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HEAT KERNEL ESTIMATES AND ASYMPTOTIC OF $\mathscr{W}$-ENTROPY ON STOCHASTICALLY COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we establish heat kernel upper bounds via ultracontractive estimates for heat diffusion semigroups on an n-dimensional complete Riemannian manifold M. This result is extended, via monotonicity property of the $\mathscr{W}$-entropy functional, to the case when M is stochastically complete. We also prove the large time asymptotic of the entropy for the stochastic complete heat kernel. This provides an alternative proof for Ni's large time asymptotic of the entropy.

Keywords. Heat kernel; Monotonicity; Entropy functional; Ultracontractive estimates.

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1. Introduction

Let (M,g) be an n-dimensional complete Riemannian manifold (compact or noncompact) equipped with Riemannian measure $d\mu$. Let $\Delta = \Delta_g$ be the Laplace-Beltrami operator on M

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(g^{ij}\sqrt{\det g}\frac{\partial}{\partial x^j}\Big),$$

where $\det g$ and $g^{ij} = (g_{ij})^{-1}$ are the determinant and inverse of metric matrix g respectively. We denote by $H(t;x,y)$ the heat diffusion kernel on M such that a C^∞-function $u(t,x) = H(t;x,y)$

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is the unique positive minimal solution to the heat equation

$$\begin{cases} \Big(\frac{\partial}{\partial t} - \Delta_g\Big)u(t,x) = 0, \quad t>0, x\in M, \\ \lim_{t\to 0} u(t,x) = \delta_y(x). \end{cases} \tag{1}$$

In the case $M = \mathbb{R}^n$, the (Gauss-Weierstrass) heat kernel admits an explicit value

$$H_{\mathbb{R}^n}(t;x,y) = (4\pi t)^{-\frac{n}{2}} \exp(-|x-y|^2/4t), \;\; t>0. \tag{2}$$

Meanwhile, there is no explicit formula for the heat kernel on M in general but by the asymptotic property we can write as $t\to 0$ [1, Theorem 2.1]

$$H(t;x,y) - (4\pi t)^{-\frac{n}{2}} \exp\Big(-\frac{d^2(x,y)}{4t}\Big)\sum_{j=0}^{k} t^j u(t,x,y) = w_k(t,x,y)$$

with

$$w_k(t,x,y) = O(t^{k+1-\frac{n}{2}} \exp(-\delta|x-y|^2/4t)),$$

where $d(x,y)$ is the distance function on M and $\delta = \delta(n) > 0$ as $t\to 0$ uniformly for all x,y. The function $u(t,x,y)$ can be chosen so that $u(0,x,x) = 1$. In particular, the result of Cheeger and Yau [2] implies that if (M,g) is a complete Riemannian manifold with nonnegative Ricci curvature, then

$$H(t;x,y)) \geq (4\pi t)^{-\frac{n}{2}} \exp\Big(-\frac{d^2(x,y)}{4t}\Big). \tag{3}$$

The first part of this paper (Section 3) proves ultracontractive estimates

$$\|e^{-t\Delta}f\|_{L^\infty(M)} \leq \|f\|_{L^1(M)}, \;\; \forall f\in L^1(M),$$

where $e^{-t\Delta}$ is the semigroup generated by $-\Delta$. Recall that it can be shown that if $f\in Dom(-\Delta)\subset L^2(M,d\mu)$, then $e^{-t\Delta}f(x)\in Dom(-\Delta)$ and that

$$f(t,x) = e^{-t\Delta}f(x) = \int_M H(t;x,y)f(y)d\mu(y) \tag{4}$$

solves the heat equation (1). Precisely, Theorem 2.8 proves that if M admits logarithmic Sobolev inequality, then upper heat kernel bound

$$H(t;x,y) \leq C(n)t^{-\frac{n}{2}}, \;\; \forall\, x,y\in M, \;\; t>0 \tag{5}$$

holds with sharp constant depending on optimal constant in the Euclidean log-Sobolev inequality

$$K(n,2)=\sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}},$$

where $\omega_n=\pi^{n/2}/\Gamma(n/2+1)$, the volume of Euclidean unit ball. For details on ultracontractivity and its equivalence to the Sobolev inequalities, see, for examples, [3, 4] and the references therein.

