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ON A MULTIPOINT FRACTIONAL BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITIONS

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Abstract. In this paper, we study a nonlinear higher order fractional differential equation with initial and integral conditions. By constructing the lower and upper solutions and applying the Schauder fixed point theorem, we prove the existence of positive solutions.

Keywords. Fractional differential equation; Boundary value problem; Positive solution; Fixed point theorem; Lower and upper solutions method.

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1. Introduction

Fractional differential equations as generalization of differential equations of integer order can describe many phenomena in different fields of applied sciences and engineering such as viscoelasticity, rheology, thermodynamics, biosciences, bioengineering, etc, see [1, 2] and the references therein. Several methods are involved in the study of the existence of solutions, we can mention the upper and lower solutions method, the theory of Mawhin and the method of

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successive approximation. In particular, some fixed point theorems, such the Schauder fixed point theorem, nonlinear alternative of the Leray-Schauder and the Guo-Krasnoselski theorem are used in the study of the existence of solutions or positive solutions for boundary value problems for nonlinear fractional differential equations; see [3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein.

The upper and lower solutions method allows us to prove not only the existence of a solution of the considered problem but to get also information on its localization. In fact, by the use of this method, we ensure the existence of the solution between the lower and the upper solutions. So the most important step of this method is to find two well-ordered functions that satisfy some appropriate inequalities.

This work concerns the existence of positive solutions for the following boundary value problem for nonlinear fractional differential equation**.** Let

$$D^{\alpha}y(t)+f(t,y(t))=0, t\in(0,1) \tag{1.1}$$

and

$$\begin{aligned} &y^{(i)}(0)=0, \quad i=0,...,n-2,\\ &y(1)=\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}y(s)\,ds, \end{aligned} \tag{1.2}$$

where $f\in C((0,1),\mathbb{R}_+)$ is a given function, $n-1\leq\alpha\leq n$, $n\in\mathbb{N}$, $n\geq 2$, $0<\eta_k<1$, $\lambda_k>0$, $k=0,...,m$. Note $\xi=1-\frac{1}{\alpha}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}>0$.

The proofs are based on the lower and upper solutions and the Schauder fixed point theorem. The organization of the paper is as follows. In Section 2, we give some definitions and lemmas to prove our main results. In Section 3, we construct the lower and upper solutions and establish the existence of at least one positive solution for boundary value problem (1.1)-(1.2) between these two functions. The obtained results are illustrated by an example.

2. Preliminaries

For the convenience of the reader, we give some background materials from the fractional calculus theory to facilitate the analysis of problem (1.1)-(1.2), that can be found in [2, 11].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $h : (0,+\infty) \to \mathbb{R}$ is given by $I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds$ provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $h : (0,+\infty) \to \mathbb{R}$ is given by

$$D^{\alpha} h(t) = \frac{1}{\Gamma(\alpha-1)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}} ds = \left(\frac{d}{dt}\right)^n I_{0^+}^{n-\alpha} h(t),$$

provided that the right side is pointwise defined on $(0,+\infty)$, where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of α.

Lemma 2.1. *Let $\alpha > 0$. Then the fractional differential equation $D^{\alpha} u(t) = 0$ has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} + \ldots + C_n t^{\alpha-n}$, $C_i \in \mathbb{R}, i = 1,2,\ldots,n$ as a solution.*

Lemma 2.2. *Assume that $h \in C(0,1) \cap L^1(0,1)$ and $n-1 \le \alpha \le n$, $n \ge 2$. Then the solution to boundary value problem*

$$D^{\alpha} y(t) + h(t) = 0, \quad t \in (0,1), \tag{2.1}$$

$$\begin{aligned} & y^{(i)}(0) = 0, \quad i = 0,\ldots,n-2, \\ & y(1) = \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} y(s)\, ds,\ \lambda_k > 0 \end{aligned} \tag{2.2}$$

is given by

$$y(t) = \int_0^1 G(t,s) h(s)\, ds + \frac{t^{\alpha-1}}{\xi} \sum_{k=0}^{m} \lambda_k \int_0^1 H(\eta_k, s) h(s)\, ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right], & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

$$H(t,s) = \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[t^{\alpha}(1-s)^{\alpha-1} - (t-s)^{\alpha}\right], & 0 \le s \le t \le 1, \\ t^{\alpha}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Proof. Let y be a solution of problem (2.1)-(2.2). By Lemma 2.1, we have

