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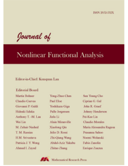

GLOBAL ANALYSIS OF A RATIO-DEPENDENT PREDATOR-PREY SYSTEM INCORPORATING A PREY REFUGE

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Abstract. In this paper, we consider a ratio-dependent predator-prey system incorporating a prey refuge. The qualitative behavior of the system at the origin in the interior of the first quadrant is studied. Using the prey refuge m as control, the global qualitative analysis of the system and conditions of the existence and the non-existence of limit cycles for the system is obtained.

Keywords. Predator-prey system; Ratio-dependent response; Attractor; Refuge; Divergency criterion.

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1. Introduction

The traditional prey-dependent predator-prey models have been challenged by several biologists, such as, Arditi and Ginzburg [1], Akcakaya [2], Gutierrez [3], based on the fact that functional responses over typical ecological time scales ought to depend on the densities of both prey and predators, especially when predators have to search for food and therefore they have to share or compete for food. Such a functional response is called a ratio-dependent response function. Based on the Michaelis-Menten function, Arditi and Ginzburg [1] proposed

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the following ratio-dependent predator-prey model

$$\dot{x} = x(a - bx) - \frac{cxy}{ny + x}, \quad \dot{y} = y\Big(-d + \frac{fx}{ny + x}\Big), \tag{1.1}$$

where $x(t)$ and $y(t)$ represent population densities of the prey and the predator at time t, respectively. $\frac{a}{b} > 0$ is the carrying capacity of the prey, $d > 0$ is the death rate of the predator, and a, c, n and f are positive constants that stand for the prey intrinsic growth rate, capturing rate, half-saturation constant, and conversion rate, respectively.

There are some arguments (see Berry [18], Lundberg and Fryxell [19]) about the advantages of the ratio-dependent versus prey-dependent models. Differing from the prey-dependent predator-prey models, the ratio-dependent predator-prey systems have two principal predictions: (a) equilibrium abundances are positively correlated along a gradient of enrichment (see Arditi and Ginzburg [1] and (b) the ”paradox of enrichment” (see Rosenzweig [20]) either completely disappears or enrichment is linked to the stability in a more complex way.

However, system (1.1) is not well-defined at the origin $(0,0)$ and thus cannot be linearized at $(0,0)$. It sets up a challenging issue regarding its dynamics near the origin since it is not well-defined there. Because the Jacobian matrix at $(0,0)$ is a zero matrix, there are some difficulties to analyze the singularity at the origin of system (1.1). Several researchers have studied system (1.1) recently. Freedman and Mathsen [4] restricted their analysis to parameter values that ensure the equilibrium $(0,0)$ behaves like a saddle point and established conditions for the persistence of the model. Jost, Arino and Arditi [5] studied the analytical behavior at $(0,0)$ for a general ratio-dependent predator-prey model and showed that this equilibrium can be either a saddle point or an attractor for certain trajectories. Kuang and Beretta [6] considered the global behaviors of solutions of system (1.1). Hsu, Hwang and Kuang [8] obtained a complete classification of the asymptotic behavior of the solutions of the Michaelis-Menten type ratio-dependent predator-prey model. Motivated by the above work (Freedman and Mathsen [4], Jost, Arino and Arditi [5], Kuang and Beretta [6]), Xiao and Ruan [7] redefined system (1.1) at $(0,0)$) as

$$\begin{aligned} &\dot{x} = x(a - bx) - \frac{cxy}{ny + x}, \\ &\dot{y} = y\Big(-d + \frac{fx}{ny + x}\Big), \\ &\dot{x} = \dot{y} = 0 \ \text{ when } \ (x, y) = (0,0) \end{aligned} \tag{1.2}$$

and they found that $(0,0)$ has its own basin of the attraction at the first quadrant, which is important for the global behavior of the model and observed the interesting dynamic behavior such as deterministic extinction, the existence of multiple attractors, and the existence of a limit cycle.

In this paper, we extend system (1.2) by incorporating a refuge protecting mx of the prey, where $m \in [0,1)$ is a constant number of the prey using refuges. This leaves $(1-m)x$ of the prey available to the predator, and modifies system (1.2) accordingly yielding the system:

$$\begin{aligned} \dot{x} &= x(a-bx) - \frac{c(1-m)xy}{ny+(1-m)x}, \\ \dot{y} &= y\Big(-d + \frac{f(1-m)x}{ny+(1-m)x}\Big), \\ \dot{x} &= \dot{y} = 0 \ \text{ when } \ (x,y) = (0,0). \end{aligned} \tag{1.3}$$

We study system (1.3) based on the techniques, which had been well taken in Xiao and Ruan [7], Hwang [9, 10]. Many scientists have paid more attention to the effects of a prey refuge to a predator-prey system. McNair [12] showed that several kinds of refuges could exert a locally destabilizing effect and create stable, large amplitude oscillations which could damp out if no refuge was present. Even now, prey refuges are widely believed to prevent prey extinction and damp predator-prey oscillations. For example, Kar [11] and Huang, Chen and Li *et al.* [13] considered an Lotka-Volterra type predator-prey system incorporating a constant proportion of the prey using refuges mx, which protects mx of prey from predation, with the Holling type II response function and the Holling type III response function, respectively. They all pointed that increasing the amount of the refuge could increase prey densities and lead to population outbreaks. For more biological backgrounds and results on the effects of a prey refuge, one refers to [14, 15, 16] and the references therein.

This paper is organized as follows. In Section 2, the dynamical analysis and boundedness of system (1.3) are obtained. We also study the singularity $(0,0)$ of system (1.3) and give all possibilities for the orbits of system (1.3) to approach $(0,0)$ as $t \to +\infty$ or $t \to -\infty$ depending on all parameters in the interior of the first quadrant. In Section 3, the existence and the stability of the equilibria of system (1.3) except $(0,0)$ are discussed, which contains that the local asymptotic stability of the positive steady state implies the nonexistence of nontrivial positive periodic solutions. It also contains the uniqueness of limit cycles. In Section 4, the global qualitative analysis of system (1.3) is carried out, which contains some results on the global stability of the positive steady state and the existence of multiple attractors and a limit cycle of system (1.3). In Section 5, the global analysis of system (1.3) is summarized and classified into a table depending on the parameters m and f, and some numerical simulations illustrate the established results. Concluding remarks are presented in Section 6.

2. Dynamical analysis of system (1.3)

Letting $I^+ = \{(x,y)|x > 0, y > 0\}$, for practical biological meaning, we simply study system (1.3) in I^+ or $\bar{I^+}$. From the first equation of system (1.3), it is easy to derive $\limsup_{t\longrightarrow\infty} x(t) \leq \frac{a}{b}$.

Lemma 2.1. *The solution $x(t), y(t)$ of system (1.3) with initial values $x(0) > 0, y(0) > 0$ are positive and bounded for all $t \geq 0$.*

Proof. Obviously the solution $x(t), y(t)$ of system (1.3) with initial values $x(0) > 0, y(0) > 0$ are positive for all $t \geq 0$. Define the function $\omega(x,y) = x + \frac{c}{f}y$. Given any $\varepsilon > 0, x(t) \leq \frac{a}{b} + \varepsilon$ for t sufficiently large, one finds from system (1.3) that

$$\dot{\omega} = x(a - bx) - \frac{cdy}{f} \leq -\min\{a, \frac{df}{c}\}\omega + 2a(\frac{a}{b} + \varepsilon).$$

Then $\omega \leq \frac{2a(\frac{a}{b}+\varepsilon)}{min\{a,\frac{df}{c}\}}$, for t sufficiently large. This completes the proof.

In the following, we study the orbits of system (1.3) to approach $(0,0)$. Our method is the same one elegantly applied by Xiao and Ruan [7] on system (1.2) and their success inspires us.

We can make a time scale change $dt = (ny + (1-m)x)d\tau$ such that system (1.3) is equivalent to the following system in I^+

$$\begin{aligned} \dot{x} &= a(1-m)x^2 + [an - (1-m)c]xy - nbx^2y - b(1-m)x^3 \equiv P_2(x,y) + \Phi(x,y), \\ \dot{y} &= (f-d)(1-m)xy - dny^2 \equiv Q_2(x,y), \end{aligned} \tag{2.1}$$

where P_2 and Q_2 are homogeneous polynomials in x and y of degree 2 and $\Phi(x,y) = -nbx^2y - b(1-m)x^3$. The equilibrium $(0,0)$ of system (2.1) is an isolated critical point of higher order. Obviously, system (2.1) is analytic in a neighborhood of the origin. By Theorem 3.10 on page

79 of [8], any orbit of (2.1) tending to the origin must tend to it spirally or along a fixed direction, which depends on the characteristic equation of system (2.1).

