure, J. Math. Anal. Appl. 274 (2002), 667-684.

[15] F. Chen, W. Chen, Y. Wu, Z. Ma, Permanence of a stage-structured predator-prey system, Appl. Math. Comput. 219 (2013), 8856-8862.

[16] Y. Kuang, Delay Differential Equations: with Applications in Population Dynamics, Academic Press, 1993.

[17] K. Gopalsamy, Stability and Oscillations in Delay Differential Equation of Population Dynamics, Springer Netherlands, 1992.