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EXISTENCE OF POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR FRACTIONAL ORDER DIFFERENTIAL EQUATIONS ON AN UNBOUNDED DOMAIN

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Abstract. In this paper, we consider the existence of positive bounded solution for a boundary value problem on an unbounded domain for differential equation involving the Riemann-Liouville fractional order derivative. The results are based on Krasnosel'skii's fixed point theorem in a cone combined with the diagonalization method.

Keywords. Fractional differential equation; Positive solution; Fixed point theorem; Diagonalization method.

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1. Introduction

In this paper, we consider the existence of positive bounded solutions for following boundary value problem on an unbounded domain for differential equations involving the Riemann-Liouville fractional derivative

$$D_{0+}^{\alpha}u(t) = f(t,u(t)),\ t \in J := [0,\ \infty),\ u(0) = 0, u \textit{ is bounded on } J, \tag{1.1}$$

where $1 < \alpha \leq 2$ and $f : J \times R \to R$ is continuous, and D_{0+}^{α} is the standard Riemann-Liouville derivative.

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Due to the development of the theory of fractional calculus and its applications, many work on fractional calculus, fractional order differential equations have been done (see, for example, [1-7]). Recently, some authors investigated the existence and multiplicity of solutions (or positive solutions) of boundary value problems for nonlinear fractional differential equations (see [8-17] and references therein). In [13], Bai and Lü established the existence results of positive solutions for the following problem

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0\leq t\leq 1,\ u(0)=0, u(1)=\beta u(\eta),\ \eta\in(0,1).$$

In [15], Li, Luo and Zhou considered the three-point boundary value problem of a coupled system of nonlinear fractional differential equations

$$D_{0+}^{\alpha}u(t)=f(t,\ v(t),\ D^{p}v(t)),\ D_{0+}^{\beta}v(t)=f(t,\ u(t),\ D^{q}u(t)),\ 0\leq t\leq 1,$$

$$u(0)=0, u(1)=\gamma u(\eta),\ v(0)=0, v(1)=\gamma u(\eta),$$

under the conditions $0<\gamma\eta^{\alpha-1}<1$, $0<\gamma\eta^{\beta-1}<1$. By using the Schauder fixed point theorem, they obtained at least one solution.

The theory of boundary value problems on infinite intervals has been found in many applications, see [18] and the references therein. Many results have been obtained on the solutions of nonlinear differential equations subject to certain boundary conditions, see [19-21] and the references therein. However, few papers deal with the boundary value problems of fractional order differential equations on infinite intervals. Recently, Agarwal *et al.* [22] established existence results of solutions for a class of boundary value problems involving the Riemann-Liouville fractional derivative on the half line by using the nonlinear alternative of the Leray Schauder type conbined with the diagonalization process. Arara *et al.* [23] considered boundary value problems involving the Caputo fractional derivative on the half line

$${}^{c}D^{\alpha}u(t)=f(t,u(t),\ t\in J:=[0,\ \infty),\ u(0)=u_0, u \textit{ is bounded on } J.$$

By using the Schauder fixed point theorem combined with the diagonalization process, they obtained the existence of solutions. However, to our best knowledge, there is no paper concerned with the existence of positive solutions to boundary value problems of fractional differential equations on infinite intervals by using the fixed point theorem combined with the diagonalization process. The goal of this paper is to fill the gap. Motivated by the corresponding results announced in [22] and [23], we consider the existence of positive solutions of boundary value problems for fractional order differential equations on an unbounded domain.

2. Preliminaries

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $u(t) : (0,\infty) \to R$ is given by

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$$

provided that the right side is point-wise defined on $(0,\infty)$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a continuous function $u(t) : (0,\infty) \to R$ is given by

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha]+1$, provided that the right side is point-wise defined on $(0,\infty)$.

Lemma 2.3. *Let $\alpha > 0$. If we assume $u \in C(0,1)\cup L(0,1)$, then the fractional differential equation $D^{\alpha}_{0+}u(t) = 0$ has a solution*

$$u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} + ... + C_Nt^{\alpha-N}$$

for some $C_i \in R,\ i = 1,\ 2,\ \cdots,\ N$, where N is the smallest integer greater than or equal to α.

Lemma 2.4. *Assume that $u \in C(0,1)\cup L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1)\cup L(0,1)$. Then $I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + ... + C_Nt^{\alpha-N}$, for some $C_i \in R,\ i = 1,\ 2,\ \cdots,\ N$.*

Lemma 2.5. [31] *Let E be a Banach space and let $K \subset E$ be a cone. Assume that Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $A : K\cap(\Omega_2\setminus\overline{\Omega}_1) \to K$ be a completely continuous operator such that*

$$\|Au\| \le \|u\|, u \in K\cap\partial\Omega_1, and\ \|Au\| \ge \|u\|, u \in K\cap\partial\Omega_2$$

or

$$\|Au\| \ge \|u\|, u \in K\cap\partial\Omega_1, and\ \|Au\| \le \|u\|, u \in K\cap\partial\Omega_2.$$

Then A has a fixed point in $K\cap(\Omega_2\setminus\overline{\Omega}_1)$.

