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CAPUTO-TYPE FRACTIONAL BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH MULTIPLE FRACTIONAL DERIVATIVES

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Abstract. In this paper, we study a new class of single-valued and multi-valued boundary value problems involving multiple fractional derivatives of the Caputo type and the Riemann-Liouville type fractional integral boundary conditions. The existence results for the single valued case are based on the contraction mapping principle, nonlinear alternative of the Leray-Schauder type and the Krasnoselski's fixed point theorem, while the results for the multivalued case are obtained by applying the Leray-Schauder nonlinear alternative and the Covitz-Nadler fixed point theorem. Examples illustrating the main results are also presented. Some generalizations involving the Riemann-Liouville type integral and discrete multipoint boundary conditions are also addressed.

Keywords. Fractional differential equation; Fractional differential inclusion; Caputo fractional derivative; Fixed point.

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1. Introduction

Fractional differential equations have gained considerable importance due to their widespread applications in various disciplines of social and natural sciences, and engineering. In recent years, there has been a significant development in fractional calculus and fractional differential

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equations, for instance, see the monographs by Kilbas *et al.* [1], Lakshmikantham *et al.* [2], Miller and Ross [3], Podlubny [4], Samko *et al.* [5], Diethelm [6] and the papers [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

The aim of the present work is to study a new class of nonlinear differential equations and inclusions with multiple fractional derivatives and Caputo-Liouville type integro-differential boundary conditions. We consider the following single-valued boundary value problem of nonlinear fractional differential equations:

$$(1) \qquad \begin{cases} D^{\alpha}[D^{\beta}x(t)-g(t,x(t))]=f(t,x(t)),\ t\in J:=[0,T], \\ x(0)=0,\ (D^{\gamma}x)(T)=\lambda(I^{\delta}x)(T), \end{cases}$$

where D^{χ} is Caputo fractional derivative of order $\chi\in\{\alpha,\beta,\gamma\}$, $0<\alpha,\beta,\gamma<1$, I^{δ} is the Riemann-Liouville fractional integral of order δ, $f,g:J\times\mathbb{R}\to\mathbb{R}$ are given functions and $\lambda\neq\dfrac{\Gamma(\beta+\delta+1)}{T^{\gamma+\delta}\Gamma(\beta-\gamma+1)}$.

The existence of solutions for the problem (1) is established by applying the Leray-Schauder nonlinear alternative [17] and the Krasnoselskii's fixed point theorem [18]. The uniqueness result for the problem (1) is obtained by means of a celebrated fixed point theorem due to Banach. We also focus our study on the multivalued problem given by

$$(2) \qquad \begin{cases} D^{\alpha}[D^{\beta}x(t)-g(t,x(t))]\in F(t,x(t)),\ t\in J:=[0,T], \\ x(0)=0,\ (D^{\gamma}x)(T)=\lambda(I^{\delta}x)(T), \end{cases}$$

where $F:J\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a multivalued map ($\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$).

Using topological methods, we discuss the cases when F is convex valued (upper semicontinuous case) as well as non-convexed valued (Lipschitz case) multivalued maps. The nonlinear alternative of the Leray-Schauder type is applied to study the upper semicontinuous case, while the Lipschitz case is investigated by means of a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. In Section 2, we recall some useful preliminaries related to our work. Section 3 contains the existence results for problem (1). Section 4 is devoted to the study of problem (2). Examples are given in Section 5 for illustration of the main results.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 4] and present preliminary results.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to \mathbb{R}$ is defined by*

$$I^{\alpha} g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the Gamma function.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0,\infty) \to \mathbb{R}$ is defined by*

$${}^{RL}D^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds, \quad n-1 < \alpha < n,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. *The Caputo derivative of order q for a function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$$D^q f(t) = {}^{RL}D^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < q < n.$$

Remark 2.4. *If $f(t) \in C^n[0,\infty)$, then*

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t),\ t > 0,\ n-1 < q < n.$$

Lemma 2.5. *If $\alpha+\beta > 1$, then the equation $(I^{\alpha} I^{\beta} u)(t) = (I^{\alpha+\beta} u)(t), t \in J$ is satisfied for $u \in L^1(J,\mathbb{R})$.*

Lemma 2.6. *Let $\beta > \alpha$. Then the equation $(D^{\alpha} I^{\beta} u)(t) = (I^{\beta-\alpha} u)(t), t \in J$ is satisfied for $u \in C(J,\mathbb{R})$.*

Lemma 2.7. *Let $n = [\alpha]+1]$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$. Then the following relations hold:*

(i) *for $k \in \{0,1,2,\dots,n-1\}$, $D^{\alpha} t^k = 0$;*

(ii) *if $\beta > n$ then $D^{\alpha} t^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$;*

(iii) $I^{\alpha} t^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}$.

Lemma 2.8. *For $q > 0$, the general solution of the fractional differential equation $D^q x(t) = 0$ is given by $x(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$ ($n = [q]+1$).*

In view of Lemma 2.8, one has

$$I^q\, D^q x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}, \tag{3}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ ($n = [q]+1$).

The following lemma is concerned with a linear variant of problem (1).

Lemma 2.9. *Let $h \in C(J,\mathbb{R})$ and*

$$\Lambda = \frac{T^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} - \lambda \frac{T^{\beta+\delta}}{\Gamma(\beta+\delta+1)} \neq 0. \tag{4}$$

The function $x \in C^2(J,\mathbb{R})$ is a solution of the problem

$$\begin{cases} D^\alpha[D^\beta x(t) - g(t,x(t))] = h(t),\ t \in J := [0,T], \\ x(0) = 0,\ (D^\gamma x)(T) = \lambda (I^\delta x)(T), \end{cases} \tag{5}$$

if and only if

$$\begin{aligned} x(t) &= I^\beta g(s,x(s))(t) + I^{\alpha+\beta} h(t) + \frac{t^\beta}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) \\ &+ \lambda I^{\alpha+\beta+\delta} h(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} h(T) \Big\}. \end{aligned} \tag{6}$$

Proof. In view of (3), the solution of fractional differential equation in (5) can be written as $D^\beta x(t) - g(t,x(t)) = I^\alpha h(t) + c_1$, where $c_1 \in \mathbb{R}$ is an arbitrary constant and moreover

$$x(t) = I^\beta g(s,x(s))(t) + I^{\alpha+\beta} h(t) + \frac{t^\beta}{\Gamma(1+\beta)} c_1 + c_2, \tag{7}$$

where $c_2 \in \mathbb{R}$ is an arbitrary constant. Using the condition $x(0) = 0$ in (7) implies that $c_2 = 0$. Applying the operators D^γ and I^δ to (7), we get

$$D^\gamma x(t) = I^{\beta-\gamma} g(s,x(s))(t) + I^{\alpha+\beta-\gamma} h(t) + \frac{t^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} c_1,$$

and

$$I^\delta x(t) = I^{\beta+\delta} g(s,x(s))(t) + I^{\alpha+\beta+\delta} h(t) + \frac{t^{\beta+\delta}}{\Gamma(\beta+\delta+1)} c_1.$$