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \ldots + c_n t^{\alpha-n} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds. \tag{2.3}$$

Taking conditions (2.2) into account, it yields $c_2 = c_3 = ... = c_n = 0$, and

$$\begin{aligned} y(1) &= c_1 - I^{\alpha}h(1) = \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} y(s)\,ds = \sum_{k=0}^{m} \lambda_k \left(-I^{\alpha+1}h(\eta_k) + \tfrac{c_1}{\alpha}\eta_k^{\alpha}\right) \\ &= -\sum_{k=0}^{m} \lambda_k I^{\alpha+1} h(\eta_k) + c_1 \sum_{k=0}^{m} \tfrac{\lambda_k}{\alpha}\eta_k^{\alpha}, \end{aligned}$$

which implies

$$c_1 = \frac{1}{\xi\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\alpha-1} h(s)\,ds - \frac{1}{\alpha}\sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^{\alpha} h(s)\,ds\right).$$

Hence the solution of problem (2.1)-(2.2) is

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds \\ &+ \frac{t^{\alpha-1}}{\xi\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-1} h(s)\,ds - \frac{1}{\alpha}\sum_{k=0}^{m}\lambda_k \int_0^{\eta_k} (\eta_k - s)^{\alpha} h(s)\,ds\right]. \end{aligned}$$

This implies that

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds \\ &+ \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + t^{\alpha-1}\frac{\frac{1}{\alpha}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}}{\Gamma(\alpha)\left(1 - \frac{1}{\alpha}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)}\right)\int_0^1 (1-s)^{\alpha-1} h(s)\,ds \\ &- \frac{t^{\alpha-1}\sum_{k=0}^{m}\lambda_k}{\xi\alpha\Gamma(\alpha)}\int_0^{\eta_k} (\eta_k - s)^{\alpha} h(s)\,ds \\ = &\frac{1}{\Gamma(\alpha)}\int_0^t \left(t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right) h(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1} h(s)\,ds \\ &+ \frac{t^{\alpha-1}}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}\left(\eta_k^{\alpha}(1-s)^{\alpha-1} - (\eta_k - s)^{\alpha}\right) h(s)\,ds \\ &+ \frac{t^{\alpha-1}}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\int_{\eta_k}^1 \eta_k^{\alpha}(1-s)^{\alpha-1} h(s)\,ds \\ = &\int_0^1 G(t,s) h(s)\,ds + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k, s) h(s)\,ds. \end{aligned}$$

The proof is completed.

We analyze functions G and H to get their following properties:

Lemma 2.3. *The functions G and H are continuous nonnegative on $[0,1]\times[0,1]$ and satisfy $G(t,s)\leq\frac{1}{\Gamma(\alpha)}$ and $H(t,s)\leq\frac{t^{\alpha}}{\Gamma(\alpha+1)}$, for all $t,s\in[0,1]$.*

Now we define the concept of upper and lower solutions for fractional boundary value problem (1.1)-(1.2).

Definition 2.3. A function $\beta\in C[0,1]$ is called a lower solution of the fractional boundary value problem (1.1)-(1.2), if

$$-D^{\alpha}\beta(t)\leq f(t,\beta(t)),\quad t\in(0,1),$$

and

$$\begin{aligned}&\beta^{(i)}(0)\leq 0,\quad i=0,\dots,n-2,\\&\beta(1)\leq\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}\beta(s)\,ds.\end{aligned}$$

Definition 2.4. A function $\gamma\in C[0,1]$ is called a upper solution of the fractional boundary value problem (1.1)-(1.2), if

$$-D^{\alpha}\gamma(t)\geq f(t,\gamma(t)),\quad t\in(0,1),$$

and

$$\begin{aligned}&\gamma^{(i)}(0)\geq 0,\quad i=0,\dots,n-2,\\&\gamma(1)\geq\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}\gamma(s)\,ds,\ \lambda_k>0.\end{aligned}$$