First of all, we introduce the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and define

$$G(\theta) = \cos\theta Q_2(\cos\theta, \sin\theta) - \sin\theta P_2(\cos\theta, \sin\theta).$$

Then the characteristic equation of system (2.1) takes the form

$$G(\theta) = \sin\theta\cos\theta[(f-d-a)(1-m)\cos\theta + ((1-m)c - n(a+d))\sin\theta]. \tag{2.2}$$

Clearly, either $G(\theta) = 0$ has a finite number of real roots or $G(\theta) \equiv 0$. By the results in section II. 2 in [8], we know that no orbit of system (2.1) can tend to the critical point $(0,0)$ spirally.

In the following, we discuss three cases according to the number of real roots to the characteristic equation (2.2) in $0 \le \theta \le \frac{\pi}{2}$.

2.1. $f-d-a=0$ and $(1-m)c-n(a+d)=0$

In this case, $G(\theta) \equiv 0$. Performing the Brio-Bouquet transformation [17] $y = ux$, system (2.1) in I^+ is transformed into

$$\begin{aligned} \dot{x} &= a(1-m)x^2 + [an-(1-m)c]x^2u - bx^3(1-m+nu), \\ \dot{u} &= bx^2u(1-m+nu). \end{aligned} \tag{2.3}$$

On the (u,x)-plane system (2.3) can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u}x + \frac{a}{bu} - \frac{(1-m)c}{b(1-m+nu)}. \tag{2.4}$$

Equation (2.4) has a general solution as follows

$$x = \frac{an-(1-m)c}{nb} + ku + \frac{(1-m)^2c}{n^2bu}\ln(1-m+nu),$$

where k is an arbitrary constant. So the general solution of system (2.1) in I^+ is

$$x = \frac{an-(1-m)c}{nb} + k\frac{y}{x} + \frac{(1-m)^2cx}{bn^2y}\ln\frac{ny+(1-m)x}{x},$$

when $f-d-a=0$ and $(1-m)c-n(a+d)=0$. The topological structure of the orbits of system (1.3) in I^+ is sketched in Figure 3 (a), which consists of an elliptic sector and a parabolic sector.

2.2. $(f-d-a)[(1-m)c-n(a+d)]=0$ but one of them is not zero

In this case, equation (2.2) has two roots in $0 \le \theta \le \frac{\pi}{2}$, $\theta_1 = 0$, and $\theta_2 = \frac{\pi}{2}$, and

$$\begin{aligned} G'(\theta) &= \sin\theta\cos\theta[((1-m)c - n(a+d))\cos\theta - (f-d-a)(1-m)\sin\theta] \\ &\quad + \cos 2\theta[(f-d-a)(1-m)\cos\theta + ((1-m)c - n(a+d))\sin\theta], \\ G''(\theta) &= 2\cos 2\theta[((1-m)c - n(a+d))\cos\theta - (f-d-a)(n-m)\sin\theta] \\ &\quad - 5\sin\theta\cos\theta[((1-m)c - n(a+d))\sin\theta + (f-d-a)(1-m)\cos\theta], \\ H(\theta) &= \sin\theta Q_2(\cos\theta, \sin\theta) + \cos\theta P_2(\cos\theta, \sin\theta) \\ &= a(1-m)\cos^3\theta - dn\sin^3\theta + \sin\theta\cos\theta[(f-d)(1-m)\sin\theta \\ &\quad + (an - (1-m)c)\cos\theta]. \end{aligned} \tag{2.5}$$

Case 1. $f-d-a \ne 0$ and $(1-m)c - n(a+d) = 0$.

In this case, θ_1 is a simple root of equation (2.2) and θ_2 is a multiple root with multiplicity 2 of equation (2.2). We have

Theorem 2.2. *Suppose that* $f-d-a \ne 0$ *and* $c(1-m) - n(a+d) = 0$. *Then*

(a) *there exist* $\varepsilon_1 > 0$ *and* $r_1 > 0$ *such that*

(i) *if* $f-d-a > 0$, *all orbits of system (2.1) in* $\{(\theta, r):\ 0 \le \theta < \varepsilon_1,\ 0 < r < r_1\}$ *tend to* $(0,0)$ *along* θ_1 *as* $t \to -\infty$;

(ii) *if* $f-d-a < 0$, *there exists a unique orbit of system (2.1) in* $\{(\theta, r):\ 0 \le \theta < \varepsilon_1,\ 0 < r < r_1\}$ *that tends to* $(0,0)$ *along* θ_1 *as* $t \to -\infty$; *and*

(b) *there exist* $\varepsilon_2 > 0$ *and* $r_2 > 0$ *such that all orbits of system (2.1) in* $\{(\theta, r):\ 0 \le \frac{\pi}{2} - \theta < \varepsilon_2,\ 0 < r < r_2\}$ *tend to* $(0,0)$ *along* θ_2 *as* $t \to +\infty$.

Proof. Note that $G'(\theta_1) = (f-d-a)(1-m)$ and $H(\theta_1) = a(1-m)$. If $f-d-a > 0$, then $G'(\theta_1)H(\theta_1) > 0$, by Theorem 4.4 of [8] there exist $\varepsilon_1 > 0$ and $r_1 > 0$ such that all orbits of system (3.1) in $\{(\theta, r):\ 0 \le \theta < \varepsilon_1,\ 0 < r < r_1\}$ tend to (0,0) along θ_1 as $t \to -\infty$. If $f-d-a < 0$, then $G'(\theta_1)H(\theta_1) < 0$. The conclusion follows from Theorem 4.7 of [8].

On the other hand, when $c(1-m) - n(a+d) = 0$ than $G'(\theta_2) = 0$ and $G''(\theta_2) = 2(f-d-a)(1-m)$, $H(\theta_2) = -dn$, that is $G''(\theta_2)H(\theta_2) \ne 0$, one finds from Theorem 4.8 [8] that there exist $\varepsilon_2 > 0$ and $r_2 > 0$ such that all orbits of system (2.1) in $\{(\theta, r):\ 0 \le \frac{\pi}{2} - \theta < \varepsilon_2,\ 0 < r < r_2\}$ tend to $(0,0)$ along θ_2 as $t \to +\infty$.

Case 2. $(f-d-a) = 0$ and $[c(1-m) - n(a+d)] \ne 0$.

In this case, θ_2 is a simple root of equation (2.2) and θ_1 is a multiple root with multiplicity 2 of equation (2.2). Similarly, we have the following.

Theorem 2.3. *Suppose that* $f-d-a=0$ *and* $c(1-m)-n(a+d)\neq 0$. *Then*

(a) *there exist* $\varepsilon_3>0$ *and* $r_3>0$ *such that all orbits of system (2.1)* $\{(\theta,r):\ 0\le\theta<\varepsilon_3,\ 0<r<r_3\}$ *tend to* $(0,0)$ *along* θ_1 *as* $t\to-\infty$*; and*

(b) *there exist* $\varepsilon_4>0$ *and* $r_4>0$ *such that*

(i) *if* $c(1-m)-n(a+d)>0$, *all orbits of system (2.1) in* $\{(\theta,r):\ 0\le\frac{\pi}{2}-\theta<\varepsilon_4,\ 0<r<r_4\}$ *tend to* $(0,0)$ *along* θ_2 *as* $t\to+\infty$*;*

(ii) *if* $c(1-m)-n(a+d)<0$, *there exists a unique orbit of system (2.1) in* $\{(\theta,r):\ 0\le\frac{\pi}{2}-\theta<\varepsilon_4,\ 0<r<r_4\}$ *tends to* $(0,0)$ *along* θ_2 *as* $t\to+\infty$.

2.3. $(f-d-a)[c(1-m)-n(a+d)]\neq 0$

In this case, we discuss two subcases because we only consider (2.2) in $0\le\theta\le\frac{\pi}{2}$.

(A) If $(f-d-a)[c(1-m)-n(a+d)]>0$, then equation (2.2) has two simple roots: $\theta_1=0$, $\theta_2=\frac{\pi}{2}$.

(B) If $(f-d-a)[c(1-m)-n(a+d)]<0$, then equation (2.2) has three simple roots: θ_1, θ_2 and $\theta_3=\arctan\frac{(a+d-f)(1-m)}{(1-m)c-n(a+d)}$. For the simple roots of (A) and (B): θ_1 and θ_2, we have the following theorem according to Theorems 4.4 and 4.7 in [18].