Using boundary condition $(D^\gamma x)(T) = \lambda (I^\delta x)(T)$, one has

$$c_1 = \frac{1}{\Lambda} \left\{ \lambda I^{\beta+\delta} g(s,x(s))(T) + \lambda I^{\alpha+\beta+\delta} h(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} h(T) \right\}.$$

Inserting the values of c_1 and c_2 in (7) we get the solution (6). The converse follows by direct computation. This completes the proof. □

Let $C(J,\mathbb{R})$ denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|,\ t \in J\}$. Also by $L^1(J,\mathbb{R})$ we denote the space of functions $x : J \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

In the following, for brevity, we use the notations:

$$\Lambda_1 = \frac{T^\beta}{\Gamma(\beta+1)}\left(1 + \frac{|\lambda|T^{\beta+\delta}}{|\Lambda|\Gamma(\beta+\delta+1)} + \frac{T^{\beta-\gamma}}{|\Lambda|\Gamma(\beta-\gamma+1)}\right), \tag{8}$$

and

$$\Lambda_2 = \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^\beta}{\Gamma(\beta+1)}\left(\frac{|\lambda|T^{\alpha+\beta+\delta}}{|\Lambda|\Gamma(\alpha+\beta+\delta+1)} + \frac{T^{\alpha+\beta-\gamma}}{|\Lambda|\Gamma(\alpha+\beta-\gamma+1)}\right). \tag{9}$$

3. Main results for problem (1)

In this section, we establish the existence and uniqueness of solutions for the BVP (1).

Theorem 3.1. *Assume that the following conditions hold:*

(A_1) *there exists a nonnegative constant k such that*

$$|g(t,u) - g(t,v)| \le k\|u - v\|, \quad \text{for } t \in J \text{ and every } u, v \in \mathbb{R};$$

(A_2) *there exists $\ell > 0$ such that*

$$|f(t,u) - f(t,v)| \le \ell\|u - v\|, \quad \text{for } t \in J \text{ and every } u, v \in \mathbb{R}.$$

If

$$k\Lambda_1 + \ell\Lambda_2 < 1, \tag{10}$$

where Λ_1, Λ_2 are defined by (8) and (9) respectively. Then there exists a unique solution for the BVP (1) on the interval J.

Proof. Let us introduce the operator $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ associated with the problem (1) as follows:

$$\begin{aligned} N(x)(t) &= I^\beta g(s,x(s))(t) + I^{\alpha+\beta} f(s,x(s))(t) \\ &\quad + \frac{t^\beta}{\Lambda\Gamma(\beta+1)}\Big\{\lambda I^{\beta+\delta} g(s,x(s))(T) + \lambda I^{\alpha+\beta+\delta} f(s,x(s))(T) \\ &\quad - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} f(s,x(s))(T)\Big\},\ t \in J. \end{aligned} \tag{11}$$

With Λ_1 and Λ_2 respectively given by (8) and (9), we fix

$$r \geq \frac{\Lambda_1 g_0 + \Lambda_2 f_0}{1 - k\Lambda_1 - \ell\Lambda_2},\ g_0 = \sup_{t\in J}|g(t,0)|,\ f_0 = \sup_{t\in J}|f(t,0)|,$$

and show that $NB_r \subset B_r$ where $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \leq r\}$. For $x \in B_r$, using (A_1) and (A_2), we get

$$\begin{aligned}
|N(x)(t)| &\leq I^{\beta}[|g(s,x(s)) - g(s,0)| + |g(s,0)|](t)\\
&\quad + I^{\alpha+\beta}[|f(s,x(s)) - f(s,0)| + |f(s,0)|](t)\\
&\quad + \frac{t^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda| I^{\beta+\delta}[|g(s,x(s)) - g(s,0)| + |g(s,0)|](T)\\
&\quad + |\lambda| I^{\alpha+\beta+\delta}[|f(s,x(s)) - f(s,0)| + |f(s,0)|](T)\\
&\quad + I^{\beta-\gamma}[|g(s,x(s)) - g(s,0)| + |g(s,0)|](T)\\
&\quad + I^{\alpha+\beta-\gamma}[|f(s,x(s)) - f(s,0)| + |f(s,0)|](T)\Big\}\\
&\leq (k\|x\| + g_0) I^{\beta} 1(T) + (\ell\|x\| + f_0) I^{\alpha+\beta} 1(T)\\
&\quad + \frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda|(k\|x\| + g_0) I^{\beta+\delta} 1(T) ds + |\lambda| I^{\alpha+\beta+\delta} 1(T)(\ell\|x\| + f_0)\\
&\quad + (k\|x\| + g_0) I^{\beta-\gamma} 1(T) + (\ell\|x\| + f_0) I^{\alpha+\beta-\gamma} 1(T)\Big\}\\
&\leq \Lambda_1(kr + g_0) + \Lambda_2(\ell r + f_0) < r.
\end{aligned}$$

Taking the norm for $t \in J$, one finds that $\|N(x)\| \leq r$. This shows that N maps B_r into itself.

In order to show that the operator N is a contraction, let $x, y \in C(J,\mathbb{R})$. Using (8) and (9), we get

$$\begin{aligned}
&|N(x)(t) - N(y)(t)|\\
&\leq I^{\beta}|g(s,x(s)) - g(s,y(s))|(T) + I^{\alpha+\beta}|f(s,x(s)) - f(s,y(s))|(T)\\
&\quad + \frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda| I^{\beta+\delta}|g(s,x(s)) - g(s,y(s))|(T)\\
&\quad + |\lambda| I^{\alpha+\beta+\delta}|f(s,x(s)) - f(s,y(s))|(T)\\
&\quad + I^{\beta-\gamma}|g(s,x(s)) - g(s,y(s))|(T) + I^{\alpha+\beta-\gamma}|f(s,x(s)) - f(s,y(s))|(T)\Big\}\\
&\leq \frac{T^{\beta}}{\Gamma(\beta+1)}\left\{1 + \frac{|\lambda| T^{\beta+\delta}}{|\Lambda|\Gamma(\beta+\delta+1)} + \frac{T^{\beta-\gamma}}{|\Lambda|\Gamma(\beta-\gamma+1)}\right\} k\|x - y\|\\
&\quad + \left\{\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\left(\frac{|\lambda| T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} + \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma)}\right)\right\}\ell\|x - y\|
\end{aligned}$$

$$= (k\Lambda_1 + \ell\Lambda_2)\|x-y\|.$$

Consequently $\|N(x)-N(y)\| \leq (k\Lambda_1+\ell\Lambda_2)\|x-y\|$. In view of (10), one finds that N is a contraction. Hence N has a unique fixed point by Banach's contraction principle. This, in turn, shows that problem (1) has a unique solution on J. The proof is completed. □

The next existence result is based on the Leray-Schauder nonlinear alternative.