In the second part (Section 4) we prove that at large time $\mathscr{W}(H,t)$-entropy

$$\mathscr{W}(H,t)=\int_M (t|\nabla f|^2+f-n)H d\mu, \quad t>0,$$

for heat kernel H converges to $\log(\theta/\omega_n)$, $\theta>0$, on a stochatically complete manifold M with nonnegative Ricci curvature and maximum volume growth property. This part provides alternative proof for Ni's large time asymptotics for $\mathscr{W}$ in [5]. Ni's results [6] say that $\mathscr{W}(H,t)$ is monotonically decreasing on nonnegative Ricci curvature manifold and moreover, for some $t>0$, $\mathscr{W}(H,t)\geq 0$, if and only if M is isometric to $\mathbb{R}^n$. By the monotonicity of $\mathscr{W}$, one can indeed obtain many useful geometric and topological consequences. For examples, the monotonicity of $\mathscr{W}$ yields the usual L^2-Sobolev inequality, ultracontractivity and isomperimetric inequality. It also implies that M has finite fundamental group. In fact, Ni [7] can show that M is of maximum volume growth if and only if the entropy is uniformly bounded for heat kernel H and $\forall\, t>0$.

The next section first introduces the background materials needed and then states the main theorems that will be proved in sections 3 and 4.

2. Preliminaries and main results

Now we give some definitions to explain certain geometric quantities appearing in the results that will follow. Some of these definitions are linked with Bishop volume comparison theorem when the Ricci curvature of M is bounded from below by $(n-1)k$, $k\geq 0$.

Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. For a point $p\in M$, denote $Vol(B_r(p))$ the volume of the open geodesic ball in M with centre p and radius r. Observe that $Vol(B_r(p))$ is positive and finite for all $x\in M$ and $r>0$ since M is complete.

Definition 2.1 (Properties of heat kernel). *The heat kernel of M is a C^∞-positive function $H(t,x,y)$ on $(0,\infty)\times M\times M$, such that*

$$e^{-t\Delta}f(x)=\int_M H(t;x,y)f(y)d\mu(y),\ \ f\in C^\infty(M)$$

and

$$(\partial_t-\Delta)H(t;x,\cdot)=0,\ \ \lim_{t\mapsto 0}H(t;x,y)=\delta_y(x)$$

in the sense of distribution, where $\delta_y(\cdot)$ is the dirac mass concentrated at y and $\int_M H(t;x,y)dv\leq 1$.

The last inequality shows that the heat kernel semigroup is contractive on L^p for any $1\leq p\leq \infty$. Also by semigroup and symmetry properties we have

$$H(t_1+t_2;x,y)=\int_M H(t_1;x,z)H(t_2;y,z)d\mu(z)$$
$$H(t;x,y)=H(t;y,x),\forall x,y\in M,\ t>0.$$

By the condition $\int_M H(t;x,y)d\mu=1$ we have taken H to be the limit of a sequence of Neumann heat kernel, i.e., if $H_k(t;x,y)$ is the fundamental solution to the heat equation on relatively compact sub-domain $\Omega_k,\ k=1,2,...,$ with smooth boundary in M, then H is the corresponding limit of H_k. The books [8] by Grigor'yan and [9] by Li provide detail account of this.

Definition 2.2 (Stochastic Completeness). *A Riemannian manifold is said to be stochastically complete if*

$$\int_M H(t,x,y)d\mu(y)=1$$

for $x\in M$ and $t>0$.

It has been shown that even a geodesically complete manifold may not be stochastically complete.

Theorem 2.3 (Grigor'yan [10]). *Let M be a geodesically complete Riemannian manifold. Assume that for some point $x\in M$*

$$\int_0^\infty \frac{rdr}{\ln Vol(B_R(x))}.$$

Then M is stochastically complete.

In particular, a geodesically complete manifold with bounded below Ricci curvatures is s-tochastically complete.

Definition 2.4. *We say that M satisfies volume doubling property if there exists a constant $\eta > 0$ such that*

$$Vol(B_{2r}(p)) \leq \eta Vol(B_r(p)), \qquad \forall\, p \in M,\ r > 0.$$

Here η is no less than 2^n and doubling constant 2^n implies M is isometric to $\mathbb{R}^n$.