Theorem 2.4. *Assume that* $(f-d-a)[c(1-m)-n(a+d)]\neq 0$. *Then*

(a) *there exist* $\varepsilon_5>0$ *and* $r_5>0$ *such that*

(i) *if* $f-d-a>0$, *all orbits of system (2.1) in* $\{(\theta,r):\ 0\le\theta<\varepsilon_5,\ 0<r<r_5\}$ *tend to* $(0,0)$ *along* θ_1 *as* $t\to-\infty$*;*

(ii) *if* $f-d-a<0$, *there exists a unique orbit of system (2.1) in* $\{(\theta,r):\ 0\le\theta<\varepsilon_5,\ 0<r<r_5\}$ *tends to (0,0) along* θ_1 *as* $t\to-\infty$*; and*

(b) *there exist* $\varepsilon_6>0$ *and* $r_6>0$ *such that*

(i) *if* $c(1-m)-n(a+d)>0$, *all orbits of system (2.1) in* $\{(\theta,r):\ 0\le\frac{\pi}{2}-\theta<\varepsilon_6,\ 0<r<r_6\}$ *tend to (0,0) along* θ_2 *as* $t\to+\infty$*;*

(ii) *if* $c(1-m)-n(a+d)<0$, *there exists a unique orbits of system (2.1) in* $\{(\theta,r):\ 0\le\frac{\pi}{2}-\theta<\varepsilon_6,\ 0<r<r_6\}$ *that tends to (0,0) along* θ_2 *as* $t\to+\infty$.

In the following, we consider θ_3.

Theorem 2.5. *Suppose that* $f-d-a>0$ *and* $c(1-m)-n(a+d)<0$. *Then*

(a) *there exist* $\varepsilon_7>0$ *and* $r_7>0$ *such that there exists a unique orbit of system (2.1) in* $\{(\theta,r):\ 0\le|\theta-\theta_3|<\varepsilon_7,\ 0<r<r_7\}$ *that tends to (0,0) along* θ_3 *as* $t\to-\infty$ *when one of the following conditions holds*

(i) $a+d<f$ *and* $c(1-m)\le an$ *or*

(ii) $a+d<f<\frac{c(1-m)d}{c(1-m)-an}$ *and* $an<c(1-m)<n(a+d)$*; and*

(b) *there exist* $\varepsilon_8>0$ *and* $r_8>0$ *such that all orbits of system (2.1) in* $\{(\theta,r):\ 0\le|\theta-\theta_3|<\varepsilon_8,\ 0<r<r_8\}$ *that tend to (0,0) along* θ_3 *as* $t\to+\infty$ *when* $f\ge\frac{c(1-m)d}{c(1-m)-an}$ *and* $an<c(1-m)<n(a+d)$.

Proof. We apply the Briot-Bouquet transformation

$$x=x,\ \ y=ux,\ \ \mathrm{d}\tau=x\mathrm{d}t$$

to transform system (2.1) into

$$\begin{aligned}\frac{dx}{d\tau}&=a(1-m)x+[an-c(1-m)]ux-bx^2(1-m+nu),\\ \frac{\mathrm{d}u}{\mathrm{d}\tau}&=(f-d-a)(1-m)u+[c(1-m)-n(a+d)]u^2+bxu(1-m+nu).\end{aligned}\tag{2.6}$$

Similar to the discussion in [7], we only consider the equilibria of system (2.6) in the u-axis. In the u-axis system (2.6) has two equilibria $(0,0)$ and $\left(0,\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}\right)$. $(0,0)$ is an unstable node. In the following, we consider the equilibrium $\left(0,\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}\right)$.

Let $x_1=x$, $x_2=u-\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}$. Then system (2.6) becomes

$$\begin{aligned}\frac{dx_1}{d\tau}=&\frac{(1-m)[(an-c(1-m))f+c(1-m)d]}{n(a+d)-c(1-m)}x_1+[an-c(1-m)]x_1x_2\\ &-\frac{b(1-m)(nf-c(1-m))}{n(a+d)-c(1-m)}x_1^2-bnx_1^2x_2,\\ \frac{dx_2}{d\tau}=&\frac{b(1-m)^2(f-d-a)(nf-c(1-m))}{[n(a+d)-c(1-m)]^2}x_1-(f-d-a)(1-m)x_2\\ &+\frac{b(1-m)(2nf-n(a+d)-c(1-m))}{n(a+d)-c(1-m)}x_1x_2+[c(1-m)-n(a+d)]x_2^2+bnx_1x_2^2.\end{aligned}\tag{2.7}$$

We define the functions

$$K=\frac{(1-m)[(an-c(1-m))f+c(1-m)d]}{n(a+d)-c(1-m)},L=\frac{b(1-m)(nf-c(1-m))}{n(a+d)-c(1-m)},$$

$$M=\frac{b(1-m)^2(f-d-a)(nf-c(1-m))}{[n(a+d)-c(1-m)]^2}, N=\frac{b(1-m)(2nf-n(a+d)-c(1-m))}{n(a+d)-c(1-m)}.$$

The Jacobian matrix at the equilibrium is given by

$$\begin{pmatrix} Kx_1-2Lx_1-2bnx_1x_2 & [an-c(1-m)]x_1-2bnx_1x_2 \\ M+Nx_2+bnx_2^2 & -(f-d-a)(1-m)+Nx_1+2[c(1-m)-n(a+d)]x_2+2bnx_1x_2 \end{pmatrix}.$$

The Jacobian matrix at the equilibrium $(0,0)$ is given by

$$\begin{pmatrix} \frac{(1-m)[(an-c(1-m))f+c(1-m)d]}{n(a+d)-c(1-m)} & 0 \\ \frac{b(1-m)^2(f-d-a)(nf-c(1-m))}{[n(a+d)-c(1-m)]^2} & -(f-d-a)(1-m) \end{pmatrix}.$$

The Jacobian matrix at the equilibrium $\left(0,\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}\right)$ is given by

$$\begin{pmatrix} \frac{(1-m)[(an-c(1-m))f+c(1-m)d]}{n(a+d)-c(1-m)} & 0 \\ M-b(1-m)^2(f-d-a)(nf-c(1-m))+bn\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}^2 & -3(f-d-a)(1-m) \end{pmatrix}.$$

equilibrium (0,0) and $\left(0,\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}\right)$ is saddle if any one of the following conditions holds (a) $a+d<f$ and $c(1-m)\le an$, (b) $a+d<f<\frac{c(1-m)d}{c(1-m)-an}$ and $an<c(1-m)<n(a+d)$. Condition (a) (b) indicate that m is large enough, then both equilibrium (0,0) and $\left(0,\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}\right)$ exist, so that predator extinction or survive is depending on m. And there exists a unique separatrix of this equilibrium in I^+ of system (2.6), which tends to $\left(0,\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}\right)$ as $t\to-\infty$. By the inverse Briot-Bouquet transformation, there exist $\varepsilon_7>0$ and $r_7>0$ such that there exists a unique orbit of system (2.1) in $\{(\theta,r):\ 0\le|\theta-\theta_3|<\varepsilon_7,\ 0<r<r_7\}$ which tends to $(0,0)$ along θ_3 as $t\to-\infty$.

Similarly, when $f=\frac{cd(1-m)}{c(1-m)-an}$ and $an<c(1-m)<n(a+d)$, one sees that equilibrium $(0,0)$ of system (2.7) is a saddle-node. In this case, the equilibrium $\left(0,\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}\right)$ of system (2.6) is a saddle-node, and the stable node part is in I^+ of system (2.6). When $\frac{cd(1-m)}{c(1-m)-an}<f$ and $an<c(1-m)<n(a+d)$, the equilibrium (0,0) of system (2.7) is a saddle-node. Hence, the equilibrium $\left(0,\frac{(f-d-a)(1-m)}{n(a+d)-c(1-m)}\right)$ of system (2.6) is a stable node. For both cases, we use the inverse Briot-Bouquet transformation to obtain the result: there exist $\varepsilon_8>0$ and $r_8>0$ such that all orbits of system (2.1) in $\{(\theta,r):\ 0\le|\theta-\theta_3|<\varepsilon_8,\ 0<r<r_8\}$ tends to $(0,0)$ along θ_3 as $t\to+\infty$. This completes the proof of the theorem.

Using a similar method as in the proof of Theorem 2.4, we obtain the following.