Lemma 3.2. *(Nonlinear alternative for single valued maps)* [17] . *Let E be a Banach space and let C be a closed, convex subset of E. Let U be an open subset of C and let $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 3.3. *Assume that the following hypotheses hold:*

(H_1) *$f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions;*
(H_2) *there exist constants $d_1 < 1/\Lambda_1$ and $d_2 \geq 0$ such that*

$$|g(t,u)| \leq d_1\|u\| + d_2, \ \ t \in J, \ u \in \mathbb{R};$$

(H_3) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C(J,\mathbb{R}^+)$ such that*

$$|f(t,u)| \leq p(t)\psi(\|u\|) \text{ for each } (t,u) \in J \times \mathbb{R};$$

(H_4) *there exists a constant $M > 0$ such that*

$$\frac{(1-d_1\Lambda_1)M}{d_2\Lambda_1 + \Lambda_2\|p\|\psi(M)} > 1.$$

Then the BVP (1) has at least one solution on J.

Proof. Consider the operator $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by (11). We show that N is continuous and completely continuous.

Step 1. N is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C(J,\mathbb{R})$. Then

$$
\begin{aligned}
& |N(x_n)(t) - N(x)(t)| \\
& \leq I^{\beta}|g(s,x_n(s)) - g(s,x(s))|(T) + I^{\alpha+\beta}|f(s,x_n(s)) - f(s,x(s))|(T) \\
& + \frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda| I^{\beta+\delta}|g(s,x_n(s)) - g(s,x(s))|(T) \\
& + |\lambda| I^{\alpha+\beta+\delta}|f(s,x_n(s)) - f(s,x(s))|(T) + I^{\beta-\gamma}|g(s,x_n(s)) - g(s,x(s))|(T) \\
& + I^{\alpha+\beta-\gamma}|f(s,x_n(s)) - f(s,x(s))|(T)\Big\} \\
& \leq \Lambda_1\|g(t\cdot,x_n) - g(\cdot,x)\| + \Lambda_2\|f(t\cdot,x_n) - f(\cdot,x)\|.
\end{aligned}
$$

Since f, g are continuous functions, therefore, we have

$$
\|N(x_n) - N(x)\| \leq \Lambda_1\|g(t\cdot,x_n) - g(\cdot,x)\| + \Lambda_2\|f(t\cdot,x_n) - f(\cdot,x)\| \to 0, \ as\ n \to \infty.
$$

Step 2. N maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Indeed, it is enough to show that for any $\theta > 0$ there exists a positive constant $\tilde{\ell}$ such that for each $x \in B_\theta = \{x \in C(J,\mathbb{R}) : \|x\| \leq \theta\}$, we have $\|N(x)\| \leq \tilde{\ell}$. By (H_2) and (H_3), for each $t \in J$, we have

$$
\begin{aligned}
|N(x)(t)| \ \leq\ & I^{\beta}|g(s,x(s))|(t) + I^{\alpha+\beta}|f(s,x(s))|(t) \\
& + \frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda| I^{\beta+\delta}|g(s,x(s))|(T) + |\lambda| I^{\alpha+\beta+\delta}|f(s,x(s))|(T) \\
& + I^{\beta-\gamma}|g(s,x(s))|(T) + I^{\alpha+\beta-\gamma}|f(s,x(s))|(T)\Big\} \\
\leq\ & I^{\beta}(d_1\|x\| + d_2)(T) + I^{\alpha+\beta}p(s)\psi(\|x\|)(T) \\
& + \frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda| I^{\beta+\delta}(d_1\|x\| + d_2)(T) + |\lambda| I^{\alpha+\beta+\delta}p(s)\psi(\|x\|)(T) \\
& + I^{\beta-\gamma}(d_1\|x\| + d_2)(T) + I^{\alpha+\beta-\gamma}p(s)\psi(\|x\|)(T)\Big\} \\
\leq\ & \Lambda_1(d_1\|x\| + d_2) + \Lambda_2\|p\|\psi(\|x\|).
\end{aligned}
$$

Thus $\|N(x)\| \leq \Lambda_1(d_1\theta + d_2) + \Lambda_2\|p\|\psi(\theta) := \tilde{\ell}$.

Step 3. N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let $t_1, t_2 \in J,\ t_1 < t_2$, B_θ be a bounded set of $C(J,\mathbb{R})$ as in Step 2, and let $x \in B_\theta$. Then

$$\begin{aligned}
&|N(x)(t_2) - N(x)(t_1)|\\
&\le \frac{1}{\Gamma(\beta)}\Big|\int_0^{t_1}(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}\Big]g(s,x(s))ds + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}g(s,x(s))ds\Big|\\
&+\frac{1}{\Gamma(\alpha+\beta)}\Big|\int_0^{t_1}(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}\Big]f(s,x(s))ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}f(s,x(s))ds\Big|\\
&+\frac{|t_2^\beta - t_1^\beta|}{\Lambda\Gamma(\beta+1)}\Big\{|\lambda| I^{\beta+\delta}|g(s,x(s))|(T) + |\lambda| I^{\alpha+\beta+\delta}|f(s,x(s))|(T)\\
&+I^{\beta-\gamma}|g(s,x(s))|(T) + I^{\alpha+\beta-\gamma}|f(s,x(s))|(T)\Big\}\\
&\le \frac{d_1\theta+d_2}{\Gamma(\beta+1)}[t_2^\beta - t_1^\beta + 2(t_2-t_1)^\beta] + \frac{\|p\|\psi(\theta)}{\Gamma(\alpha+\beta+1)}[t_2^{\alpha+\beta} - t_1^{\alpha+\beta} + 2(t_2-t_1)^{\alpha+\beta}]\\
&+\frac{|t_2^\beta - t_1^\beta|}{|\Lambda|\Gamma(\beta+1)}\Bigg\{\left(\frac{|\lambda| T^{\beta+\delta}}{\Gamma(\beta+\delta+1)} + \frac{T^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right)(d_1\theta+d_2)\\
&+\left(\frac{|\lambda| T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} + \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)}\right)\|p\|\psi(\theta)\Bigg\}.
\end{aligned}$$

Clearly the right-hand side of the above inequality tends to zero independent of x as $t_1 \longrightarrow t_2$.

In consequence of Steps 1 to 3, it follows by the Arzelá-Ascoli theorem that the operator $N : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ is continuous and completely continuous.