Definition 2.5 (Maximum volume growth). *Define*

$$\theta_M := \liminf_{r\to\infty} \frac{Vol(B_r(p))}{r^n \omega_n}, \quad \forall r > 0 \tag{6}$$

as the asymptotic volume growth of M. Then M is said to have maximum volume growth property when $\theta_M > 0$, i.e., $\lim_{r\to\infty} \frac{Vol(B_r(p))}{r^n} > 0,\ \forall r > 0.$

Note that the constant θ_M is a global invariant of M, that is, it is independent of the base point p. It is also easy to see that whenever M has maximum volume growth

$$Vol(B_r(p)) \geq \theta_M \omega_n r^n, \quad \forall\, p \in M, r > 0.$$

By Bishop volume comparison theorem, $\theta_M = 1$ implies that M is isometric to $\mathbb{R}^n$.

Definition 2.6 ($\mathscr{W}$-entropy functional). [5, 6, 7] *On a complete Riemannian manifold, $\mathscr{W}$ is defined by*

$$\mathscr{W}(g, f, \sigma(t)) = \int_M (\sigma|\nabla f|^2 + f - n)(4\pi\sigma)^{-n/2} e^{-f} d\mu$$

restricted to function $f = f(\sigma(t), x)$ satisfying

$$\int_M (4\pi\sigma)^{-n/2} e^{-f} d\mu = 1 \text{ and } u(t,x) = (4\pi t)^{-n/2} e^{-f}$$

is a positive solution to the heat equation.

Using the asymptotic of Euclidean heat kernel, Ni [5] has found the large time asymptotic of $\mathscr{W}$

$$\lim_{t\to\infty} \mathscr{W} = \lim_{\rho\to\infty} \frac{Vol(B_\rho(x))}{\omega\rho^n}, \quad \forall \rho > 0,$$

where $B_\rho(x)$ is a ball of radius ρ centred at x and $Vol(B_\rho(x))$ is its volume. He used the sharp pointwise bounds for the heat kernel proved by Li, Tam and Wang [11], which is closely related to the large time behavior of heat kernel by Li in [12] (also see [13]). Perelman [14] discovered $\mathscr{W}$-entropy functional originally for the ancient solution to the Ricci flow with bounded nonnegative curvature operator. He claimed that the entropy is uniformly bounded for any fundamental solution to the conjugate heat equation if and only if an ancient solution to the Ricci flow is κ-noncollapsed on manifold with nonnegative curvature operators.

Main results. We first state logarithmic Sobolev inequality [15] on M with the constant depending on the optimal Euclidean constant (i.e., $K(n,2)^2 \sim 2/\pi en$) as

$$\int_M |f|^2 \log |f|^2 d\mu \leq \frac{\varepsilon^2}{\pi} \int_M |\nabla f|^2 d\mu - n(1+\log\varepsilon)\|f\|_2^2, \tag{7}$$

where $\varepsilon > 0,\ f \in C_0^\infty(M)$ and $\int_M f d\mu = 1$.

Our first results is stated as follows:

Theorem 2.7. *Let M be a complete Riemannian manifold of dimension n. Let $f(t,x)$ be a positive solution to the heat equation satisfying (4) with the initial data $f(x) \in L^2(M)$. Suppose M admits logarithmic Sobolev inequality (7). Then the following estimates*

$$\|f(T,\cdot)\|_\infty \leq (4\pi T)^{-\frac{n}{2}} \|f(0,\cdot)\|_1 \tag{8}$$

hold. Moreover, we have the following upper estimates for the heat kernel with sharp constant

$$H(t;x,y) \leq C(n) t^{-\frac{n}{2}}, \ \ \forall\, x,y \in M,\ t > 0. \tag{9}$$

Our approach to this theorem follows from a classical idea of finding time derivative of L^p-norm, where $p = p(t)$ is a function of time. In general L^p-norm is continuously Fréchet differentiable on $L^p\backslash\{0\}$. By chain rule of differentiation, for instance, one has for $h \in L^q(M)$ since $\|h\|_p$ is a differentiable function of time, ($p \leq q$) that

$$\frac{d}{dp}\|h\|_p = p^{-1}\|h\|_p^{1-p}\Big\{|h|_p \ln|h| dv - \|h\|_p^p \ln\|h\|_p\Big\}. \tag{10}$$

(See [16, Lemma 1.1] and Weissler [15, Proposition 1]).

