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THE INDEX-JUMP PROPERTY FOR 1-HOMOGENEOUS POSITIVE MAPS AND FIXED POINT THEOREMS IN CONES

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Abstract. We present in this paper new fixed point theorems for positive maps having approximative minorant and majorant at 0 and ∞ in specific classes of operators.

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1. Introduction

The problem of seeking positive solutions for boundary value problems associated with p-Laplacian differential equations having positive nonlinearities, is usually converted to that of finding solutions in the cone of nonnegative functions C of some functional space X, to the abstract Hammerstein equation,

$$u = NFu, \tag{1.1}$$

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where N is an increasing, 1-homogeneous and completely continuous self mapping of X and $F : C \to C$ is continuous and bounded (maps bounded sets into bounded sets). Note that the mapping $T = NF$ leave invariant the cone C.

This formulation has motivated many works, where existence results of fixed point for operators leaving invariant a cone have been proved. Krasnosel'skii's theorems of compression and expansion of a cone in a Banach space (see Theorems 4.12 and 4.14 in [14] and Theorems 2.3.3 and 2.3.4 in [13]), are the most famous and the most used in the literature.

Krasnosel'skii has provided in [14] many others interesting fixed point theorems. Among these results, Theorems 4.10, 4.11 and 4.16 have attracted the attention of Amann in [1] where he generalized these results for strict set-contraction leaving invariant a cone in a Banach space. Roughtly speaking, these theorems and their generalization, state that if such an operator is approximatively linear at 0 and ∞, and the spectral radius of the linear approximations are oppositely located with respect to 1, then it has a fixed point.

In this paper, we will prove new fixed point theorems for operators leaving invariant a cone in a Banach space, and as in Krasnosel'kii's theorems, the main assumptions are on the behavior of the operator at 0 and ∞. Let $\mathcal{F}(X)$ be the class of all increasing 1-homogeneous self operator in X, more precisely, we will assume that our operator has an approximative minorant at 0 and an approximative majorant at ∞ in the class $\mathcal{F}(X)$, or conversely; existence of the fixed point is obtained under additional conditions: it is required that, the approximative minorant has the strongly index-jump property and the positive spectrums of the approximative majorant and minorant are oppositely located with respect to 1. The concepts of index-jump and the strongly index-jump will be introduced in Section 2, where we prove that a 1-homogeneous, positive and completely continuous operator has the index-jump property if and only if it has a positive eigenvalue and we present some classes of such operators having the strongly index-jump property.

The interest to the strongly index-jump property is motivated by the fact that it is conserved by limits of nondecreasing sequences of operators having the strongly index-jump property (see the proof of Theorem 3.16). In order to indicate the interest of this property, let us return to bvps. In the case where the nonlinearity has a singular weight, the operator N in formulation

(1.1) will contain this singular weight. Technically, one can see that such an operator is a limit of a nondecreasing sequence of operators (N_n) having the SIJP. This what makes interesting the above property.

The spirit of hypotheses in this work meet that in many results in the literature. Theorem 7.B in [20] state that if a positive mapping T has a linear minorant having a eigensubsolution, then T has eigensolutions. Webb in [19] has obtained fixed calculations for a positive mapping A under the condition that A has a specific linear majorant or minorant (see Theorems 4.4, 4.5 and 4.7 in [19]); he has also provided nonexistence results under similar conditions (see Theorem 4.9 in [19]). Main ideas of this work are inspired from the works in [2], [3] and [4].

The paper is organized as follows. Section 2 is devoted for the needed background. In Section 3, we present the main results and their needed preliminaries. In the last section, we prove by means of main results of Section 3, existene results for at least one positive solution to a class of p-Lapalcian bvps having regular or singular weights.

2. Abstract background

We will use extensively in this work cones and the fixed point index theory, so let us recall some facts related to these two tools. Let X be a Banach space, a nonempty closed convex subset K of X is said to be a cone if $(tC) \subset C$ for all $t \geq 0$ and $C \cap (-C) = \{0_X\}$. It is well known that a cone C induces a partial order in the Banach space X. We write for all $x, y \in X$: $x \preceq y$ if $y - x \in C$, $x \prec y$ if $y - x \in C$, $y \neq x$ and $x \npreceq y$ if $y - x \notin C$. Notations $\succeq$, $\succ$ and $\nsucceq$ denote respectively the inverse situations.

A cone C is said to be normal with a constant $n_C > 0$ if for all u, v in C, $u \preceq v$ implies $\|u\| \leq n_C \|v\|$.