Theorem 2.6. *Assume that* $f-d-a<0$ *and* $c(1-m)-n(a+d)>0$*. Then*

(a) *there exist* $\varepsilon_9 > 0$ *and* $r_9 > 0$*, such that there exists a unique orbit of system (2.1) in* $\{(\theta, r):\ 0 \le |\theta - \theta_3| < \varepsilon_9,\ 0 < r < r_9\}$ *which tends to (0,0) along* θ_3 *as* $t \to +\infty$ *if one of the following conditions holds:*

(i) $f \le d$ *and* $c(1-m) > n(a+d)$*, or*

(ii) $d < f < \frac{cd(1-m)}{c(1-m)-an}$ *and* $c(1-m) > n(a+d)$*; and*

(b) *there exist* $\varepsilon_{10} > 0$ *and* $r_{10} > 0$ *such that there exists an infinite number of orbits of system (2.1) in* $\{(\theta, r):\ 0 \le |\theta - \theta_3| < \varepsilon_{10},\ 0 < r < r_{10}\}$ *which tend to (0,0) along* θ_3 *as* $t \to -\infty$ *when* $\frac{cd(1-m)}{c(1-m)-an} \le f < a+d$ *and* $c(1-m) > n(a+d)$.

3. Equilibria of system (1.3) except $(0,0)$

In this section, we discuss the existence and stability of equilibria of system (1.3) except (0,0). We show that if the positive steady state of the system (1.3) is locally asymptotically stable. The system has no nontrivial positive periodic solution. Moreover, we show system (1.3) has at most one limit cycle for some parameters.

System (1.3) always has a boundary equilibrium $(\frac{a}{b}, 0)$ and at most one interior equilibrium (x_*, y_*), where

$$x_* = \frac{[an-(1-m)c]f + (1-m)cd}{bnf}, \quad y_* = \frac{(f-d)(1-m)}{nd}x_*.$$

The unique interior equilibrium (x_*, y_*) exists if and only if any one of the following conditions holds: (i) $f > d$ and $an \ge (1-m)c$, that is, $f > d$ and $1 - \frac{an}{c} \le m < 1$;

(ii) $d < f < \frac{(1-m)cd}{(1-m)c-an}$ and $an < (1-m)c$, that is, $f > d$ and $1 - \frac{anf}{(f-d)c} < m < 1 - \frac{an}{c}$.

Next, we discuss the stability of the equilibrium $(\frac{a}{b}, 0)$ and (x_*, y_*). The Jacobian matrix of the system (1.3) is given by

$$J(x,y) = \begin{pmatrix} a - bx - \frac{(1-m)cy}{ny+(1-m)x} - bx + \frac{(1-m)^2cxy}{[ny+(1-m)x]^2} & -\frac{(1-m)^2cx^2}{[ny+(1-m)x]^2} \\ \frac{nf(1-m)y^2}{[ny+(1-m)x]^2} & -d + \frac{f(1-m)x}{ny+(1-m)x} - \frac{nf(1-m)xy}{[ny+(1-m)x]^2} \end{pmatrix}.$$

The Jacobian matrix at the equilibrium $(\frac{a}{b}, 0)$ is given by

$$\begin{pmatrix} -a & -c \\ 0 & f-d \end{pmatrix}.$$

Clearly, $(\frac{a}{b}, 0)$ is a saddle when $f > d$ and a stable node when $f < d$. When $f = d$, the equilibrium $(\frac{a}{b}, 0)$ is saddle-node.

The Jacobian matrix at the equilibrium (x_*, y_*) is given by

$$J(x_*, y_*) = \begin{pmatrix} -bx_* + \frac{(1-m)^2 cx_* y_*}{(ny_*+(1-m)x_*)^2} & -\frac{(1-m)^2 cx_*^2}{(ny_*+(1-m)x_*)^2} \\ \frac{nf(1-m)y_*^2}{(ny_*+(1-m)x_*)^2} & -\frac{nf(1-m)x_* y_*}{(ny_*+(1-m)x_*)^2} \end{pmatrix}.$$

The determinant of $J(x_*, y_*)$ is positive and its trace is

$$trJ(x_*, y_*) = \frac{[(1-m)c - an - dn]f^2 + nd^2 f - cd^2(1-m)}{nf^2}. \tag{3.1}$$

Theorem 3.1. *Suppose that system (1.3) has a positive equilibrium* (x_*, y_*), $0 \leq m < 1$. *Then*

(1) (x_*, y_*) *is locally asymptotically stable if any one of the following conditions holds*

(i) $f > d$ *and* $c(1-m) \leq an$,

(ii) $d < f < \frac{cd(1-m)}{c(1-m)-an}$ *and* $an < c(1-m) \leq an + dn$,

(iii) $d < f < \frac{-nd^2+\sqrt{\Delta}}{2(c(1-m)-an-dn)}$ *and* $c(1-m) > an + dn$, *where* $\Delta = n^2 d^4 + 4cd^2(1-m)(c(1-m) - an - dn)$;

(2) (x_*, y_*) *is unstable if*

(iv) $\frac{-nd^2+\sqrt{\Delta}}{2(c(1-m)-an-dn)} < f < \frac{cd(1-m)}{c(1-m)-an}$ *and* $c(1-m) > an + dn$.

Proof. It is clear that (x_*, y_*) is locally asymptotically stable (unstable) if $trJ(x_*, y_*) < 0$ ($trJ(x_*, y_*) > 0$, respectively). From (3.1), we define the function $F = [c(1-m) - an - dn]f^2 + nd^2 f - cd^2(1-m)$. If the condition (i) holds, then $F = (c(1-m) - an)f^2 - ndf(f-d) - cd^2(1-m) < 0$, which implies that (x_*, y_*) is locally asymptotically stable. If the condition (ii) holds, then we consider three subcases: (a) $d < f < \frac{c(1-m)}{n}$ and $an < c(1-m) < an + dn$. (b) $\frac{c(1-m)}{n} < f < \frac{cd(1-m)}{c(1-m)-an}$ and $an < c(1-m) < an + dn$. (c) $d < f < \frac{cd(1-m)}{c(1-m)-an}$ and $c(1-m) = an + dn$. In the subcase (a), we have $F = (c(1-m) - an - dn)f^2 - d^2(c(1-m) - nf) < 0$. Hence, (x_*, y_*) is locally asymptotically stable. In subcase (b), when $f > \frac{c(1-m)}{n}$, we find from (3.1) that

$$trJ(x_*, y_*) = -bx_* + \frac{(1-m)(c(1-m) - nf)x_* y_*}{(ny_* + (1-m)x_*)^2} < 0,$$

when $f < \frac{cd(1-m)}{c(1-m)-an}$, we have $(1-m)cd > (1-m)cf - anf$. It follows that

$$F < (c(1-m) - an - dn)f^2 + nd^2 f - c(1-m)df + andf = f(f-d)(c(1-m) - an - dn) < 0.$$

Hence, in subcase (b), (x_*, y_*) is locally asymptotically stable. In subcase (c), we have $d < f < a + d$ and $F = nd^2 f - cd^2(1-m) = nd^2(f - a - d) < 0$, which imply (x_*, y_*) is locally asymptotically stable. When $c(1-m)-an-dn>0$, after some straightforward computations, we have $d<\frac{-nd^2+\sqrt{\Delta}}{2(c(1-m)-an-dn)}<\frac{cd(1-m)}{c(1-m)-an}<a+d$. We can rewrite F as follows

$$F=(c(1-m)-an-dn)\Big(f-\frac{-nd^2+\sqrt{\Delta}}{2(c(1-m)-an-dn)}\Big)\Big(f+\frac{-nd^2+\sqrt{\Delta}}{2(c(1-m)-an-dn)}\Big). \quad (3.2)$$

If the condition (iii) holds, it is clear that $F<0$, which implies that (x_*,y_*) is locally asymptotically stable. If the condition (iv) holds, we find from (3.2) that $F>0$. Hence, (x_*,y_*) is unstable. This proves the theorem.