3. Main results

Define an operator $F:C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ by

$$Fy(t)=\int_0^1 G(t,s)f(s,y(s))\,ds+\frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)f(s,y(s))\,ds.$$

Then y is a solution of problem (1.1)-(1.2) if and only if y is a fixed point of F. Let K be the cone $K=\{y\in C[0,1], y(t)\geq 0, t\in[0,1]\}$. Note that

$$\begin{aligned} p(t) &= \int_0^1 G(t,s)\,ds+\frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)\,ds,\\ &= \frac{-t^{\alpha}+t^{\alpha-1}}{\Gamma(\alpha+1)}+\frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\frac{\lambda_k}{\Gamma(\alpha+1)}\left(\frac{-\eta_k^{\alpha+1}}{\alpha+1}+\frac{\eta_k^{\alpha}}{\alpha}\right)\\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha+1)}\left[(1-t)+\frac{1}{\xi}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha}-\frac{\eta_k}{\alpha+1}\right)\right]. \end{aligned}$$

The function

$$g(t)=\int_0^1 G(t,s)f(s,p(s))\,ds+\frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)f(s,p(s))\,ds, t\in[0,1]$$

is a positive solution of the following problem

$$-D^{\alpha}g(t)=f(t,p(t)),\quad t\in(0,1), \tag{3.1}$$

and

$$\begin{aligned} &g^{(i)}(0)=0,\quad \forall i=0,...,n-2,\\ &g(1)=\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}g(s)\,ds. \end{aligned} \tag{3.2}$$

Consequently, one has

$$a_1p(t)\leq g(t)\leq a_2p(t),\qquad t\in[0,1], \tag{3.3}$$

where

$$a_1=\min\left\{1,\min_{t\in[0,1]}f(t,p(t))\right\},\ \ a_2=\max\left\{1,\max_{t\in[0,1]}f(t,p(t))\right\}. \tag{3.4}$$

We have the following result.

Theorem 3.1. *Assume that following conditions are satisfied*

$(H1)$ *There exist a function* $\varphi\in L^1([0,1],\mathbb{R}_+)$ *and a continuous nondecreasing function* $\psi:\mathbb{R}_+\to\mathbb{R}_+$, *such that*

$$f(t,y)\leq\varphi(t)\,\psi(|y|), t\in[0,1], y\in\mathbb{R}.$$

$(H2)$ *There exists a constant* $\rho > 0$ *such that*

$$\psi(\rho)\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)\|\varphi\|_{L^1} < \rho. \tag{3.5}$$

$(H3)$ $f(t,p(t)) \neq 0,$ *for* $t \in [0,1]$ *and there exists a constant* μ, $0<\mu<1$ *such that for all* k, $0<k<1$, *we have*

$$k^{\mu} f(t,u) \leq f(t,ku),\ u \in \mathbb{R}_+. \tag{3.6}$$

Then problem (1.1)-(1,2) has at least one positive solution $y \in C[0,1]$ *satisfying*

$$\beta(t) \leq y(t) \leq \gamma(t), t \in [0,1].$$

Moreover, β *and* γ *are respectively lower and upper solution of problem (1.1)-(1.2), where*

$$\beta(t) = k_1 g(t), \gamma(t) = k_2 g(t), \tag{3.7}$$

$$k_1 = \min(1,r)k_3,\ k_2 = \max(1,R)k_4, \tag{3.8}$$

$$r = \min(f(t,y(t)), t \in [0,1], \|y\| \leq \rho),$$
$$R = \max(f(t,y(t)), t \in [0,1], \|y\| \leq \rho)$$

and

$$k_3 = \min\left(\frac{1}{a_2}, a_1^{\frac{\mu}{1-\mu}}\right),\ k_4 = \max\left(\frac{1}{a_1}, a_2^{\frac{\mu}{1-\mu}}\right). \tag{3.9}$$

Proof. Let us prove that F is completely continuous operator. Letting

$$y \in B_{\rho} = \{y \in K : \|y\| < \rho\},$$

we have

$$Fy(t) \leq \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,y(s))\,ds$$
$$+ \frac{1}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\int_0^1 (1-s)^{\alpha-1} f(s,y(s))\,ds.$$

Applying Condition $(H1)$, one has

$$Fy(t) \leq \psi(\rho)\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)\int_0^1 (1-s)^{\alpha-1}\varphi(s)\,ds.$$

It follows that

$$\|Fy\| \leq \psi(\rho)\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)\|\varphi\|_{L^1}$$

and consequently F is uniformly bounded on B_ρ and $F\left(B_\rho\right)\subset B_\rho$.