A function $f : \Omega \subset X \to X$ is said to be bounded, if it maps bounded sets into bounded sets and it is said to be completely continuous, if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.1. Let C be a cone in X and $N : X \to X$ a continuous map. N is said to be

a) positive, if $N(C) \subset C$,

b) strongly positive, if C has a nonempty interior ($intC \neq \emptyset$) and $N(C \setminus \{0_X\}) \subset intC$,

c) increasing, if for all $u, v \in X,\ u \preceq v$ implies $Nu \preceq Nv$,

d) strictly increasing, if for all $u, v \in X,\ u \prec v$ implies $Nu \prec Nv$.

e) 1-homogeneous, if for all $u \in X$ and $t \in \mathbb{R}$, $N(tu) = tN(u)$ and

f) subadditive on C if for all $u, v \in C$, $N(u+v) \preceq Nu + Nv$.

Definition 2.2. Let C be a cone in X and let $N_1, N_2 : X \to X$ be positive maps. We write $N_1 \preceq N_2$ if for all $x \in C,\ N_1 x \preceq N_2 x$.

Definition 2.3. Let C be a cone in X and $N : X \to X$ a positive map. N is said to be lower bounded on C if there exists a positive constant m such that for all $u \in C,\ \|Nu\| \geq m\|u\|$. For such an operator N, we denote

$$N_C^- = \inf\{\|Nu\| / \|u\|,\ u \in C \setminus \{0_X\}\}.$$

Let $H(X)$ be the set of all continuous and 1-homogeneous self mapping on X, for $N \in H(X)$, we set $\|N\| = \sup_{\|u\|=1} \|Nu\|$. Arguing as for the space of linear continuous self mapping, we obtain the following lemma.

Lemma 2.4. *The pair* $(H(X), \|\cdot\|)$ *is a Banach space.*

The concept of positive eigenvalue will be extensively evoked in this work, it is introduced in the following definition.

Definition 2.5. Let C be a cone in X and let $N \in H(X)$ be a positive operator. A nonnegative constant μ is said to be a positive eigenvalue of N if there exist $u \in C \setminus \{0_X\}$ such that $Nu = \mu u$.

Lemma 2.6. *Let C be a cone in X and for all integer $n \geq 1,\ N_n$ is a positive completely continuous mapping in $H(X)$, having a positive eigenvalue λ_n. If $N_n \to N$ in operator norm and $\lambda_n \longrightarrow \lambda > 0$ as $n \longrightarrow +\infty$ then λ is a positive eigenvalue of N.*

Proof. Let ϕ_n be a normalized eigenvector associated with λ_n and $\psi_n = N(\phi_n)$. Since N is completely continous, there exists a subsequence denote also (ϕ_n), such that (ψ_n) converges to some $\psi \in K$. Thus, we have the following estimates:

$$\begin{aligned} \|\lambda_n \phi_n - \psi\| &= \|N_n(\phi_n) - \psi\| \\ &\leq \|N_n(\phi_n) - N(\phi_n)\| + \|N(\phi_n) - \psi\| \\ &\leq \|N_n - N\| + \|\psi_n - \psi\|. \end{aligned}$$

So,

$$\lim \lambda_n \phi_n = \psi \text{ and } \|\psi\| = \lim \|\lambda_n \phi_n\| = \lim \lambda_n = \lambda > 0.$$

Also, we have

$$\left\|N_n(\phi_n) - \tfrac{1}{\lambda} N(\psi)\right\| = \left\|\tfrac{1}{\lambda_n} N_n(\lambda_n \phi_n) - \tfrac{1}{\lambda} N(\psi)\right\| \leq$$
$$\left\|\tfrac{1}{\lambda_n} N_n(\lambda_n \phi_n) - \tfrac{1}{\lambda} N_n(\lambda_n \phi_n)\right\| + \left\|\tfrac{1}{\lambda} N_n(\lambda_n \phi_n) - \tfrac{1}{\lambda} N(\lambda_n \phi_n)\right\| + \left\|\tfrac{1}{\lambda} N(\lambda_n \phi_n) - \tfrac{1}{\lambda} N(\psi)\right\|$$
$$\leq \left|\tfrac{1}{\lambda_n} - \tfrac{1}{\lambda}\right| |\lambda_n| \|N_n\| + \tfrac{1}{\lambda} |\lambda_n| \|N_n - N\| + \tfrac{1}{\lambda} \|N(\lambda_n \phi_n) - N(\psi)\|.$$

This together with the continuity of N, implies that $\lim N(\lambda_n \phi_n) = N(\psi)$. Leting $n \to \infty$ in $N_n(\phi_n) = \lambda_n \phi_n$, we get that $N(\psi) = \lambda \psi$. This completes the proof.

We will use extensively in this work the fixed point index theory. For sake of completeness, let us recall some lemmas providing fixed point index computations. Let C be a cone in X. Let for $R > 0$, $C_R = C \cap B(0_X, R)$ where $B(0_X, R)$ is the open ball of radius R centred at 0_X, ∂C_R be its boundary and consider a compact mapping $f : \overline{C_R} \to C$.