Theorem 3.2. *If (x_*,y_*) is locally asymptotically stable, then system (1.3) has no nontrivial positive periodic orbit in I^+.*

Proof. Let $\Gamma(t)=(x(t),y(t))$ be any one nontrivial positive periodic orbit of system (1.3) with period $T>0$. It is sufficient to show that

$$\int_0^T trJ(x(t),y(t))\mathrm{d}t<0. \quad (3.3)$$

Since

$$\int_0^T (a-bx(t)-\frac{c(1-m)y(t)}{ny(t)+(1-m)x(t)}\Big)\mathrm{d}t=\int_0^T\frac{x'(t)}{x(t)}\mathrm{d}t=0,$$

and

$$\int_0^T\Big(-d+\frac{f(1-m)x(t)}{ny(t)+(1-m)x(t)}\Big)\mathrm{d}t=\int_0^T\frac{y'(t)}{y(t)}\mathrm{d}t=0,$$

we have

$$\int_0^T trJ(x_*,y_*))\mathrm{d}t=\int_0^T\Big[-bx(t)+(1-m)(c(1-m)-nf)\frac{x(t)y(t)}{(ny(t)+(1-m)x(t))^2}\Big]\mathrm{d}t. \quad (3.4)$$

If $c(1-m)\le nf$, then $trJ(x(t),y(t))<0$ for all $t\ge 0$. Hence (3.3) holds. Thus we assume $c(1-m)>nf$ in the following. Notice that

$$a-bx_*-\frac{(1-m)cy_*}{ny_*+(1-m)x_*}=0,\quad -d+\frac{f(1-m)x_*}{ny_*+(1-m)x_*}=0. \quad (3.5)$$

From (1.3) and (3.5), we have

$$\begin{aligned}\frac{x'(t)}{x(t)}&=bx_*+\frac{(1-m)cy_*}{ny_*+(1-m)x_*}-bx(t)-\frac{(1-m)cy(t)}{ny(t)+(1-m)x(t)}\\&=(x(t)-x_*)\Big[-b+\frac{(1-m)^2cy_*}{ny_*+(1-m)x_*}\cdot\frac{1}{ny(t)+(1-m)x(t)}\Big]\\&\quad+\frac{(1-m)^2cx_*}{ny_*+(1-m)x_*}\cdot\frac{y_*-y(t)}{ny(t)+(1-m)x(t)},\end{aligned} \quad (3.6)$$

$$\frac{y'(t)}{y(t)} = (x(t)-x_*)\frac{(f-d)(1-m)}{ny(t)+(1-m)x(t)} + \frac{dn(y_*-y(t))}{ny(t)+(1-m)x(t)}. \tag{3.7}$$

From (3.7), $\frac{y_*-y(t)}{ny(t)+(1-m)x(t)}$ can be solved and substituted into (3.6). After some straightforward computations with (3.5), we have

$$x_* - x(t) = \frac{1}{b}\cdot\frac{x'(t)}{x(t)} - \frac{c(1-m)}{fnb}\cdot\frac{y'(t)}{y(t)}. \tag{3.8}$$

Using (1.3), (3.4), (3.5), (3.8) and the Green's theorem, we have

$$\begin{aligned}
&\int_0^T trJ(x(t),y(t))\mathrm{d}t\\
&= \int_0^T \Big[-bx(t) + \frac{(1-m)c-nf}{f}\cdot\frac{y'(t)}{ny(t)+(1-m)x(t)}\\
&\quad + \frac{d((1-m)c-nf)}{(1-m)fc}\cdot\frac{(1-m)cy(t)}{ny(t)+(1-m)x(t)}\Big]\mathrm{d}t\\
&= \int_0^T \Big[\frac{ad((1-m)c-nf)}{(1-m)fc} - b\Big(1+\frac{d((1-m)c-nf)}{(1-m)fc}\Big)x_*\Big]\mathrm{d}t\\
&\quad + \int_0^T b\Big(1+\frac{d((1-m)c-nf)}{(1-m)fc}\Big)(x_*-x(t))\mathrm{d}t\\
&\quad - \frac{d((1-m)c-nf)}{(1-m)fc}\int_0^T \frac{x'(t)}{x(t)}\mathrm{d}t + \frac{(1-m)c-nf}{f}\int_0^T \frac{y'(t)}{ny(t)+(1-m)x(t)}\mathrm{d}t\\
&= T\cdot trJ(x_*,y_*) - \frac{(1-m)[(1-m)c-nf]}{f}\iint_{\Omega}\frac{\mathrm{d}x\mathrm{d}y}{(ny(t)+(1-m)x(t))^2} < 0,
\end{aligned}$$

where Ω is the bounded region enclosed by Γ. Hence, the divergency criterion [9] implies all the positive solutions must be orbitally stable. This is impossible since the (x_*,y_*) is locally asymptotically stable. Thus, system (1.3) has no positive periodic orbit in I^+, which complete the proof.

Now under the assumption Theorem 3.1 (iv) and through the change of variables $\mathrm{d}t = (ny+(1-m)x)\mathrm{d}\tau$, system (1.3) becomes

$$\begin{aligned}
&\dot{x} = \varphi(x)(h(x)-u),\\
&\dot{u} = \psi(x)u, \\
&\dot{x} = \dot{u} = 0 \ \text{ when } \ (x,u)=(0,0),
\end{aligned} \tag{3.9}$$

where

$$\sigma(x) = \Big(\frac{(1-m)c-n(a-bx)}{x}\Big)^l, l = \frac{dn}{(1-m)c-na} > 0,$$

$$
\begin{aligned}
\varphi(x) &= \frac{x[(1-m)c-n(a-bx)]}{\sigma(x)}, h(x) = \frac{(1-m)(a-bx)x\sigma(x)}{(1-m)c-n(a-bx)},\\
\psi(x) &= \left[(f-d)x - \frac{dn(a-bx)x}{(1-m)c-n(a-bx)}\right]\\
&= (1-m)\cdot\frac{x[nbfx+((1-m)c-na)f-(1-m)cd]}{(1-m)c-n(a-bx)}.
\end{aligned}
\tag{3.10}
$$

Conversely, the change of variables $y = u/\sigma(x)$, $\mathrm{d}t = (ny+(1-m)x)\mathrm{d}\tau$ converts system (3.9) to system (1.3). Hence, system (1.3) has a unique limit cycle in I^+ if and only if system (3.9) does. Notice that the set $\Omega = (0, \frac{a}{b}) \times R_+ \subseteq I^+$ is positively invariant and any trajectory must intersect it from the exterior to the interior provided $x(0) \geq \frac{a}{b}$. So, the limit cycle must lie in Ω. From (3.5), we have $\frac{c(1-m)}{n} > \frac{\frac{c(1-m)}{n}\cdot ny_*}{ny_*+(1-m)x_*} = a-bx_*$, that is

$$
c(1-m) > n(a-bx_*). \tag{3.11}
$$

In view of the expression of x_*, we have $\psi(x_*) = 0$ and consequently, $\psi(x_*)$ can be written as

$$
\psi(x) = (1-m)\frac{nbfx(x-x_*)}{c(1-m)-n(a-bx)}. \tag{3.12}
$$

Clearly, $h(\frac{a}{b}) = 0$ and

$$
h'(x) = \frac{(1-m)\sigma(x)}{x[(1-m)c-n(a-bx)]}\cdot\left[x(a-2bx) - \frac{nx(a-bx)(d+bx)}{(1-m)c-n(a-bx)}\right]. \tag{3.13}
$$

System (3.9) has equilibria $E_1(\frac{a}{b}, 0)$ and $E_*(x_*, h(x_*))$. The local stability of E_1 and E_* are determined by the eigenvalues of the matrix $\overline{J}(E_1)$ and $\overline{J}(E_*)$, respectively, where $\overline{J}(x,u)$ is the variational matrix of system (3.9) and is given by

$$
\overline{J}(x,u) = \begin{pmatrix} \varphi'(x)(h(x)-u)+\varphi(x)h'(x) & -\varphi(x)\\ \psi'(x)u & \psi(x)\end{pmatrix}. \tag{3.14}
$$

At E_1, we have

$$
\overline{J}(E_1) = \begin{pmatrix} -\frac{(1-m)a^2}{b} & \left(\frac{a}{b}\right)^{1+l}((1-m)c)^{1-l}\\ 0 & \psi(\frac{a}{b})\end{pmatrix},
$$

which gives that E_1 is a saddle point. At E_*, we have

$$
\overline{J}(E_*) = \begin{pmatrix} \varphi(x_*)h'(x_*) & -\varphi(x_*)\\ \psi'(x_*)h(x_*) & 0\end{pmatrix}.
$$

Since the determinant of $\bar{J}(E_*)$ is positive, $tr\bar{J}(E_*) = \varphi(x_*)h'(x_*)$. From the assumption Theorem 3.1 (iv) together with (3.1), (3.5), (3.11) and (3.13), we have

$$h'(x_*) = \frac{\sigma(x_*)(ny_* + (1-m)x_*)}{x_*((1-m)c - n(a-bx_*))} trJ(x_*, y_*) > 0. \tag{3.15}$$

So, E_* is unstable.