We also prove a similar bound on heat kernel of a stochastically complete Riemannian manifold. This we do via the monotonicity of $\mathscr{W}$-entropy, namely,

Theorem 2.8. *Let M be a stochastically complete Riemannian manifold of dimension n and nonnegative Ricci curvature. Let $f(t,x)$ be a positive solution to the heat equation. Then the estimate*

$$H(t;x,y) \leq e^{\mathscr{B}} (4\pi t)^{-\frac{n}{2}} \tag{11}$$

holds, where $\mathscr{B} = \mathscr{B}(\theta_M, n) > 0$ and θ_M is defined bythe maximum volume growth rate (6).

Lastly, we state the result on large time convergence of the $\mathscr{W}$-entropy that will be proved in the last section.

Theorem 2.9. *Let M be a stochastically complete Riemannian manifold with nonnegativee Ricci curvatures. Suppose M has maximum volume growth property (20). Then*

$$\lim_{t\to\infty}\mathscr{W}(H,t)=\log\frac{\theta}{\omega_n}, \tag{12}$$

where $H=H(t;x,y)$ is the heat kernel satisfying the condition $\int_M H(t;x,y)d\mu=1$.

3. Ultracontractivity for heat diffusion semigroups

It is well known that $-\Delta$ is an unbounded self-adjoint operator on $L^2(M,d\mu)$ and associated with a self-adjoint diffusion semigroup of operator $e^{-t\Delta}$, $t>0$ on $L^2(M,d\mu)$; see, for detail, Davies [3] and Grigor'yan [8] and the references therein. Recall that $P_t=e^{-t\Delta}$ is positivity preserving and has some qualitative smoothing effect. It is a contraction on $L^1(M)$ and $L^\infty(M)$ for all $t>0$. Hence, by Riez-Thorin interpolation theorem, it is bounded on $L^p(M)$, for any $p\in[1,\infty]$. The aim of this section therefore is to prove estimates of the form

$$\|e^{-t\Delta}f\|_\infty\le \; Const.\,\|f\|_1, \quad \forall\, f\in L^1(M). \tag{13}$$

This type of estimate is popularly called ultracontractive estimate for heat kernel. Note that this type of estimate is stronger than hypercontractive estimates of Nelson [17]. Nelson hypercontractive estimates say that for all $f\in L^q(M)$

$$\|e^{-t\Delta}f\|_p\le\|f\|_q, \quad t>0, \tag{14}$$

where $1<q<p<\infty$. Gross [16] has proved that the above estimates are equivalent to logarithmic Sobolev inequality; see Davies [3], Davies and Simon [4], Weissler [15].

Proof of Theorem 2.7. Set $\mathscr{A}=\|f\|_{p(t)}^{p(t)}$, where $\|f\|_{p(t)}=(\int_M|f|^{p(t)}\,d\mu)^{\frac{1}{p(t)}}$ and define a logarithmic entropy

$$\mathscr{F}(t)=\ln\|f\|_{p(t)}=\frac{1}{p(t)}\ln\int_M|f|^{p(t)}\,d\mu,$$

where $p(t):[0,T]\to[1,\infty]$ is a continuously differentiable increasing function such that $p(0)=1$ and $p(T)=\lim_{t\to T}p(t)=\infty$ for $T>0$. Now taking time derivative of $\mathscr{F}(t)$ and applying integration by parts formula, we have