Let $t_1,t_2\in[0,1], t_1<t_2$. Then

$$\begin{aligned}
|Fy(t_2)-Fy(t_1)| \leq\ & \frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,y(s))\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]f(s,y(s))\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))\,ds\\
&+\frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)f(s,y(s))\,ds.
\end{aligned}$$

Thanks to Condition $(H1)$, we obtain

$$\begin{aligned}
&|Fy(t_2)-Fy(t_1)|\\
&\leq \psi(\rho)\|\varphi\|_{L^1}\left(\frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\Gamma(\alpha)}+\frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\Gamma(\alpha)}+\frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\right.\\
&\quad\left.+\frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)\\
&\leq\frac{\psi(\rho)\|\varphi\|_{L^1}\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\Gamma(\alpha)}\left(2+\frac{1}{\alpha\xi}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)+\frac{\psi(\rho)\|\varphi\|_{L^1}(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)},
\end{aligned}$$

that tends to 0 as $t_2\to t_1$. Hence $F\left(B_\rho\right)$ is equicontinuous. By means of the the Arzela-Ascoli Theorem, F is completely continuous. Applying the Schauder fixed point Theorem, it follows that F has a fixed point $y\in B_\rho$. Let us remark that the solution y satisfies

$$rp(t)\leq y(t)\leq Rp(t),\ \forall t\in(0,1).$$

Now we prove that $\beta(t)\leq y(t)\leq\gamma(t)$, $t\in[0,1]$. Combining (3.3) and (3.7), we get the following estimates for $t\in(0,1)$

$$k_1a_1\leq\frac{\beta(t)}{p(t)}\leq k_1a_2, \tag{3.10}$$

$$\frac{1}{k_2a_2} \leq \frac{p(t)}{\gamma(t)} \leq \frac{1}{k_2a_1}. \tag{3.11}$$

Furthermore (3.8) implies

$$k_3a_2 \leq 1,\ k_4a_1 \geq 1. \tag{3.12}$$

Let $t \in (0,1)$. From (3.6), (3.10)-(3.12), we obtain

$$\frac{\beta(t)}{p(t)} \leq k_1a_2 \leq \min(1,r)\,k_3a_2 \leq r,$$

and

$$\frac{p(t)}{\gamma(t)} \leq \frac{1}{k_2a_1} \Rightarrow \frac{\gamma(t)}{p(t)} \geq k_2a_1 = \max(R,1)\,k_4a_1 \geq R.$$

Hence

$$\beta(t) \leq rp(t),\ \gamma(t) \geq Rp(t) \text{ for any } t \in [0,1]. \tag{3.13}$$

From (3.7) and (3.13), one has

$$\beta(t) \leq y(t) \leq \gamma(t) \text{ for any } t \in [0,1].$$

Finally, we shall prove that $\beta(t) = k_1g(t),\ \gamma(t) = k_2g(t)$ are respectively lower and upper solutions of problem (1.1)-(1.2). Thanks to (3.10), we get the following estimates

$$(k_3a_1)^{\mu} \geq k_3 \text{ and } (k_4a_2)^{\mu} \leq k_4,$$

$$(k_1a_1)^{\mu} \geq k_1 \text{ and } (k_2a_2)^{\mu} \leq k_2. \tag{3.14}$$

Using (3.6), (3.10) and (3.14), we get, for any $t \in (0,1)$, that

$$\begin{aligned} f(t,\beta(t)) &= f\left(t, \tfrac{\beta(t)}{p(t)}p(t)\right) \\ &\geq \left(\tfrac{\beta(t)}{p(t)}\right)^{\mu} f(t,p(t)) \\ &\geq (k_1a_1)^{\mu} f(t,p(t)) \\ &\geq k_1 f(t,p(t)), \end{aligned} \tag{3.15}$$

and

$$
\begin{aligned}
k_2 f(t,p(t)) &= k_2 f\left(t, \tfrac{p(t)}{\gamma(t)}\gamma(t)\right) \\
&\geq k_2 \left(\tfrac{p(t)}{\gamma(t)}\right)^{\mu} f(t,\gamma(t)) \\
&\geq k_2 (k_2 a_2)^{-\mu} f(t,\gamma(t)) \\
&\geq f(t,\gamma(t)).
\end{aligned} \tag{3.16}
$$