Lemma 3.3 ([10]). *Suppose that* $h'(\lambda) > 0$ *and (A1)-(A5) hold and, moreover, there exist* $\alpha, \beta \geq 0$ *and not all zero such that*

$$\frac{d}{dx}\left(\frac{\varphi(x)h'(x)}{\psi(x)(\alpha + \beta h(x))}\right) \leq 0$$

for all $x\varepsilon(0, K) - \{\lambda\}$. *Then system (1.3) possesses at most one limit cycle, and if it exists then it is stable.*

Theorem 3.4. *If* $\frac{-nd^2+\sqrt{\Delta}}{2[(1-m)c-an-dn]} < f < \frac{(1-m)cd}{(1-m)c-an}$, *and* $(1-m)c - an - dn > 0$, *where* Δ *is defined in Theorem* 4.1, *that is,* $trJ(x_*, y_*) > 0$, *then system (1.3) has at most one limit cycle in* $\Omega = (0, \frac{a}{b}) \times R_+ \subseteq I^+$. *Moreover, if it exists then it is stable.*

Proof. From lemma 3.3, we only prove

$$\frac{\mathrm{d}}{\mathrm{d}x}(\varphi(x)h'(x)/\psi(x)) < 0, \text{ for } x \in (0, \frac{a}{b}) - \{x_*\}.$$

From (2.10) and (2.13), we have

$$\begin{aligned}\varphi(x)h'(x) &= (1-m)\left[x(a-2bx) - \frac{nx(a-bx)(d+bx)}{(1-m)c - n(a-bx)}\right] \\ &= (1-m)x(a-2bx) - \frac{(1-m)(f-d)}{d}x(d+bx) + \frac{bx+d}{d}\psi(x) \\ &= -\frac{b(1-m)(f+d)}{d}x^2 + (1-m)(a+d-f)x + \frac{bx+d}{d}\psi(x) \qquad (3.16) \\ &\equiv q(x) + \frac{bx+d}{d}\psi(x). \qquad (3.17)\end{aligned}$$

Notices that $q(x)$ is defined by (3.16) and (3.17). From (3.11), (3.12) and (3.15), we have

$$q(x_*) = \varphi(x_*)h'(x_*) > 0. \tag{3.18}$$

and

$$\begin{aligned}q(x) - q(x_*) = &-\frac{(f+d)[(1-m)c - n(a-bx)]}{ndf}\psi(x) \\ &+ \left[(a+d-f) - \frac{b(f+d)}{d}x_*\right](1-m)(x-x_*).\end{aligned}$$

Division of $\varphi(x)h'(x)$ by $\psi(x)/d$ yields

$$\begin{aligned} d\cdot\frac{\varphi(x)h'(x)}{\psi(x)} &= \frac{d}{\psi(x)}\Big[q(x)-q(x_*)+q(x_*)+\frac{bx+d}{d}\psi(x)\Big] \\ &= -\frac{bd}{f}x+d-\frac{(f+d)[(1-m)c-na]}{nf}+\frac{-b(f+d)x_*+(a+d-f)d}{f} \\ &\quad \frac{dq(x_*)}{(1-m)f}\Big[1+\frac{(1-m)c-na}{nbx_*}\Big]\cdot\frac{1}{x-x_*}. \end{aligned}$$

Hence,

$$d\cdot\frac{d}{dx}\Big(\frac{\varphi(x)h'(x)}{\psi(x)}\Big)=-\frac{bd}{f}-\frac{dq(x_*)}{(1-m)f}\Big[1+\frac{(1-m)c-na}{nbx_*}\Big]\frac{1}{(x-x_*)^2}<0$$

for $x\in(0,\frac{a}{b})-\{x_*\}$. This completes the theorem.

4. Global analysis of system (1.3)

In this section, we summarize the results in Sections 2 and 3, and classify the global dynamics of system (1.3) depending on all parameters.

Theorem 4.1. *System (1.3) has no interior equilibrium and (0,0) is a global attractor of system (1.3) in* I^+ *if any one of the following conditions holds*

(i) $f-a-d>0$ *and* $(1-m)c-an-dn\geq 0$*, that is,* $f>a+d$ *and* $m\leq 1-\frac{n(a+d)}{c}$*;*

(ii) $f-a-d=0$ *and* $(1-m)c-an-dn\geq 0$*, that is,* $f=a+d$ *and* $m\leq 1-\frac{n(a+d)}{c}$*;*

(iii) $a+d>f\geq\frac{(1-m)cd}{(1-m)c-an}$ *and* $(1-m)c-an-dn>0$*, that is,* $d<f<a+d$ *and* $m\leq 1-\frac{anf}{c(f-d)}$*;*

(iv) $f\geq\frac{(1-m)cd}{(1-m)c-an}$ *and* $an<(1-m)c<an+dn$*, that is,* $1-\frac{n(a+d)}{c}\leq m\leq 1-\frac{anf}{c(f-d)}$.

Proof. By the sufficient and necessary conditions of the existence of an interior equilibrium of system (1.3) in Section 3, we see that system (1.3) has no interior equilibrium for all case in Theorem 4.1. Clearly, any case in Theorem 4.1 implies $f>d$. Thus, $(\frac{a}{b},0)$ is a saddle. According to the theorems in Section 2, there exist φ_0 and r_0 such that all orbits of system (1.3) in $\{(\theta,r):\ 0\leq\frac{\pi}{2}-\theta<\varphi_0,\ 0<r<r_0\}$ tend to $(0,0)$ along θ_2 as $t\to\infty+\infty$ if any one of conditions (i), (ii) and (iii) holds. However, if condition (iv) holds, then there exist $\overline{\varphi}$ and $\bar{r}$ such that all orbits of system (1.3) in $\{(\theta,r):\ 0\leq|\theta-\theta_3|<\overline{\varphi},\ 0<r<\bar{r}\}$ tend to $(0,0)$ along θ_3 as $t\to+\infty$. This complete the proof.

Theorem 4.2. *If $f \leq d$ and $(1-m)c - an - dn < 0$, that is, $f \leq d$ and $m > 1 - \frac{n(a+d)}{c}$, then system (1.3) has no interior equilibrium and the equilibrium $(\frac{a}{b}, 0)$ is a global attractor in I^+.*

Proof. Obviously, system (1.3) has no interior equilibrium an d$(\frac{a}{b}, 0)$ is a stable node when $f \leq d$ and $(1-m)c - an - dn < 0$. By Theorem 3.3, there exists a unique orbit of system (1.3) tending to $(0,0)$ along $\theta_1(\theta_2)$ as $t \to \infty - \infty$ ($t \to +\infty$, respectively), i.e, the positive x-axis (y-axis). Moreover, no other orbits tend to $(0,0)$. This completes the proof.

Theorem 4.3. *If $f \leq d$ and $(1-m)c - an - dn \geq 0$, that is, $f \leq d$ and $m \leq 1 - \frac{n(a+d)}{c}$, then system (1.3) has no interior equilibrium and $(0,0)$, $(\frac{a}{b}, 0)$ are attractors of system (1.3) in I^+.*

Proof. It is clear that $(\frac{a}{b}, 0)$ is a stable node in I^+ and system (1.3) has no interior equilibrium under the assumption. On the other hand, when $f \leq d$ and $(1-m)c - an - dn = 0$, we see that there exists an infinite number of orbits (a unique orbit) of system (1.3) tending to $(0,0)$ along θ_2 (θ_1, respectively) as $t \to +\infty$ ($t \to -\infty$, respectively), and no other orbit tends to (0,0) as $t \to +\infty$ or $t \to -\infty$ by Theorem 3.1. When $f \leq d$ and $c(1-m) > an + dn$, there exists an infinite number of orbits (a unique orbit) of system (1.3) tending to $(0,0)$ along θ_2 (θ_1, θ_3, respectively) as $t \to +\infty$ ($t \to -\infty$, $t \to +\infty$, respectively), and no other orbit tends to $(0,0)$ by Theorems 3.3 and 3.5. This completes the proof.

Theorem 4.4. *System (1.3) has a unique interior equilibrium, which is a global attractor in I^+ if any one of the following conditions holds*

(i) *$d < f < a + d$ and $(1-m)c - an - dn < 0$, that is, $d < f < a + d$ and $m > 1 - \frac{n(a+d)}{c}$;*

(ii) *$f - a - d = 0$ and $(1-m)c - an - dn < 0$, that is, $f = a + d$ and $m > 1 - \frac{n(a+d)}{c}$;*

(iii) *$f - a - d > 0$ and $(1-m)c \leq an$, that is, $f > a + d$ and $m \geq 1 - \frac{an}{c}$;*

(iv) *$a + d < f < \frac{(1-m)cd}{(1-m)c - an}$ and $an < (1-m)c < an + dn$, that is, $f > a + d$ and $1 - \frac{anf}{c(f-d)} < m < 1 - \frac{an}{c}$.*

Proof. In any one of the cases (i)-(iv), system (1.3) has a unique interior equilibrium (x_*, y_*) and it is locally asymptotically stable by Theorem 3.1. Moreover, $(\frac{a}{b}, 0)$ is a saddle.

On the other hand, in case (i), only the positive x-axis and y-axis tend to (0,0) as $t \to \infty - \infty$ and $t \to +\infty$, respectively, no other orbits approach (0,0) according to Theorem 2.3. Thus, (x_*, y_*) is a global attractor in I^+ by Theorem 3.2.