$$\begin{aligned}
\frac{d}{dt}\mathscr{F}(t) &= -\dot{p}(t)p^{-2}(t)\ln\|f\|_{p(t)}^{p(t)} + \Big(p(t)\int_M |f|^{p(t)}d\mu\Big)^{-1}\frac{\partial}{\partial t}\Big(\int_M |f|^{p(t)}\,d\mu\Big)\\
&= -\dot{p}(t)p^{-2}(t)\ln\mathscr{A} + \Big(p(t)\mathscr{A}\Big)^{-1}\Big(\dot{p}(t)\int_M |f|^{p(t)}\log|f|\,d\mu + p(t)\int_M |f|^{p(t)-1}\frac{\partial}{\partial t}fd\mu\Big)\\
&= -\dot{p}(t)p^{-2}(t)\ln\mathscr{A} + \Big(p(t)\mathscr{A}\Big)^{-1}\dot{p}(t)\int_M |f|^{p(t)}\log|f|\,d\mu + \mathscr{A}^{-1}\int_M |f|^{p(t)-1}\Delta f d\mu\\
&= \dot{p}(t)p^{-2}(t)\Big\{p(t)\mathscr{A}^{-1}\int_M |f|^{p(t)}\ln|f|\,d\mu - \ln\mathscr{A}\Big\} - (p(t)-1)\mathscr{A}^{-1}\int_M |f|^{p(t)-2}|\nabla f|^2\,d\mu.
\end{aligned}$$

We now make a change of variable $v^2 = \mathscr{A}^{-1}|f|^{p(t)}$ with $\|v\|_2 = 1$. Note that by this we have

$$4|\nabla v|^2 = p^2(t)\mathscr{A}^{-1}|f|^{p(t)-2}|\nabla f|^2$$

and

$$\int_M v^2\ln v^2 d\mu = p(t)\mathscr{A}^{-1}\int_M |f|^{p(t)}\ln|f|d\mu - \ln\|f\|_{p(t)}^{p(t)}.$$

Hence

$$\frac{d}{dt}\mathscr{F}(t) = \dot{p}(t)p^{-2}(t)\Big\{\int_M v^2\ln v^2 d\mu - \frac{4(p(t)-1)}{\dot{p}(t)}\int_M |\nabla v|^2\,d\mu\Big\}. \tag{15}$$

Comparing the RHS with Log Sobolev inequality (7), we can choose $\varepsilon^2 = \frac{4\pi(p-1)}{\dot{p}} > 0$ and obtain

$$\frac{d}{dt}\mathscr{F}(t) \le \frac{\dot{p}(t)}{p^2(t)}\Big\{-n\Big[\ln\Big(\frac{4\pi(p(t)-1)}{\dot{p}(t)}\Big)^{\frac{1}{2}} + 1\Big]\Big\}.$$

With the choice of $p(t) = T/(T-t)$ and by integrating with respect to t from 0 to T, we obtain

$$\ln\frac{\|f(T,\cdot)\|_{p(T)}}{\|f(0,\cdot)\|_{p(0)}} \le -\frac{n}{2T}\int_0^T \log\frac{4\pi t(T-t)}{T}dt - n = \frac{n}{2}\log(4\pi T),$$

which implies $\|f(T,\cdot)\|_\infty \le (4\pi T)^{-\frac{n}{2}}\|f(0,\cdot)\|_1$. Since $f(T,x) = \int_M H(T;x,y)f(T,y)d\mu(y)$, one has $H(T;x,y) \le C'T^{-\frac{n}{2}}$. This completes the proof. □

Remark 3.1. Let $f(t,x)$ be a positive solution to the heat equation, then we write

$$f(t,x) = \int_M H(t;x,y)f(0,y)d\mu(y)$$

and

$$\sup_{f\neq 0}\frac{\|f(t,x)\|_\infty}{\|f(0,x)\|_1} = \sup_{x,y\in M} H(t;x,y),$$

which is equivalent to the estimates that we derived.

Remark 3.2. In disguise, C' depends on the optimal constant $K(n,2)$ in the Euclidean space since it can be calculated in form of π and π in turn can be computed in terms of $K(n,2)$; see the book [18] by Lieb and Loss for another way to obtain sharp constant by using sharp Young's inequality.

We will quickly show a similar bound on heat kernel of a stochastically complete Riemannian manifold (Theorem 2.8). This we do via the monotonicity of $\mathscr{W}$-entropy. We need the following Proposition in the proof.