First we consider the problem on a bounded domain

$$D^{\alpha}_{0+}u(t)+f(t,\ u(t))=0,\ t\in[0,\ n],\ u(0)=0,u'(n)=0. \tag{2.1}$$

Lemma 2.6. *Given $y(t)\in C[0,1]$. The problem*

$$D^{\alpha}_{0+}u(t)+y(t)=0,\ u(0)=0,\ u'(n)=0 \tag{2.2}$$

is equivalent to $u(t)=\int_0^n G(t,\ s)y(s)ds$, where

$$G(t,s)=\begin{cases}\frac{1}{\Gamma(\alpha)}-(t-s)^{\alpha-1}+n^{2-\alpha}t^{\alpha-1}(n-s)^{\alpha-2}t\geq s,\\ n^{2-\alpha}t^{\alpha-1}(n-s)^{\alpha-2}t\leq s\end{cases}$$

Furthermore, $G(t,\ s)$ is continuous on $[0,\ n]\times[0,\ n]$ and satisfies the conditon $G(t,\ s)>0,\ t,\ s\in[0,\ n]$.

Proof. From Lemma 2.3, we get that problem (2.2) is equivalent to

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+C_1t^{\alpha-1}+C_2t^{\alpha-2}.$$

The boundary condition $u(0)=0$ induces $C_2=0$. Considering the boundary condition $u'(n)=0$, we have

$$C_1=\frac{n^{2-\alpha}}{\Gamma(\alpha)}\int_0^n(n-s)^{\alpha-2}y(s)ds.$$

Thus

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\frac{n^{2-\alpha}t^{\alpha-1}}{\Gamma(\alpha)}\int_0^n(n-s)^{\alpha-2}y(s)ds.$$

It is obvious that $G(t,\ s)$ is continuous on $[0,\ n]\times[0,\ n]$. Furthermore, $G(t,\ s)\geq 0$ for $t\leq s$. For $t\geq s$, one has

$$\begin{aligned}G(t,\ s)&=-(t-s)^{\alpha-1}+n^{2-\alpha}t^{\alpha-1}(n-s)^{\alpha-2}\\&=t^{\alpha-1}[(1-\frac{s}{n})^{\alpha-2}-(1-\frac{s}{t})^{\alpha-1}]\\&>0.\end{aligned}$$

Lemma 2.7. *The function $G(t,s)$ satisfies the following conditions:*

(1) $G(t,s)\leq G(s,s),\ t,s\in[0,1]$,

(2) there exist a positive function $\gamma(s)\in C(0,\ 1)$ such that

$$\min_{s\in[\frac{n}{4},\ \frac{3n}{4}]}G(t,\ s)\leq\gamma G(s,\ s),\ 0<s<1.$$

Proof (1) For given $s \in [0,\ n]$, $G(t,\ s)$ is decreasing with respect to t for $s \le t$ and increasing with respect to t for $t \le s$. Thus one can easily check that $G(t,s) \le G(s,s),\ t,s \in [0,1]$.

(2) Setting

$$g_1(t,\ s) = \frac{-(t-s)^{\alpha-1} + n^{2-\alpha} t^{\alpha-1}(n-s)^{\alpha-2}}{\Gamma(\alpha)}, g_2(t,\ s) = \frac{n^{2-\alpha} t^{\alpha-1}(n-s)^{\alpha-2}}{\Gamma(\alpha)},$$

One has

$$\min_{n/4 \le t \le 3n/4} G(t,s) = \begin{cases} g_1(\frac{3n}{4},\ s), & 0 < s \le \frac{n}{4}, \\ \min\{g_1(\frac{3n}{4},\ s),\ g_2(\frac{n}{4},\ s),\} & \frac{n}{4} \le s \le \frac{3n}{4}, \\ g_2(\frac{n}{4},\ s), & \frac{3n}{4} \le s < 1, \end{cases}$$

$$= \begin{cases} g_1(\frac{3n}{4},\ s), & 0 < s \le r, \\ g_2(\frac{n}{4},\ s), & r \le s < 1, \end{cases}$$

$$= \begin{cases} \dfrac{-(\frac{3n}{4}-s)^{\alpha-1} + n^{2-\alpha}(\frac{3n}{4})^{\alpha-1}(n-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 < s \le r, \\ \dfrac{n^{2-\alpha}(\frac{n}{4})^{\alpha-1}(n-s)^{\alpha-2}}{\Gamma(\alpha)}, & r \le s < 1, \end{cases}$$

where $\frac{n}{4} < r < \frac{3n}{4}$ is the unique solution of the equation

$$-(\frac{3n}{4}-s)^{\alpha-1} + n^{2-\alpha}(\frac{3n}{4})^{\alpha-1}(n-s)^{\alpha-2} = n^{2-\alpha}(\frac{n}{4})^{\alpha-1}(n-s)^{\alpha-2}.$$