In case (ii), there exists an infinite number of orbits of system (1.3) tending to $(0,0)$ along θ_1 as $t \to -\infty$ and a unique orbit of system (1.3) tending to $(0,0)$ along θ_2 as $t \to +\infty$ by Theorem 2.2. Hence, (x_*,y_*) is a global attractor in I^+ by Theorem 3.2.

In cases (iii) and (iv), from Theorem 3.3, the same statements hold for θ_1 and θ_2. Furthermore, the exists a unique orbit of system (1.3) tending to $(0,0)$ along θ_3 as $t \to -\infty$ by Theorem 3.4. Hence, (x_*,y_*) is a global attractor in I^+ by Theorem 4.2. The proof of this theorem is completed.

Theorem 4.5. *Suppose that one of the following conditions holds*

(i) $a+d > f > d$ *and* $(1-m)c - an - dn = 0$, *that is,* $d < f < a+d$ *and* $m = 1 - \frac{n(a+d)}{c}$*;*

(ii) $d < f < \frac{-nd^2+\sqrt{\Delta}}{2[(1-m)c-an-dn]}$ *and* $(1-m)c - an - dn > 0$, *where* Δ *is defined in Theorem 4.1, that is,* $f > d$ *and* $1 - \frac{nf(af+df-d^2)}{c(f^2-d^2)} < m < 1 - \frac{n(a+d)}{c}$, Δ *is defined in Theorem 4.1.*

Then system (1.3) has a unique interior equilibrium (x_*,y_*) *and no limit cycle, both* $(0,0)$ *and* (x_*,y_*) *are attractors of system (1.3) in* I^+.

Proof. In both cases, system (1.3) has a unique interior equilibrium (x_*,y_*), which is locally asymptotically stable by Theorem 3.1. Hence, system (1.3) has no limit cycle according to Theorem 3.2. In case (i), there exists an infinite number of orbits (a unique orbit) of system (1.3) tending to $(0,0)$ along θ_2 (θ_1, respectively) as $t \to +\infty$ ($t \to -\infty$, respectively), and no other orbit approaches $(0,0)$ as $t \to +\infty$ or $t \to -\infty$ by Theorem 3.1. In case (ii), from Theorem 3.3, there exists a unique orbit (an infinite number of orbits) of system (1.3) tending to $(0,0)$ along θ_1 (θ_2, respectively) as $t \to -\infty$ ($t \to +\infty$, respectively), and there exists a unique orbit of system (1.3) tending to $(0,0)$ along θ_3 as $t \to +\infty$ by Theorem 3.5. This completes the proof of the theorem.

Based on Theorems 3.1 and 3.3, we have the following.

Theorem 4.6. *Suppose that* $\frac{-nd^2+\sqrt{\Delta}}{2[(1-m)c-an-dn]} < f < \frac{(1-m)cd}{(1-m)c-an}$ *and* $(1-m)c > an+dn$, *where* Δ *is defined in Theorem 4.1, that is,* $1 - \frac{anf}{c(f-d)} < m < 1 - \frac{nf(af+df-d^2)}{c(f^2-d^2)}$. *Then system (1.3) has a unique unstable interior equilibrium and can have a unique limit cycle in* I^+*, which is stable if it exists.*

5. The influence of *m* and numerical simulations

(1) The influence of prey refuge on predator densities.

If $f > d$, $\frac{dy_*}{dm} = \frac{(f-d)[-an+2c(1-m)-2cd(1-m)]}{bdn^2f}$ for all $1-\frac{an}{c} \le m < 1$.

Case 1. If $2c-2cd \le 0$, then $\frac{dy_*}{dm} < 0$ for all $1-\frac{an}{c} \le m < 1$, and y_* is the strictly decreasing function of parameter m. That is, increasing the amount of prey refuge can decrease the densities of predator species. In this cases, y_* obtains its maximum value $\frac{(f-d)[(an-c)f+d]}{bdn^2f}$ at $m = 0$.

Case 2. If $2c-2cd > 0$ holds and define $m^* = 1-\frac{an}{2c-2cd}$, it follows that $\frac{dy_*}{dm} > 0$ for all $1-\frac{an}{c} \le m < m^*$ and $\frac{dy_*}{dm} < 0$ for all $m^* < m < 1$. y_* obtains its maximum $\frac{a^2(f-d)[(2-2d)f+d-f]}{(2-2d)^2cbdf}$ at the point $m = m^*$. The above analysis shows that on the interval $(1-\frac{an}{c}, m^*)$, increasing the amount of prey refuge can increase the densities of predator species. While for m larger than m^*, i.e., as the prey refuge becomes large enough, increasing the amount of prey refuge can decrease the predator densities. The predator species obtain their maximum densities at the point $m = m^*$. Because of the positive equilibrium of global stability, it ensures the prey densities will not increase infinitely, and the predator species will not become extinct, which is contrary to the predator-prey model with the Holling type II incorporating a prey refuge [11]. Kar observed that the predator will be extinct with a large refuge parameter $m(m\varepsilon(B,1))$ when $c\beta > a\gamma$, that is, increasing the amount of refuge can increase prey densities and lead to population outbreak.

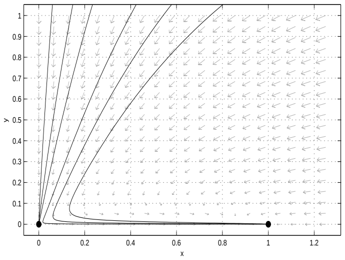
(2) The influence of prey refuge on the dynamics of system (1.3).

[7] classified the dynamics of system depending on all parameters. Our results in Section 4 and Section 5 show that the existence and stability properties of the equilibrium for system (1.3) depend on all parameters especially the refuge parameter m. In the following, we classified the existence and stability properties of the equilibrium for system (1.3) using m as control when $f \le d$, $d < f < a+d$, $f = a+d$ and $f > a+d$, respectively into the following table 1.

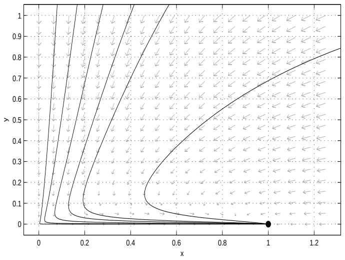
Let $A_1 = 1-\frac{anf}{c(f-d)}$, $A_2 = 1-\frac{nf(af+df-d^2)}{c(f^2-d^2)}$, and $A_3 = 1-\frac{n(a+d)}{c}$.

Table 1 Property of equilibria for system (1.3) with m when $f \le d$, $d < f < a+d$, $f = a+d$ and $f > a+d$, respectively.

Parameters		$(0,0)$	$\left(\frac{a}{b},0\right)$	(x_*,y_*)	Phase portrait
$f \leq d$	$m \in [0,A_3]$	Attractor	Attractor	None	Fig1(a)
	$m \in (A_3,1)$	Unstable	Global attractor	None	Fig1(b)
$d<f<a+d$	$m \in [0,A_1]$	Global attractor	saddle	None	Fig 2(a)
	$m \in (A_1,A_2)$	Attractor	saddle	Limit cycle	Fig 2(b), Fig2(c)
	$m \in (A_2,A_3]$	Attractor	saddle	Attractor	Fig 2(d)
	$m \in (A_3,1)$	Unstable	saddle	Global attractor	Fig 2(e)
$f=a+d$	$m \in [0,A_3]$	Global attractor	saddle	None	Fig 3(a)
	$m \in (A_3,1)$	Unstable	saddle	Global attractor	Fig 3(b)
$f>a+d$	$m \in [0,A_1]$	Global attractor	saddle	None	Fig 4(a)
	$m \in (A_1,1)$	Unstable	saddle	Global attractor	Fig 4(b)

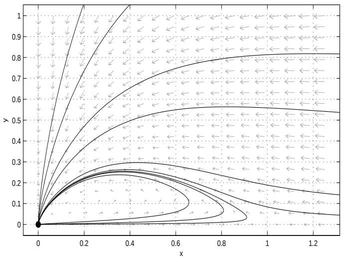


(a) Both (0,0) and (1,0) are attractors, where $a=b=n=d=1$, $c=2.5$ and $f=0.5$, $m=0.1$

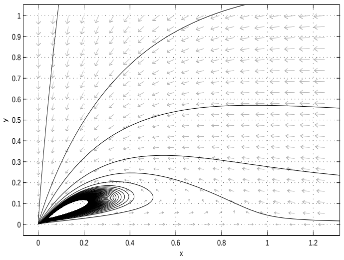


(b) (0,0) is unstable, and (1,0) is a global attractor, where $a=b=n=d=1$, $c=2.5$ and $f=0.5$, $m=0.3$

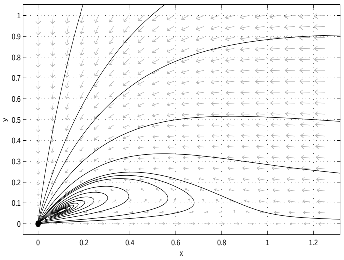
FIGURE 1. The phase portrait of system (1.3) for different values of m with $f=0.5$ in the case of $f \leq d$



(a) (0,0) is a global attractor, and (1,0) is a saddle, where $a = b = n = d = 1$, $c = 2.5$ and $f = 1.7$, $m = 0.025$.