Step 4. We show that there exists an open set $U \subseteq C(J,\mathbb{R})$ with $x \neq \mu N(x)$ for $\mu \in (0,1)$ and $x \in \partial U$.

Let $x \in C(J,\mathbb{R})$ and $x = \mu N(x)$ for some $0 < \mu < 1$. Thus, for each $t \in J$, we have

$$\begin{aligned}
x(t) \;=\; & \mu I^\beta g(s,x(s))(t) + \mu I^{\alpha+\beta} f(s,x(s))(t) + \mu\frac{t^\beta}{\Lambda\Gamma(\beta+1)}\Big\{\lambda I^{\beta+\delta}g(s,x(s))(T)\\
& +\lambda I^{\alpha+\beta+\delta}f(s,x(s))(T) - I^{\beta-\gamma}g(s,x(s))(T) - I^{\alpha+\beta-\gamma}f(s,x(s))(T)\Big\}.
\end{aligned}$$

As in Step 2, for each $t \in J$, it can be established that $|x(t)| \le \Lambda_1(d_1\|x\| + d_2) + \Lambda_2\|p\|\psi(\|x\|)$, which can be expressed as $\frac{(1-d_1\Lambda_1)\|x\|}{d_2\Lambda_1 + \Lambda_2\|p\|\psi(\|x\|)} \le 1$. In view of (H_4), there exists M such that $\|x\| \neq$ M. Let us set $U = \{x \in C(J,\mathbb{R}) : \|x\| < \text{M}\}$. Note that the operator $N : \overline{U} \to C(J,\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \mu N x$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.2), we deduce that N has a fixed point $x \in \overline{U}$ which is a solution of the problem (1). This completes the proof. □

Our next existence result is based on the Krasnoselskii's fixed point theorem.

Lemma 3.4. *(Krasnoselskii's fixed point theorem)* [18]. *Let S be a closed, bounded, convex and nonempty subset of a Banach space X. Let $\mathscr{Y}_1, \mathscr{Y}_2$ be the operators such that (a) $\mathscr{Y}_1 s_1 + \mathscr{Y}_2 s_2 \in S$ whenever $s_1, s_2 \in S$; (b) $\mathscr{Y}_1$ is compact and continuous; (c) $\mathscr{Y}_2$ is a contraction mapping. Then there exists $s_3 \in S$ such that $s_3 = \mathscr{Y}_1 s_3 + \mathscr{Y}_2 s_3$.*

Theorem 3.5. *Assume that (A_1) and (H_1) hold. In addition we assume that:*

(H_5) $|g(t,x)| \leq \phi(t),\ |f(t,x)| \leq q(t), \quad \forall (t,x) \in J \times \mathbb{R}$, *and* $q, \phi \in C(J, \mathbb{R}^+)$.

Then the BVP (1) has at least one solution on J, provided

$$k\Lambda_1 < 1. \tag{12}$$

Proof. Let us split the operator $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by (11) as $N = A + B$, where A and B are given by

$$A(x)(t) = I^{\beta} g(s,x(s))(t) + \frac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) - I^{\beta-\gamma} g(s,x(s))(T) \Big\}$$

and

$$\begin{aligned} B(x)(t) \;=\; & I^{\alpha+\beta} f(s,x(s))(t) + \frac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\alpha+\beta+\delta} f(s,x(s))(T) \\ & - I^{\alpha+\beta-\gamma} f(s,x(s))(T) \Big\}. \end{aligned}$$

Setting $\sup_{t\in J} |\phi(t)| = \|\phi\|, \sup_{t\in J} |q(t)| = \|q\|$ and choosing $\rho \geq \|\phi\|\Lambda_1 + \|q\|\Lambda_2$, we consider $B_\rho = \{x \in C(J,\mathbb{R}) : \|x\| \leq \rho\}$. For any $x, y \in B_\rho$, we have

$$\begin{aligned} |Ax(t) + By(t)| \;\leq\; & I^{\beta}|g(s,x(s))|(t) + I^{\alpha+\beta}|f(s,x(s))|(t) \\ & + \frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)} \Big\{ |\lambda| I^{\beta+\delta}|g(s,x(s))|(T) + |\lambda| I^{\alpha+\beta+\delta}|f(s,x(s))|(T) \\ & + I^{\beta-\gamma}|g(s,x(s))|(T) + I^{\alpha+\beta-\gamma}|f(s,x(s))|(T) \Big\} \\ \leq\; & \|\phi\|\Lambda_1 + \|q\|\Lambda_2 \leq \rho. \end{aligned}$$

Hence $\|Ax + By\| \leq \rho$, which shows that $Ax + By \in B_\rho$. In view of (12), it is easy to show that A is a contraction mapping. Continuity of f implies that the operator B is continuous. Also, B is uniformly bounded on B_ρ as $\|Bx\| \le \|q\|\Lambda_2$. Finally we prove the compactness of the operator B. For that we define $\sup_{(t,x)\in J\times B_\rho}|f(t,x)| = \bar{f} < \infty$. Then, for $t_1, t_2 \in J$, $t_1 < t_2$, we have

$$\begin{aligned}
|Bx(t_2) - Bx(t_1)| &\le \frac{1}{\Gamma(\alpha+\beta)}\Big|\int_0^{t_1}(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}\Big] f(s,x(s))ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1} f(s,x(s))ds\Big| \\
&\quad + \frac{|t_2^\beta - t_1^\beta|}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda| I^{\alpha+\beta+\delta}|f(s,x(s))|(T) + I^{\alpha+\beta-\gamma}|f(s,x(s))|(T)\Big\} \\
&\le \frac{\bar{f}}{\Gamma(\alpha+\beta+1)}[t_2^{\alpha+\beta} - t_1^{\alpha+\beta} + 2(t_2-t_1)^{\alpha+\beta}] \\
&\quad + \frac{|t_2^\beta - t_1^\beta|\bar{f}}{|\Lambda|\Gamma(\beta+1)}\left\{\frac{|\lambda|T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} + \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)}\right\},
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, B is equicontinuous. So B is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, B is compact on B_ρ. Thus all the assumptions of Lemma 3.4 are satisfied. So the conclusion of Lemma 3.4 implies that the problem (1) has at least one solution on J. □

4. Main results for problem (2)

In this section, we establish the existence results for the BVP (2). We outline first some basic concepts of multivalued analysis [19, 20].