Consequently, one has

$$
\begin{cases}
-D^{\alpha}\beta(t) = k_1 f(t,p(t)) \leq f(t,\beta(t)), & t \in (0,1), \\
-D^{\alpha}\gamma(t) = k_2 f(t,p(t)) \geq f(t,\gamma(t)), & t \in (0,1),
\end{cases}
$$

$$
\begin{aligned}
&\beta^{(i)}(0) \leq 0, \quad \forall i = 0,\dots,n-2, \\
&\beta(1) \leq \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} \beta(s)\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\gamma^{(i)}(0) \geq 0, \quad \forall i = 0,\dots,n-2, \\
&\gamma(1) \geq \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} \gamma(s)\,ds.
\end{aligned}
$$

Thus $\beta(t) = k_1 g(t)$, $\gamma(t) = k_2 g(t)$ are respectively lower and upper solutions of problem (1.1)-(1.2). The proof of Theorem 3.1 is completed.

Now, we give an example to illustrate the obtained results.

Example 3.1. Consider fractional boundary value problem (1.1)-(1.2) with

$$
\begin{aligned}
\alpha &= 2.5, \lambda_1 = 0.25, \lambda_2 = 0.75, \eta_1 = 0.5, \eta_2 = 0.25, \\
f(t,y) &= 1 + t + \frac{1}{100}\left(\frac{\Gamma(\alpha+1)|y|}{(1-t) + \frac{1}{\xi}\sum_{k=0}^{m} \lambda_k \eta_k^{\alpha}\left(\frac{1}{\alpha} - \frac{\eta_k}{\alpha+1}\right)}\right)^{\frac{1}{2}}.
\end{aligned}
$$

We denote by (P). Then

$$
\begin{aligned}
p(t) &= 0.3009 t^{\frac{3}{2}} (1.019595 - t), \\
f(t,p(t)) &= 1 + t + 0.01 t^{\frac{3}{4}}, \\
\xi &\approx 0.97295 > 0.
\end{aligned}
$$

For $\mu=\frac{1}{2}$ and for $0<k<1$, it is easy to verify that

$$\begin{aligned} k^{\frac{1}{2}}f(t,y) = &\ k^{\frac{1}{2}}+k^{\frac{1}{2}}t+\tfrac{1}{100}\left(\frac{\Gamma(\alpha+1)k|y|}{(1-t)+\frac{1}{\xi}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha}-\frac{\eta_k}{\alpha+1}\right)}\right)^{\frac{1}{2}} \\ &\le 1+t+\tfrac{1}{100}\left(\frac{\Gamma(\alpha+1)k|y|}{(1-t)+\frac{1}{\xi}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha}-\frac{\eta_k}{\alpha+1}\right)}\right)^{\frac{1}{2}} \\ &\le f(t,ky). \end{aligned}$$

Moreover

$$\begin{aligned} f(t,y(t)) &\le 2\left(1+\frac{1}{200}\left(\frac{\Gamma(\alpha+1)}{(1-t)+\frac{1}{\xi}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha}-\frac{\eta_k}{\alpha+1}\right)}\right)^{\frac{1}{2}}|y|^{\frac{1}{2}}\right) \\ &\le 2\left(1+0.0751886\,|y|^{\frac{1}{2}}\right)=\varphi(t)\,\psi(|y|). \end{aligned}$$

It follows that

$$\varphi(t)=2,\psi(|y|)=\left(1+0.0751886\,|y|^{\frac{1}{2}}\right).$$

If we choose $\rho=2$, then

$$\psi(\rho)\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)\|\varphi\|_{L^1}-\rho=-0.3008<0.$$