Proposition 3.3. ([6, Proposition 4.2]) *Let M be a complete Riemannian manifold of dimension n and nonnegative Ricci curvature. M has maximum volume growth if and only if there exists $\mathscr{B} = \mathscr{B}(\theta_M, n) > 0$ such that $\mathscr{W}(H,t) \geq -\mathscr{B}(\theta_M, n)$, for $t > 0$.*

Proof of Theorem 2.8. We follow the argument of the last theorem up to (15).

Set $v^2 = (4\pi t)^{-n/2} e^{-\widetilde{f}(t,x,y)} := H(t,x,y)$ the heat kernel. By stochastic completeness $\int_M v^2 d\mu = 1$ and

$$\int_M (v^2 \ln v^2 - 4t|\nabla v|^2) d\mu = -\mathscr{W}(g,t,\widetilde{f}) - n - \frac{n}{2}\ln(4\pi t),$$

one find from (15) that

$$\frac{d}{dt}\mathscr{F}(t) = \dot{p}(t)p^{-2}(t)\Big\{-\mathscr{W}(g, \frac{p-1}{\dot{p}}, v) - n - \frac{n}{2}\ln\Big(4\pi\frac{p-1}{\dot{p}}\Big)\Big\} \tag{16}$$

$$\leq T^{-1}\Big\{B(\theta_M,n) - n - \frac{n}{2}\ln\Big(4\pi\frac{(T-t)t}{T}\Big)\Big\}. \tag{17}$$

Since M has maximum volume growth, we have combined the monotonicity property of $\mathscr{W}$ and Proposition 3.3 to obtain the last inequality. Integrating with respect to t from 0 to T, we obtain

$$\ln\frac{\|f(T,\cdot)\|_\infty}{\|f(0,\cdot)\|_1} \leq \mathscr{B}(\theta_M,n) - n - \frac{n}{2}(\log(4\pi T - 2),$$

which implies

$$\|f(T,\cdot)\|_\infty \leq e^{\mathscr{B}(\theta_M,n)}(4\pi T)^{-\frac{n}{2}}\|f(0,\cdot)\|_1.$$

This completes the proof. □

4. Large time asymptotic of the entropy

Let $H(t;x,y)$ be the heat kernel on M for fixed $y \in M$, we defined Perelman's entropy [14] by

$$\mathscr{W}(H,t) = \int_M (t|\nabla f|^2 + f - n)H d\mu, \quad t > 0, \tag{18}$$

where $\int_M H d\mu = 1 = \int_M (4\pi t)^{-\frac{n}{2}} e^{-f} d\mu$. Ni's results [6] say that $\mathscr{W}(H,t)$ is monotonically decreasing on nonnegative Ricci curvature manifold, precisely

$$\frac{d\mathscr{W}(H,t)}{dt} = -2t \int_M \Big(\Big| \nabla\nabla \log H - \frac{1}{2t} g \Big|^2 + Rc\Big(\nabla \log H, \nabla \log H\Big)\Big) H d\mu. \tag{19}$$

Moreover, for some $t > 0$, $\mathscr{W}(H,t) \geq 0$, if and only if M is isometric to $\mathbb{R}^n$.

In this section, we want to prove that at large time the $\mathscr{W}$-entropy converges to $\log(\theta/\omega_n)$, where $\theta > 0$, on a stochastically complete manifold having maximum volume growth. In fact, this is a consequence of Li's large time asymptotics for heat kernel [12] and provides alternative proof for Ni's large time asymptotics for $\mathscr{W}$ [6]. Recall that a complete Riemannian manifold with nonnegative Ricci curvature is said to have maximum volume growth property if

$$\lim_{\rho\to\infty} \frac{Vol(B_\rho(x))}{\rho^n} = \theta > 0, \quad \forall\, x \in M,\ \rho > 0 \tag{20}$$

Proof of Theorem 2.9. In [12, Theorem 1], Li has proved that

$$\lim_{t\to\infty} Vol(B_{\sqrt{t}}(x)) H(t;x,y) = \omega_n (4\pi)^{-\frac{n}{2}}, \tag{21}$$

from where we observe by using the volume growth condition (20) that

$$\lim_{t\to\infty} t^{n/2} H(t;x,y) = \frac{\omega_n}{\theta}(4\pi)^{-\frac{n}{2}} \ \text{ and } \ H(t;x,y) \leq \frac{\omega_n}{\theta}(4\pi t)^{-\frac{n}{2}} \exp\Big(-\frac{d^2}{4t}\Big).$$