For a normed space $(X, \|\cdot\|)$, let $\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is closed}\}$, $\mathscr{P}_b(X) = \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\}$, $\mathscr{P}_{cl,b}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is closed and bounded}\}$, $\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact}\}$, and $\mathscr{P}_{cp,c}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G : X \to \mathscr{P}(X)$:

- (i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$.
- (ii) is *bounded* on bounded sets if $G(Y) = \cup_{x\in Y} G(x)$ is bounded in X for all $Y \in \mathscr{P}_b(X)$ (i.e. $\sup_{x\in Y}\{\sup\{|y| : y \in G(x)\}\} < \infty$).
- (iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0) \subseteq N$.
- (iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap Y \neq \emptyset\}$ is open for any open set Y in X.

(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathscr{P}_b(X)$; If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

(vi) is said to be *measurable* if for every $y \in X$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

4.1. **The Carathéodory case.** In this subsection we consider the case when F has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

Definition 4.1. *A multivalued map $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,x)$ *is measurable for each* $x \in \mathbb{R}$*;*

(ii) $x \longmapsto F(t,x)$ *is upper semicontinuous for almost all* $t \in J$*;*

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) *for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in J$.

For each $x \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \text{ on } J\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 4.2. *(*[19, Proposition 1.2]*) If $G : X \to \mathscr{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 4.3. *([21]) Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathscr{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator $\Theta \circ S_F : C(J,X) \to \mathscr{P}_{cp,c}(C(J,X))$, $x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(J,X) \times C(J,X)$.*

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 4.4. *(Nonlinear alternative for Kakutani maps)* [17]. *Let E be a Banach space and let C be a closed convex subset of E. Let U be an open subset of C and let $0 \in U$. Suppose that $F : \overline{U} \to \mathscr{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Definition 4.5. *A function $x \in C^2(J,\mathbb{R})$ is a solution of the problem (2) if $x(0) = 0$, $(D^\gamma x)(T) = \lambda(I^\delta x)(T)$, and there exists function $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,x(t))$ a.e. on J and*

$$x(t) = I^\beta g(s,x(s))(t) + I^{\alpha+\beta} v(t) + \frac{t^\beta}{\Lambda\Gamma(\beta+1)}\Big\{\lambda I^{\beta+\delta} g(s,x(s))(T) + I^{\alpha+\beta+\delta} v(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} v(T)\Big\}, \tag{13}$$

where $\Lambda \neq 0$ is defined by (4).

Theorem 4.6. *Assume that (H_2) and (H_4) hold. In addition, we suppose that:*

(B_1) *$F : J \times \mathbb{R} \to \mathscr{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*
(B_2) *there exists a continuous nondecreasing function $\phi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1(J,\mathbb{R}^+)$ such that $\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\phi(\|x\|)$ for each $(t,x) \in J \times \mathbb{R}$;*
(B_3) *there exists a constant $M > 0$ such that*

$$\frac{(1-d_1\Lambda_1)M}{d_2\Lambda_1 + \Lambda_2\|p\|\phi(M)} > 1.$$

Then the BVP (2) has at least one solution on J.

Proof. Associated with the problem (2), we introduce an operator $\mathscr{F}: C(J,\mathbb{R}) \longrightarrow \mathscr{P}(C(J,\mathbb{R}))$ as follows:

$$\mathscr{F}(x) = \left\{ \begin{array}{l} h \in C(J,\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} I^{\beta} g(s,x(s))(t) + I^{\alpha+\beta} v(t) \\ + \dfrac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) + I^{\alpha+\beta+\delta} v(T) \\ - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} v(T) \Big\}, \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$. It is obvious that the fixed points of $\mathscr{F}$ are solutions of the boundary value problem (2). We next show that $\mathscr{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative (Lemma 4.4). The proof consists of several steps.

Step 1. $\mathscr{F}(x)$ is convex for each $x \in C(J,\mathbb{R})$.

This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Step 2. $\mathscr{F}$ maps bounded sets (balls) into bounded sets in $C(J,\mathbb{R})$.

For a positive number r, let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \leq r\}$ be a bounded ball in $C(J,\mathbb{R})$. Then, for each $h \in \mathscr{F}(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= I^{\beta} g(s,x(s))(t) + I^{\alpha+\beta} v(t) + \frac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) \\ &\quad + I^{\alpha+\beta+\delta} v(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} v(T) \Big\},\ t \in J. \end{aligned}$$

Then, for $t \in J$, one can easily obtain that $\|h\| \leq \Lambda_1(d_1\theta + d_2) + \Lambda_2 \|p\| \phi(\theta) := \tilde{\ell}$.

Step 3. $\mathscr{F}$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then, for each $h \in \mathscr{F}(x)$, it follows by the assumptions (H_2) and (B_2) that

$$\begin{aligned} |h(t_2) - h(t_1)| &\leq \frac{d_1\theta + d_2}{\Gamma(\beta+1)} [t_2^{\beta} - t_1^{\beta} + 2(t_2 - t_1)^{\beta}] \\ &+ \frac{\|p\|\phi(\theta)}{\Gamma(\alpha+\beta+1)} [t_2^{\alpha+\beta} - t_1^{\alpha+\beta} + 2(t_2 - t_1)^{\alpha+\beta}] \\ &+ \frac{|t_2^{\beta} - t_1^{\beta}|}{|\Lambda|\Gamma(\beta+1)} \left\{ \left(\frac{|\lambda| T^{\beta+\delta}}{\Gamma(\beta+\delta+1)} + \frac{T^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \right) (d_1\theta + d_2) \right. \\ &\left. + \left(\frac{|\lambda| T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} + \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)} \right) \|p\|\phi(\theta) \right\}. \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathscr{F}: C(J,\mathbb{R}) \to \mathscr{P}(C(J,\mathbb{R}))$

is completely continuous. Since $\mathscr{F}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph. Thus, in our next step, we show that

Step 4. $\mathscr{F}$ has a closed graph.

Let $x_n \to x_*, h_n \in \mathscr{F}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathscr{F}(x_*)$. Associated with $h_n \in \mathscr{F}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$,

$$\begin{aligned} h_n(t) &= I^{\beta} g(s,x(s))(t) + I^{\alpha+\beta} v_n(t) + \frac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) \\ &\quad + \lambda I^{\alpha+\beta+\delta} v_n(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} v_n(T) \Big\},\ t \in J. \end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in J$,

$$\begin{aligned} h_*(t) &= I^{\beta} g(s,x(s))(t) + I^{\alpha+\beta} v_*(t) + \frac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) \\ &\quad + \lambda I^{\alpha+\beta+\delta} v_*(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} v_*(T) \Big\},\ t \in J. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(v)(t) &= I^{\beta} g(s,x(s))(t) + I^{\alpha+\beta} v(t) + \frac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) \\ &\quad + I^{\alpha+\beta+\delta} v(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} v(T) \Big\},\ t \in J. \end{aligned}$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$, as $n \to \infty$, and thus, it follows by Lemma 4.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= I^{\beta} g(s,x(s))(t) + I^{\alpha+\beta} v_*(t) + \frac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) \\ &\quad + \lambda I^{\alpha+\beta+\delta} v_*(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} v_*(T) \Big\},\ t \in J, \end{aligned}$$

for some $v_* \in S_{F,x_*}$.