Considering the monotonicity of $G(t,\ s)$, we have

$$\max_{0 \le t \le 1} G(t,\ s) = G(s,\ s) = \frac{n^{2-\alpha} s^{\alpha-1}(n-s)^{\alpha-2}}{\Gamma(\alpha)}.$$

Thus, setting

$$\gamma(s) = \begin{cases} \dfrac{-(\frac{3n}{4}-s)^{\alpha-1} + n^{2-\alpha}(\frac{3n}{4})^{\alpha-1}(n-s)^{\alpha-2}}{n^{2-\alpha} s^{\alpha-1}(n-s)^{\alpha-2}}, & 0 < s \le r, \\ (\frac{n}{4s})^{\alpha-2}, & r \le s < 1, \end{cases}$$

we find that

$$\min_{s \in [\frac{n}{4},\ \frac{3n}{4}]} G(t,\ s) \le \gamma G(s,\ s),\ 0 < s < 1.$$

This completes the proof.

3. Main results

In this section, we give the existence of positive solutions for problem (1.1). We divide it into two steps. First, we obtain the existence of positive solutions of the problem on the bounded domain (2.1). Then, the existence results of positive solutions of (1.1) is obtained by using the diagonalization method.

Let $E = C[0,\ n]$ be endowed with the maximum norm, $\|u\| = \max_{0\leq t\leq n}$. Define the cone $P \subset E$ by $P = \{u \in E|u(t) \geq 0\}$,

Theorem 3.1. *Let $T : P \to E$ be the operator defined by $Tu(t) = \int_0^n G(t,\ s)f(s,\ u(s))ds$. Then $T : P \to P$ is completely continuous.*

Proof It is obvious that operator $T : P \to P$ is continuous. Let $\Omega \subset P$ be bounded, that is, there exists a positive constant $M_1 > 0$ such that $\|u\| \leq M_1$ for all $u \in \Omega$. Then for $u \in \Omega$, we have

$$Tu(t) = \int_0^n G(t,\ s)f(s,\ u(s))ds \leq M_2 \int_0^n G(s,\ s)ds,$$

where $M_2 = \max_{0\leq t\leq n,\ 0\leq u\leq M_1}$. It follows that T is bounded on the bounded subset of E. On the other hand, for each $u \in \Omega,\ t_1,\ t_2 \in [0,\ n],\ t_1 < t_2$ and $|t_2 - t_1| \leq \delta$, one has

$$\begin{aligned}|Tu(t_2) - Tu(t_1)| = |&\int_0^n (G(t_1,\ s) - G(t_2,\ s))f(s,\ u(s))ds| \\ &\leq \int_0^n |G(t_1,\ s) - G(t_2,\ s)|f(s,\ u(s))ds \\ &\leq M_2 \int_0^n |G(t_1,\ s) - G(t_2,\ s)ds.\end{aligned}$$

Then the continuity of function $G(t,\ s)$ implies that T is equicontinuity on the bounded subset of E. This completes the proof.

Theorem 3.2. *Let $f(t,\ u)$ is continuous on $[0,\ n]\times[0,\ +\infty)$. Assume that there exist two positive constant $r_2 > r_1 > 0$ such that*
(A1) $f(t,\ u) \leq Mr_2,\ (t,\ u) \in [0,\ n]\times[0,\ r_2]$,
(A2) $f(t,\ u) \geq Nr_2,\ (t,\ u) \in [0,\ n]\times[0,\ r_1]$,
where

$$M = (\int_0^n G(s,\ s)ds)^{-1},\ (\int_{\frac{n}{4}}^{\frac{3n}{4}} \gamma(s)G(s,\ s)ds)^{-1},$$

then problem (2.1) has at least one positive solution u such that $r_1 \leq \|u\| \leq r_2$.