Now, we consider

$$\begin{aligned} \lim_{t\to\infty} \log(t^{n/2} H(t;x,y)) &= \lim_{t\to\infty} \Big\{ \log H(t;x,y) + \frac{n}{2}\log t \Big\} \\ &= \log \frac{\omega_n}{\theta} - \frac{n}{2}\log(4\pi). \end{aligned} \tag{22}$$

We now compute $\lim_{t\to\infty} \mathscr{W}(H,t)$. Notice that

$$\mathscr{W}(H,t) = \int_M \Big(t\frac{|\nabla H|^2}{H} - H \log H \Big) d\mu - \Big(\frac{n}{2}\log(4\pi t) + n\Big), \quad \int_M H d\mu = 1.$$

By the Li-Yau Harnack inequality [19] which says

$$\frac{|\nabla H|^2}{H^2} - \frac{\Delta H}{H} - \frac{n}{2t} \leq 0, \ \ \forall x \in M, t > 0,$$

we have

$$\int_M t \frac{|\nabla H|^2}{H} d\mu \leq \frac{n}{2}.$$

This follows from the fact that $\int_M \Delta H dv = 0$. Then

$$\begin{aligned}\mathscr{W} &\leq -\int_M H\log H d\mu - \Big(\frac{n}{2}+\frac{n}{2}\log(4\pi t)\Big)\\ &= -\int_M \Big(H\log H + \frac{n}{2}\log t\Big)d\mu - \Big(\frac{n}{2}+\frac{n}{2}\log(4\pi)\Big).\end{aligned}$$

Taking the limit and using (22), we obtain

$$\begin{aligned}\lim_{t\to\infty}\mathscr{W} &\leq -\lim_{t\to\infty}\Big\{\log H + \frac{n}{2}\log t\Big\}\int_M H d\mu - \Big(\frac{n}{2}+\frac{n}{2}\log(4\pi)\Big)\\ &= -\Big\{\log\frac{\omega_n}{\theta} - \frac{n}{2}\log(4\pi)\Big\} - \frac{n}{2}\log(4\pi e),\end{aligned}$$

which implies

$$\lim_{t\to\infty}\mathscr{W} \leq \log\frac{\theta}{\omega_n}. \tag{23}$$

On the other hand, we know from [6] that the monotonicity of $\mathscr{W}(H,t)$ implies $\lim_{t\mapsto\infty}\mathscr{W}(H,t) = \lim_{t\mapsto\infty}\mathscr{N}(H,t)$, where $\mathscr{N}(H,t)$ is the so called normalized Nash entropy defined by

$$\mathscr{N}(H,t) = -\int_M H\log H d\mu - \frac{n}{2}\log(4\pi t) - \frac{n}{2}. \tag{24}$$

Now using the following upper and lower bounds for H

$$H \leq \frac{\omega_n}{\theta}(4\pi t)^{-\frac{n}{2}}\exp\Big(-\frac{d^2(x,y)}{4t}\Big) \text{ and } H \geq (4\pi t)^{-\frac{n}{2}}\exp\Big(-\frac{d^2(x,y)}{4t}\Big)$$

in (24), one has

$$\begin{aligned}\mathscr{N} &\geq -\int_M H\log\frac{\omega_n}{\theta}d\mu + \int_M H\Big(\frac{d^2}{4t}\Big)d\mu - \frac{n}{2}\\ &\geq \log\frac{\theta}{\omega_n} + (4\pi t)^{-\frac{n}{2}}\int_M \frac{d^2}{4t}\exp\Big(-\frac{d^2}{4t}\Big)dv - \frac{n}{2},\end{aligned}$$

which implies

$$\lim_{t\to\infty}\mathscr{W} = \lim_{t\to\infty}\mathscr{N} \geq \log\frac{\theta}{\omega_n}. \tag{25}$$

In the last inequality we have claimed that (the claim will be justified later)

$$\lim_{t\to\infty}(4\pi t)^{-\frac{n}{2}}\int_M \frac{d^2}{4t}\exp\Big(-\frac{d^2}{4t}\Big)d\mu = \frac{n}{2}. \tag{26}$$