Step 5. We show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $x \notin \mu\mathscr{F}(x)$ for any $\mu \in (0,1)$ and all $x \in \partial U$.

Let $\mu \in (0,1)$ and $x \in \mu\mathscr{F}(x)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in J$, we have

$$\begin{aligned} x(t) &= \mu I^{\beta} g(s,x(s))(t) + \mu I^{\alpha+\beta} f(s,x(s))(t) + \mu \frac{t^{\beta}}{\Lambda\Gamma(\beta+1)} \Big\{ \lambda I^{\beta+\delta} g(s,x(s))(T) \\ &\quad + \lambda I^{\alpha+\beta+\delta} f(s,x(s))(T) - I^{\beta-\gamma} g(s,x(s))(T) - I^{\alpha+\beta-\gamma} f(s,x(s))(T) \Big\}. \end{aligned}$$

As before (Step 2), we can find that $|x(t)| \le \Lambda_1(d_1\|x\| + d_2) + \Lambda_2\|p\|\phi(\|x\|)$, which yields

$$\frac{(1-d_1\Lambda_1)\|x\|}{d_2\Lambda_1 + \Lambda_2\|p\|\phi(\|x\|)} \le 1.$$

In view of (H_3), there exists $\widehat{M}$ such that $\|x\| \ne \widehat{M}$. Let us set $V = \{x \in C(I,\mathbb{R}) : \|x\| < \widehat{M}\}$. Note that the operator $\mathscr{F} : \overline{V} \to \mathscr{P}(C(I,\mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of V, there is no $x \in \partial V$ such that $x \in \mu\mathscr{F}(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4.4), we deduce that $\mathscr{F}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (2). This completes the proof. □

Remark 4.7. *In case F is not necessarily convex valued (lower semicontinuous case), we can combine the nonlinear alternative of Leray Schauder type and a selection theorem due to Bressan and Colombo* [22] *for lower semi-continuous maps with decomposable values to prove the existence of solutions for the problem (2). We can formulate this result as follows: "Let $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a nonempty compact-valued multivalued map such that (a) $(t,x) \longmapsto F(t,x)$ is $\mathscr{L} \otimes \mathscr{B}$ measurable and (b) $x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in J$. In addition we assume that (B_2) and (B_3) hold. Then the problem (2) has at least one solution on J". Here we recall that a subset of $J \times \mathbb{R}$ is $\mathscr{L} \otimes \mathscr{B}$ measurable if it belongs to the $\sigma-$algebra generated by all sets of the form $\mathscr{J} \times \mathscr{D}$, where $\mathscr{J}$ is Lebesgue measurable in J and $\mathscr{D}$ is Borel measurable in $\mathbb{R}$. One can find more details about such results in* [10].

4.2. **The Lipschitz case.** In this subsection we prove the existence of solutions for the problem (2) with a not necessary nonconvex valued right hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [23].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathscr{P}_{cl,b}(X), H_d)$ is a metric space (see [24]).

Definition 4.8. *A multivalued operator $N : X \to \mathscr{P}_{cl}(X)$ is called*

(a) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x),N(y)) \le \gamma d(x,y) \text{ for each } x,y \in X;$$

(b) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 4.9. *(*[23]*) Let (X,d) be a complete metric space. If $N : X \to \mathscr{P}_{cl}(X)$ is a contraction, then $FixN \ne \emptyset$.*

Theorem 4.10. *Let (A_1) and the following assumptions holds:*

(C_1) $F : J\times\mathbb{R}\to\mathscr{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot,x) : J\to\mathscr{P}_{cp}(\mathbb{R})$ *is measurable for each* $x\in\mathbb{R}$.

(C_2) $H_d(F(t,x),F(t,\bar{x}))\le m(t)|x-\bar{x}|$ *for almost all* $t\in J$ *and* $x,\bar{x}\in\mathbb{R}$ *with* $m\in L^1(J,\mathbb{R}^+)$ *and* $d(0,F(t,0))\le m(t)$ *for almost all* $t\in J$.

Then the boundary value problem (2) has at least one solution on J if

$$\varpi := k\Lambda_1 + I^{\alpha+\beta}m(T) + \frac{T^\beta}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda|I^{\alpha+\beta+\delta}m(T) + I^{\alpha+\beta-\gamma}m(T)\Big\} < 1. \tag{14}$$

Proof. Consider the operator $\mathscr{F}$ defined at the begin of the proof of Theorem 4.6. Observe that the set $S_{F,x}$ is nonempty for each $x\in C(J,\mathbb{R})$ by the assumption (C_1), so F has a measurable selection (see Theorem III.6 [25]). Now we show that the operator $\mathscr{F}$ satisfies the assumptions of Lemma 4.9. We show that $\mathscr{F}(x)\in\mathscr{P}_{cl}((CJ,\mathbb{R}))$ for each $x\in C(J,\mathbb{R})$. Let $\{u_n\}_{n\ge 0}\in\mathscr{F}(x)$ be such that $u_n\to u$ $(n\to\infty)$ in $C(J,\mathbb{R})$. Then $u\in C(J,\mathbb{R})$ and there exists $v_n\in S_{F,x_n}$ such that, for each $t\in J$,

$$\begin{aligned} u_n(t) &= I^\beta g(s,x(s))(t) + I^{\alpha+\beta}v_n(t) + \frac{t^\beta}{\Lambda\Gamma(\beta+1)}\Big\{\lambda I^{\beta+\delta}g(s,x(s))(T) \\ &\quad + \lambda I^{\alpha+\beta+\delta}v_n(T) - I^{\beta-\gamma}g(s,x(s))(T) - I^{\alpha+\beta-\gamma}v_n(T)\Big\},\ t\in J. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus, $v\in S_{F,x}$ and for each $t\in J$, we have

$$\begin{aligned} u_n(t)\to v(t) &= I^\beta g(s,x(s))(t) + I^{\alpha+\beta}v(t) + \frac{t^\beta}{\Lambda\Gamma(\beta+1)}\Big\{\lambda I^{\beta+\delta}g(s,x(s))(T) \\ &\quad + I^{\alpha+\beta+\delta}v(T) - I^{\beta-\gamma}g(s,x(s))(T) - I^{\alpha+\beta-\gamma}v(T)\Big\},\ t\in J, \end{aligned}$$

which implies that $u\in\mathscr{F}(x)$.