Hence, (3.5) is satisfied. Since $f(t,p(t))$ is increasing in t, one has

$$\begin{aligned} \min_{t\in[0,1]} f(t,p(t)) &= \min_{t\in[0,1]}\left(1+t+10^{-2}t^{\frac{3}{4}}\right)=1, \\ \max_{t\in[0,1]} f(t,p(t)) &= \max_{t\in[0,1]}\left(1+t+10^{-2}t^{\frac{3}{4}}\right)=2.01. \end{aligned}$$

Hence, $f(t,p(t))\neq 0$. Since all conditions of Theorem 3.1 are satisfied, one sees that the fractional boundary value problem (P) has at least one positive solution such that $\beta(t)\le y(t)\le \gamma(t)$, $t\in[0,1]$. Let us find the explicit forms of the functions g, β and γ. By computation, we get

$$\begin{aligned} a_1 &= 1,\ a_2=2.01, r=1, R=2.184174, k_1=k_3=0.49751, \\ k_4 &= 2.01,\ k_2=4.390189, \end{aligned}$$

$$g(t)=0.36506t^{1.5}-0.75225t^{2.5}\left(0.4+0.11429t+0.84263\times 10^{-3}t^{0.75}\right),$$

$$\begin{aligned}\beta(t) &= 0.181621t^{1.5}-0.374251t^{2.5}\left(0.11429t+1.4746\times 10^{-3}t^{0.75}+0.4\right),\\ \gamma(t) &= 1.602682t^{1.5}-3.302519t^{2.5}\left(0.11429t+1.4746\times 10^{-3}t^{0.75}+0.4\right).\end{aligned}$$

Finally, we show that the functions $\beta(t)=k_1g(t)$ and $\gamma(t)=k_2g(t)$ are respectively lower and upper solutions of problem (P). In fact, since function g is solution of problem (3.1)-(3.2), we find from (3.15) that

$$-D^{\alpha}\beta(t)=-k_1D^{\alpha}g(t)=k_1f(t,p(t))\leq f(t,\beta(t))$$

and

$$\begin{aligned}&\beta^{(i)}(0)=k_1g^{(i)}(0)=0,\quad \forall i=0,...,n-2,\\ &\beta(1)=k_1g(1)=\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}k_1g(s)\,ds=\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}\beta(s)\,ds.\end{aligned}$$

Similarly, by using (3.16), we show that $\gamma(t)=k_2g(t)$ is an upper solution of problem (P).

Finally, we remark that if y is an approximate solution of problem (P), then the error is approximative 0.4. This shows that the upper and lower solutions β and γ are good approximations of the solution of problem (P).

References

[1] K. Diethelm, The Analysis of Fractional Differential Equations, vol. 2004 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 2010.

[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[3] M. Benchohra, J.J. Nieto, A. Ouahab, Existence results for functional integral inclusions of Volterra type, Dynam. Systems Appl. 14 (2005), 57-69.

[4] C. De Coster, P. Habets, Two-Point Boundary Value Problems: Lower and Upper Solutions Elsevier, Amsterdam, 2006.

[5] D. Franco, J.J. Nieto, D. O'Regan, Upper and lower solutions for first order problems with nonlinear boundary conditions, Extracta Math. 18 (2003), 153-160.

[6] A. Guezane-Lakoud, R. Khaldi, D.F.M. Torres, On a fractional oscillator equation with natural boundary conditions, Progr. Fract. Differ. Appl. 3 (2017), 191-197.

[7] R. Khaldi, A. Guezane-Lakoud, Upper and lower solutions method for higher order boundary value problems, Progr. Fract. Differ. Appl. 3 (2017), 1-5.

[8] X. Liu, M. Jia, W. Ge, The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator, Appl. Math. Lett. 65 (2017), 56-62.

[9] J.J. Nieto, Generalized quasilinearization method for a second order ordinary differential equation with Dirichlet boundary conditions, Proc. Amer. Math. Soc. 125 (1997), 2599-2604.

[10] J.J. Nieto, R. Rodríguez-López, Monotone method for first-order functional differential equations, Comput. Math. Appl. 52 (2006), 471-484.

[11] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.