Proof From Lemmas 2.3 and 2.4, we see that problem (2.1) has a positive solution $u = u(t)$ if and only if u is a fixed point of operator T. Let $\Omega_1 = \{u \in P| \|u\| \leq r_1\}$. For $u \in \partial\Omega_1$, we have $0 \leq u(t) \leq r_1$, for all $t \in [0,\ n]$. From assumption $(A2)$, for $t \in [\frac{n}{4},\ \frac{3n}{4}]$, we have

$$Tu(t) = \int_0^n G(t,\ s)f(s,\ u(s))ds \geq \int_0^n \gamma(s)G(s,\ s)f(s,u(s))ds \geq r_1 = \|u_1\|.$$

Thus $\|Tu\| \geq \|u\|, u \in \partial\Omega_1$. Let $\Omega_2 = \{u \in P| \|u\| \leq r_2$. For $u \in \partial\Omega_2$, we have $0 \leq u(t) \leq r_2$, for all $t \in [0,\ n]$. From assumption $(A1)$, for $t \in [0,\ n]$, we have

$$Tu(t) = \int_0^n G(t,\ s)f(s,\ u(s))ds \leq Mr_2 \int_0^n G(s,\ s)ds \geq r_2 = \|u\|.$$

Thus $\|Tu\| \leq \|u\|, u \in \partial\Omega_2$. An application of Lemma 2.5 ensures the existence of at least one positive solution of problem (2.1).

Next, we denote the positive solution of problem (2.1) by u_n. We now use the diagonalization process to obtain the positive solution of problem (1.1) on the infinite intervals. For $k \in N$, setting

$$u_k(t) = \begin{cases} u_k(t), & t \in [0,\ n_k], \\ u_k(n_k), & t \in [n_k,\ \infty), \end{cases}$$

where $\{n_k\} \in N^*$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \cdots < n_k < \cdots \uparrow \infty.$$

Let $S = \{u_k\}_{k=1}^{\infty}$. We claim that

$$r_1 \leq u_k(t) \leq r_2 \ for\ t \in [0,\ n_k],\ k \in N.$$

For $k \in N$ and $t \in [0, n_1]$, we have

$$u_{n_k}(t) = \int_0^{n_1} G_{n_1}(t,\ s)f(s,\ u_{n_k}(s))ds.$$

Thus, for $k \in N$ and $t,\ x \in [0,\ n_1]$, we have

$$u_{n_k}(t) - u_{n_k}(x) = \int_0^{n_1} [G_{n_1}(t,\ s) - G_{n_1}(x,\ s)]f(s,\ u_{n_k}(s))ds.$$

Using the Arezela-Ascoli theorem, we see that there exists a subsequence N_1^* of N and a function $v_1 \in C([0,\ n_1],\ R^+)$ with $u_{n_k} \to v_1$ in $C([0,\ n_1],\ R^+)$ as $k \to \infty$ through N_1^*. Let $N_1 = N_1^* \backslash \{1\}$. Consider that

$$u_{n_k}(t) \leq r_2,\ t \in [0,\ n_2],\ k \in N.$$

Then by Arezela-Ascoli theorem, there exists a subsequence N_2^* of N and a function $v_2 \in C([0,\ n_2],\ R^+)$ with $u_{n_k} \to v_2$ in $C([0,\ n_2],\ R^+)$ as $k \to \infty$ through N_2^* and $v_1 = v_2$ *on* $[0,\ n_1]$. Let $N_2 = N_2^* \backslash \{2\}$. Proceed inductively to obtain for $m \in \{3,\ 4,\ \cdots\}$ a subsequence N_m^* of N_{m-1} and a function $v_m \in C([0,\ n_m],\ R^+)$ with $u_{n_k} \to v_m$ in $C([0,\ n_m],\ R^+)$ as $k \to \infty$ through N_m^*. Let $N_m = N_m^* \backslash \{m\}$. Define a function u as follows. Fix $t \in (0,\ \infty)$ and let $m \in N$ with $s \leq n_m$. Then define $u(t) = v_m(t)$. It follows that $u \in C([0,\ \infty),\ R^+)$ and $u(0) = 0,\ r_1 \leq u(t) \leq r_2$ for $t \in [0,\ \infty)$. Fix $t \in (0,\ \infty)$ and let $m \in N$ with $s \leq n_m$. For $n \in N_m$, we get

$$u_{n_k}(t) = \int_0^{n_m} G_{n_m}(t,\ s) f(s,\ u_{n_m}(s)) ds.$$

Letting $n_k \to \infty$ through N_m, we obtain

$$v_m(t) = \int_0^{n_m} G_m(x,\ s) f(s, z_m(s)) ds,$$

that is

$$u(t) = \int_0^{n_m} G_{n_m}(t,\ s) f(s,\ y(s)) ds.$$

We can use this method for each $x \in [0,\ n_m]$ and for each $m \in N$. Thus

$$D^\alpha u(t) = f(t,\ u(t)),\ t \in [0,\ n_m]$$

for each $m \in N$ and $\alpha \in (1,\ 2]$ and the constructed function y is a positive solution of (1.1). This completes the proof.

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