Next we show that there exists $\varpi<1$ (defined by (14)) such that $H_d(\mathscr{F}(x),\mathscr{F}(\bar{x}))\le\varpi\|x-\bar{x}\|$ for each $x,\bar{x}\in C^2(J,\mathbb{R})$. Let $x,\bar{x}\in C^2(J,\mathbb{R})$ and $h_1\in\mathscr{F}(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in J$,

$$\begin{aligned} h_1(t) &= I^\beta g(s,x(s))(t) + I^{\alpha+\beta}v_1(t) + \frac{t^\beta}{\Lambda\Gamma(\beta+1)}\Big\{\lambda I^{\beta+\delta}g(s,x(s))(T) \\ &\quad + \lambda I^{\alpha+\beta+\delta}v_1(T) - I^{\beta-\gamma}g(s,x(s))(T) - I^{\alpha+\beta-\gamma}v_1(T)\Big\},\ t\in J. \end{aligned}$$

By (C_2), we have $H_d(F(t,x),F(t,\bar{x}))\le m(t)|x(t)-\bar{x}(t)|$. So, there exists $w\in F(t,\bar{x}(t))$ such that $|v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|$, $t\in J$. Define $\widehat{U} : J\to\mathscr{P}(\mathbb{R})$ by

$$\widehat{U}(t) = \{w\in\mathbb{R} : |v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $\widehat{U}(t)\cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [25]), there exists a function $v_2(t)$ which is a measurable selection for $\widehat{U}$. So $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in J$, we have $|v_1(t)-v_2(t)|\le m(t)|x(t)-\bar{x}(t)|$. For each $t\in J$, let us define

$$\begin{aligned} h_2(t) &= I^{\beta}g(s,\bar{x}(s))(t)+I^{\alpha+\beta}v_2(t)+\frac{t^{\beta}}{\Lambda\Gamma(\beta+1)}\Big\{\lambda I^{\beta+\delta}g(s,\bar{x}(s))(T) \\ &\quad +\lambda I^{\alpha+\beta+\delta}v_2(T)-I^{\beta-\gamma}g(s,\bar{x}(s))(T)-I^{\alpha+\beta-\gamma}v_2(T)\Big\},\ t\in J. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t)-h_2(t)| &\le I^{\beta}|g(s,x(s))-g(s,\bar{x}(s))|(T) \\ &\quad +\frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda|I^{\beta+\delta}|g(s,x(s))-g(s,\bar{x}(s))|(T) \\ &\quad +I^{\beta-\gamma}|g(s,x(s))-g(s,\bar{x}(s))|(T)\Big\}+I^{\alpha+\beta}|v_1-v_2|(T) \\ &\quad +\frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda|I^{\alpha+\beta+\delta}|v_1-v_2|(T)+I^{\alpha+\beta-\gamma}|v_1-v_2|(T)\Big\} \\ &\le \frac{T^{\beta}}{\Gamma(\beta+1)}\left\{1+\frac{|\lambda|T^{\beta+\delta}}{|\Lambda|\Gamma(\beta+\delta+1)}+\frac{T^{\beta-\gamma}}{|\Lambda|\Gamma(\beta-\gamma+1)}\right\}k\|x-\bar{x}\| \\ &\quad +\Big[I^{\alpha+\beta}m(s)(T)+\frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda|I^{\alpha+\beta+\delta}m(s)(T) \\ &\quad +I^{\alpha+\beta-\gamma}m(s)(T)\Big\}\Big]\|x-\bar{x}\| \\ &\le \Big[k\Lambda_1+I^{\alpha+\beta}m(s)(T)+\frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda|I^{\alpha+\beta+\delta}m(s)(T) \\ &\quad +I^{\alpha+\beta-\gamma}m(s)(T)\Big\}\Big]\|x-\bar{x}\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|h_1-h_2\| &\le \Big[k\Lambda_1+I^{\alpha+\beta}m(s)(T)+\frac{T^{\beta}}{|\Lambda|\Gamma(\beta+1)}\Big\{|\lambda|I^{\alpha+\beta+\delta}m(s)(T) \\ &\quad +I^{\alpha+\beta-\gamma}m(s)(T)\Big\}\Big]\|x-\bar{x}\|. \end{aligned}$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain $H_d(\mathscr{F}(x),\mathscr{F}(\bar{x}))\le\varpi\|x-\bar{x}\|$, where ϖ is defined by (14). So $\mathscr{F}$ is a contraction. Therefore, it follows by Lemma 4.9 that $\mathscr{F}$ has a fixed point x which is a solution of (2). This completes the proof. □

5. Examples

In this section we give examples to illustrate the usefulness of our main results.

5.1. **Single-valued case.** Let us consider the fractional functional differential equation

$$\begin{cases} D^{3/7}[D^{4/7}x(t)-g(t,x(t))]=f(t,x(t)),\ t\in J:=[0,2], \\ x(0)=0,\ (D^{2/7}x)(2)=(I^{8/7}x)(2). \end{cases} \tag{15}$$

Here $\alpha=3/7, \beta=4/7, \gamma=2/7, \delta=8/7, \lambda=1, T=2$ (note that we can chose any λ such that $\lambda\neq\Gamma(\beta+\delta+1)/T^{\gamma+\delta}\Gamma(\beta-\gamma+1)=0.645119$), and $g(t,y)$, $f(t,y)$ will be chosen suitably for the illustration of the obtained results. Using the given data, we find that $|\Lambda|=0.745299, \Lambda_1=9.40315, \Lambda_2=10.34570$, where Λ, Λ_1 and Λ_2 are respectively given by (4), (8) and (9). Let us take

$$f(t,x)=\frac{1}{\sqrt{900+t^2}}\tan^{-1}x+e^{-t} \text{ and } g(t,x)=\frac{1}{24+e^t}\Big(\frac{|x|}{1+|x|}+t+1\Big). \tag{16}$$

It is easy to check that $f(t,x)$ and $g(t,x)$ satisfy the conditions (A_1) and (A_2) respectively with $\ell=1/30$ and $k=1/25$. Also $k\Lambda_1+\ell\Lambda_2\approx 0.72098<1$. Thus all the conditions of Theorem 3.1 are satisfied. So, by the conclusion of Theorem 3.1, the problem (15) with $f(t,x)$ and $g(t,x)$ given by (16) has a unique solution on $[0,2]$. Also the hypothesis of Theorem 3.5 holds true with $\phi(t)=4/(24+e^t)$, $q(t)=\frac{0.5\pi}{\sqrt{900+t^2}}+e^{-t}$ and $k\Lambda_1\approx 0.37613$. Thus by the conclusion of Theorem 3.5, there exists at least one solution for the problem (15) with $f(t,x)$ and $g(t,x)$ given by (16).