Using (23) and (25), we arrive at

$$\lim_{t\to\infty}\mathscr{W} \leq \ln\frac{\theta}{\omega_n}.$$

This completes the proof. □

Remark 4.1. In [5], Ni proved

$$\lim_{t\to\infty}\mathscr{W} = \ln\lim_{\rho\to\infty}\frac{Vol(B_\rho(x))}{\omega_n\rho^n}. \tag{27}$$

Combining (27) with assumption of maximal volume growth on M yields

$$\lim_{t\to\infty}\mathscr{W} = \ln\lim_{\rho\to\infty}\frac{Vol(B_\rho(x))}{\omega_n\rho^n} = \ln\frac{\theta}{\omega_n}, \tag{28}$$

which is exactly what we have shown.

In conclusion we want to justify the claim (26). Here we apply [20, Corollary 16.15] which states that for a complete manifold M with $Rc(M)\geq 0$, there exists a finite constant $C=C(n)$ depending on n such that

$$\int_M f(t;x,y)H(t;x,y)d\mu(y)\leq C(n)<\infty \tag{29}$$

for any $x\in M$, where f is differentiable, a.e. and $t>0$. Let us take $H(t;x,y)$ to be any positive solution, say $H=(4\pi t)^{-n/2}e^{-f}$. Then $f\leq\frac{d^2(x,y)}{4t}$ by Cheeger and Yau's result mentioned before in (3). Furthermore, using an upper bound from Li-Yau [19]

$$H(t;x,y)\leq\frac{C(\delta)}{Vol(B_{\sqrt{t}}(x))}\exp\Big(-\frac{d^2(x,y)}{(4+\delta)t}\Big)$$

with $C(\delta)<\infty$, $\delta\in(0,1)$, we have

$$\int_M fH dv\leq\frac{\widetilde{C}(n)}{Vol(B_{\sqrt{t}}(x))}\int_M\frac{d^2(x,y)}{4t}\exp\Big(-\frac{d^2(x,y)}{4t}\Big)d\mu(y)<\infty \tag{30}$$

for $t>0$ and $\delta\to 0$. A consequence of the above is that

$$\int_M\frac{d^2(x,y)}{4t}\exp\Big(-\frac{d^2(x,y)}{4t}\Big)d\mu(y)\leq C(n)Vol(B_{\sqrt{t}}(x))<\infty. \tag{31}$$

Indeed, if $d(x,y)$ was Euclidean, then we have

$$(4\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\frac{|x-y|^2}{4t}\exp\Big(\frac{-|x-y|^2}{4t}\Big)dy=\frac{n}{2},$$

which also holds on manifold (as $t\to 0$) by the heat kernel parametrix. Similarly, an analogue of the above formula (as $t\to\infty$) can also be carried over to manifold. An approach to evaluating the limit as $t \to \infty$ is by rescaling argument. Set $x = \varepsilon^{-1} y$, $\varepsilon > 0$ and choose $\varepsilon^2 = 4\pi t$. Then

$$\begin{aligned}
&\lim_{t\to\infty}(4\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\frac{|x-y|^2}{4t}\exp\Big(\frac{-|x-y|^2}{4t}\Big)dy \\
&\quad=\lim_{t\to\infty}(4\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\frac{\varepsilon^2}{4t}(\varepsilon^{-1}-1)^2|x|^2\exp\Big(-\frac{\varepsilon^2}{4t}(\varepsilon^{-1}-1)^2|x|^2\Big)\varepsilon dx \\
&\quad=\lim_{t\to\infty}\int_{\mathbb{R}^n}\frac{\varepsilon^2}{4t}(\varepsilon^{-1}-1)^2|x|^2\exp\Big(-\frac{\varepsilon^2}{4t}(\varepsilon^{-1}-1)^2|x|^2\Big)dx \\
&\quad=\pi\int_{\mathbb{R}^n}|x|^2e^{-\pi|x|^2}dx=\frac{n}{2}.
\end{aligned}$$

In the above calculation, we use $\varepsilon^{-1} \to 0$ as $t \to \infty$ and the standard Gauss integral identity.

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