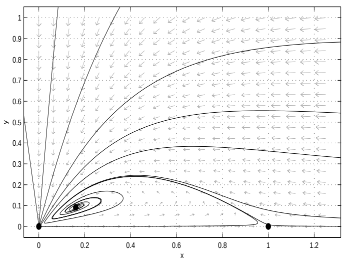


(b) (0,0) is an attractor, (1,0) is a saddle, $(0.19, 0.11)$ is an unstable focus, and there is a stable limit cycle surrounding (x^*, y^*), where $a = b = n = d = 1$, $c = 2.5$ and $f = 1.7$, $m = 0.136$.

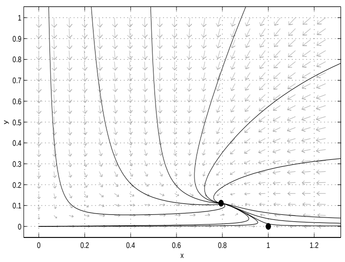


(c) (0,0) is an attractor, (1,0) is a saddle, $(0.10, 0.06)$ is an unstable focus, and the limit cycle is broken, when m decreases, where $a = b = n = d = 1, c = 2.5$ and $f = 1.7$, $m = 0.126$.

Let $a = b = n = d = 1$, $c = 2.5$. For these values of parameters, we verify the existence and stability properties of the equilibrium for the system (1.3) by choosing the appropriate refuge

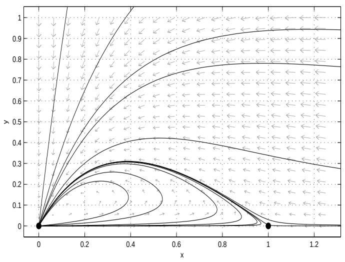


(d) Both (0,0) and $(0.16,0.09)$ are attractors, and (1,0) is a saddle, where $a=b=n=d=1$, $c=2.5$ and $f=1.7$, $m=0.185$

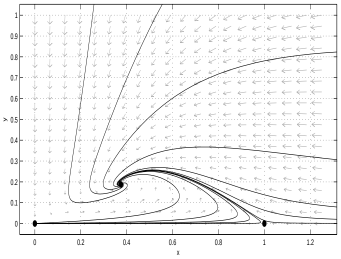


(e) (0,0) has a hyperbolic sector, (1,0) is a saddle, and $(0.79,0.11)$ is a global attractor, where $a=b=n=d=1$, $c=2.5$ and $f=1.7$, $m=0.8$

FIGURE 2. The phase portrait of system (1.3) for different values of m with $f=1.7$ in the case of $d<f<a+d$

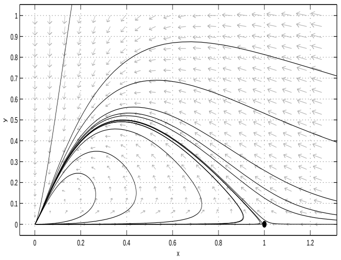


(a) (0,0) is a global attractor, and (1,0) is a saddle, where $a=b=n=d=1$, $c=2.5$ and $f=2$, $m=0.2$

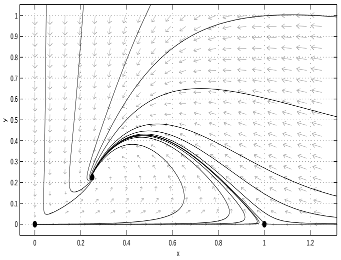


(b) (0,0) is unstable, and (1,0) is a saddle, where $a = b = n = d = 1$, $c = 2.5$ and $f = 2, m = 0.5$

FIGURE 3. The phase portrait of system (1.3) for different values of m with $f = 2$ in the case of $f = a + d$



(a) (0,0) is a global attractor, and (1,0) is a saddle, where $a = b = n = d = 1$, $c = 2.5$ and $f = 3$, $m = 0.35$



(b) (0,0) is unstable, (1,0) is a saddle, and (0.25,0.23) is a global attractor, where $a = b = n = d = 1$, $c = 2.5$ and $f = 3, m = 0.55$

FIGURE 4. The phase portrait of system (1.3) for different values of m with $f = 3$ in the case of $f > a + d$

parameter m while f is fixed according to $f \leq d$, $d < f < a+d$, $f = a+d$ and $f > a+d$ respectively.

Fig.1 (a) (b) show that for $f < d$ predators go extinct and hence, predator and prey species of system (1.3) may not exist.

Fig.2 (a) shows that for $d < f < a+d$, if $0 < m < 0.029$, predators go extinct and hence, predator and prey species of system (1.3) may not exist. Fig.2 (b) shows that if $0.029 < m < 0.137$, (x^*, y^*) is an unstable focus, and there is a stable limit cycle surrounding. Fig.2(c) shows that if $0.029 < m < 0.137$, when m decrease, the limit cycle is broken. Fig.2 (d) shows that if $0.137 < m < 0.2$, (x^*, y^*) is attractor. Fig.2 (e) shows that if $0.2 < m < 1$,(x^*, y^*) is a global attractor. That is, Fig.2 show that for $d < f < a+d$, as m increase, the limit cycle is broken and both the prey and predator population converge to their equilibrium values, which means adding a large refuge to the system prevent the oscillating behavior and drive it to a stable state.

Fig.3 (a) shows that for $f = a+d$ if $0 < m < 0.6$, predators go extinct and hence, predator and prey species of system (1.3) may not exist. Fig.3 (b) shows that if $0.6 < m < 1$, (x^*, y^*) is a global attractor. That is, Fig.3 shows that for $f = a+d$, as m increase, under the effects of predator mutual interference, predators do not go extinct and predator and prey species of system (1.3) still coexist.

Fig.4 (a) shows that for $f > a+d$ if $0 < m < 0.4$, predators go extinct and hence, predator and prey species of system (1.3) may not exist, Fig.4 (b) shows that if $0.4 < m < 1$, (x^*, y^*) is a global attractor. That is , Fig.4 shows that for $f > a+d$, as m increase, under the effects of predator mutual interference, predators do not go extinct and predator and prey species of system (1.3) still coexist.

Figures 1, 2 and 3 show that the dependence of the dynamics behavior of system (1.3) on the prey refuge m. If $d < f < a+d$, (x^*, y^*) is an unstable focus as $A_1 < m < A_2$, and (x^*, y^*) is stable as $A_2 < m < 1$. That is, as m increase, the limit cycle is broken and both the prey and predator population converge to their equilibrium values, which means adding a large refuge to the system prevent the oscillating behavior and drive it to a stable state. If $f \geq d$, (x^*, y^*) do not exist as $0 < m < \min(A_1, A_3)$, and (x^*, y^*) is global stable as $max(A_1, A_3) < m < 1$. That is, a large refuge m is possible to prevent nonexistence and drive it to a required stable state when m increase.

6. Concluding remarks

In this paper, we considered a ratio-dependent predator-prey system incorporating a prey refuge m. We classified and determined all possible nature near the origin $(0,0)$ and two other equilibrium $(\frac{a}{b},0)$ and (x_*,y_*) depending on all parameters especially the prey refuge m. Interesting dynamic behavior such as deterministic extinction, existence of multiple attractors, and existence of a limit cycle has been observed. Simulation indicates that m has a stabilizing effect on predator-prey interactions. For one region of parameter f, if adding a small refuge to the system which we considered, the refuge does not alter the dynamical stability of the neutrally stable system. Adding a large refuge to the system prevent the oscillating behavior and drive it to a stable state. The analysis of the influence of m on density values show that increasing the amount of refuge can increase prey densities, increasing the amount of refuge can increase predator densities and decrease predator densities with a large refuge. But because of the positive equilibrium of global stability, it ensures the prey densities will not increase infinitely, and the predator species will not extinct with a large refuge parameter m when $f \geq d$, which is contrary to the predator-prey model with Holling type II incorporating a prey refuge. Kar observed that the predator will be extinct with a large refuge parameter $m(m\varepsilon(B,1))$ when $c\beta > a\gamma$, that is, increasing the amount of refuge can increase prey densities and lead to population outbreak.

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