5.2. **Multivalued case.** Consider the following multivalued fractional functional boundary value problem:

$$\begin{cases} D^{3/7}\Big(D^{4/7}x(t)-g(t,x(t))\Big)\in F(t,x(t)),\ t\in[0,2], \\ x(0)=0,\ (D^{2/7}x)(2)=(I^{8/7}x)(2), \end{cases} \tag{17}$$

where $g(t,x(t))=\frac{1}{20}\sin x(t)+\frac{t}{30}$ and $F(t,x(t))$ will be chosen appropriately.

For the illustration of Theorem 4.6, we take

$$F(t,x(t))=\Big[\frac{|x|^2}{200(|x|^2+1)}+\frac{1}{250}\cos t\ ,\ \frac{e^{2-t}}{150}(|x|+1)\Big]. \tag{18}$$

Using the given data, we find that $d_1=1/20, d_2=1/15$, $\|p\|=(e^2-1)/150$, $\phi(\|x\|)=1+\|x\|$ and by the Condition (B_3), we have $M>M_1\approx 11.970818$. Thus, all the conditions of Theorem 4.6 are satisfied and consequently, there exists at least one solution for the problem (17) with $F(t,y)$ given by (18) on $[0,2]$.

In order to demonstrate the application of Theorem 4.10, let us choose

$$F(t,y(t)) = \left[0\,,\ \frac{(t+1)\tan^{-1}x+t}{40}\right]. \tag{19}$$

Clearly, $H_d(F(t,y),F(t,\bar{x})) \leq (t+1)\|x-\bar{x}\|$. Letting $m(t)=(t+1)$, it is easy to check that $d(0,F(t,0)) \leq m(t)$ holds for almost all $t \in [0,2]$ and that $\varpi \leq 0.865783$ (ϖ is given by (14)). As the hypotheses of Theorem 4.10 are satisfied, we conclude that the problem (17) with $F(t,x)$ given by (19) has at least one solution on $[0,2]$.

6. Some analogous problems

In this section, we present some problems analogue to (1) by replacing $x(0)=0$ with the nonlocal condition of the form $x(0)=\phi(y)$, $\phi : C(J,\mathbb{R}) \to \mathbb{R}$ is given function, and the condition $(D^{\gamma}x)(T)=\lambda(I^{\delta}x)(T)$ with Riemann-Liouville type integral and discrete multipoint boundary conditions of the form:

$$(i)\,D^p y(\eta) = \sum_{i=1}^{n} \lambda_i\, I^{\gamma_i} y(\xi_i),\ (ii)\ D^p y(\eta) = \sum_{i=1}^{n} \lambda_i\, y(\xi_i),\ \eta,\ \xi_i \in (0,T).$$

Precisely we consider the problems:

$$\begin{cases} D^{\alpha}[D^{\beta}y(t)-g(t,y(t))] = f(t,y(t)),\ t \in J := J, \\ y(0)=\phi(y),\ D^p y(\eta) = \sum_{i=1}^{n} \lambda_i\, I^{\gamma_i} y(\xi_i),\ 0<p<1,\ \gamma_i>0,\ \eta,\xi_i \in (0,T), \end{cases} \tag{20}$$

and

$$\begin{cases} D^{\alpha}[D^{\beta}y(t)-g(t,y(t))] = f(t,y(t)),\ t \in J := J, \\ y(0)=\phi(y),\ D^p y(\eta) = \sum_{i=1}^{n} \lambda_i\, y(\xi_i),\ 0<p<1,\ \eta,\xi_i \in (0,T). \end{cases} \tag{21}$$

Using the boundary conditions given by (20) in (7) together with the expressions:

$$\begin{aligned} I^{\gamma_i}y(\xi_i) &= I^{\beta+\gamma_i}g(s,y(s))(\xi_i) + I^{\alpha+\beta+\gamma_i}f(s,y(s))(\xi_i) + \frac{\xi_i^{\beta+\gamma_i}}{\Gamma(\beta+\gamma_i+1)}c_1 \\ &\quad + \frac{\xi_i^{\gamma_i}}{\Gamma(\gamma_i+1)}c_2 \end{aligned}$$

and

$$D^p y(\eta) = I^{\beta-p}g(s,y(s))(\eta) + I^{\alpha+\beta-p}f(s,y(s))(\eta) + \frac{\eta^{\beta-p}}{\Gamma(\beta-p+1)}c_1,$$

the solution of the problem (20) is found to be

$$
\begin{aligned}
y(t) &= I^{\beta} g(s,y(s))(t) + I^{\alpha+\beta} f(s,y(s))(t) + \phi(y) \\
&+ \frac{t^{\beta}}{\Gamma(\beta+1)} \frac{1}{\Omega} \Bigg\{ \sum_{i=1}^{n} \lambda_i \, I^{\beta+\gamma_i)} g(s,y(s))(\xi_i) + \sum_{i=1}^{n} \lambda_i \xi_i^{\gamma_i} \frac{\phi(y)}{\Gamma(1+\gamma_i)} \\
&+ \sum_{i=1}^{n} \lambda_i \, I^{\alpha+\beta+\gamma_i} f(s,y(s))(\xi_i) - I^{\beta-p} g(s,y(s))(\eta) \\
&- I^{\alpha+\beta-p} f(s,y(s))(\eta) \Bigg\},
\end{aligned} \tag{22}
$$

where

$$
\Omega := \frac{\eta^{\beta-p}}{\Gamma(1+\beta-p)} - \sum_{i=1}^{n} \frac{\lambda_i}{\Gamma(\beta+\gamma_i+1)} \xi_i^{\beta+\gamma_i} \neq 0. \tag{23}
$$

In a similar manner, we can find the solution of the problem (21) which is given by

$$
\begin{aligned}
y(t) &= I^{\beta} g(s,y(s))(t) + I^{\alpha+\beta} f(s,y(s))(t) + \phi(y) \\
&+ \frac{t^{\beta}}{\Gamma(\beta+1)} \frac{1}{\Omega_1} \Bigg\{ \sum_{i=1}^{n} \lambda_i \, I^{\beta} g(s,y(s))(\xi_i) + \phi(y) \sum_{i=1}^{n} \lambda_i \\
&+ \sum_{i=1}^{n} \lambda_i \, I^{\alpha+\beta} f(s,y(s))(\xi_i) - I^{\beta-p} g(s,y(s))(\eta) - I^{\alpha+\beta-p} f(s,y(s))(\eta) \Bigg\},
\end{aligned} \tag{24}
$$

where

$$
\Omega_1 := \frac{\eta^{\beta-p}}{\Gamma(1+\beta-p)} - \frac{1}{\Gamma(\beta+1)} \sum_{i=1}^{n} \lambda_i \xi_i^{\beta} \neq 0. \tag{25}
$$

The existence results for the problems (20) and (21) can be obtained by following the methodology employed in Section 3, while the inclusions case for the problems (20) and (21) can be studied by employing the strategy used in Section 4.

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