Lemma 2.7. *If $fx \neq \lambda x$ for all $x \in \partial C_R$ and $\lambda \geq 1$, then $i(f, C_R, C) = 1$.*

Lemma 2.8. *If $fx \ngeq x$ for all $x \in \partial C_R$, then $i(f, C_R, C) = 1$.*

Lemma 2.9. *If $fx \nleq x$ for all $x \in \partial C_R$, then $i(f, C_R, C) = 0$.*

Lemma 2.10. *If there exists $e \succ 0_X$ such that $x \neq fx + te$ for all $t \geq 0$ and all $u \in \partial C_R$, then $i(f, C_R, C) = 0$.*

For more details on the fixed point index theory, see, for instance, [10] and [13].

3. Main results

Preliminaries

In all this section E is a real Banach space, K, P are two nontrivial cones in E with $P \subset K$ and $H(E)$ denotes the Banach space of continuous 1-homogeneous self operator of E endowed with the norm, $\|N\| = \sup_{\|u\|=1} \|Nu\|$. Hereafter $\preceq$ denotes the order induced by the cone K on E and we set,

$$N_K^P(E) = \{N \in N_K(E) : N \text{ is increasing and } N(K) \subset P\}$$

and

$$Q_K^P(E) = \left\{N \in N_K^P(E) : N \text{ is completely continuous}\right\}.$$

In fact, the cone K is a naturel cone of E as the cone of nonnegative functions in functional spaces and for $N \in N_K^P(E)$, the cone P is related to the operator N and in some manner it represents the regularity of N; See for example the cone P in (4.5) and the property of lower boundness of the operator N_p in Lemma 4.1.

Now, for $N \in N_K^P(E)$ we define the subsets

$$\Lambda_P^N = \{\lambda \geq 0 : \text{ there exist } u \in P \setminus \{0_E\} \text{ such that } Nu \preceq \lambda u\},$$

$$\Theta_P^N = \{\theta \geq 0 : \text{ there exist } u \in P \setminus \{0_E\} \text{ such that } Nu \succeq \theta u\}.$$

Remark 3.1. Note that

a) $0 \in \Theta_P^N$ and if $\theta \in \Theta_P^N$ then $[0, \theta] \subset \Theta_P^N$.

b) If $\lambda \in \Lambda_P^N$ then $[\lambda, +\infty[\subset \Lambda_P^N$.

c) $\Lambda_P^N \subset \Lambda_K^N$ and $\Theta_P^N \subset \Theta_K^N$.

d) If μ is positive eigenvalue of N then $\mu \in \Theta_P^N \cap \Lambda_P^N \cap [0, \|N\|]$.

e) If $N^{-1}(0_E) \cap K = \{0_E\}$ then $\Lambda_P^N = \Lambda_K^N$ and $\Theta_P^N = \Theta_K^N$.

Also, for $N \in N_K^P(E)$, $\sigma_K(N)$ denotes the set of all positive eigenvalues of N,

$$\sigma_N^- = \inf \sigma_K(N) \text{ and } \quad \sigma_N^+ = \begin{cases} \sup \sigma_K(N) \text{ if } \sigma_K(N) \neq \emptyset, \\ 0 \qquad \text{if } \sigma_K(N) = \emptyset. \end{cases}$$

Lemma 3.2. *For all $N \in N_K^P(E)$, the subset Θ_P^N is bounded from above by $\|N\|$.*

Proof. Let $\theta > \|N\|$ and $R_\theta = \sum_{k \in \mathbb{N}} N^k/\theta^k$ and note that $R_\theta = I + R_\theta(N/\theta)$. Moreover, we have $R_\theta(P) \subset P$ since for all k, $N^k(P) \subset P$.

Now, by the contrary, suppose that there exists $u \in \partial P_1$ such that $Nu \succeq \theta u$ and set $v = \theta^{-1} Nu$. We have then the contradiction

$$R_\theta(v) \succeq R_\theta(u) = u + R_\theta(v) \succ R_\theta(v).$$

This shows that Θ_P^N is bounded from above by $\|N\|$. This completes the proof.

Lemma 3.3. *For all $N \in Q_K^P(E)$, the subset Λ_P^N is nonempty.*

Proof. Let $\lambda > \|N\|$ and $e \in P \setminus \{0_E\}$ and consider the equation

$$u = N_\lambda(u,t), \tag{3.1}$$

where for all $u \in P$ and $t \in [0,1]$, $N_\lambda(u,t) = (t/\lambda) Nu + e$. Clearly $N_\lambda(P \times [0,1]) \subset P$ and equation (3.1) has no solution in ∂P_R with $R > \max(\lambda \|e\| / (\lambda - \|N\|), \|e\|)$. Thus, by homotopy and normality properties of the fixed point index, we conclude that

$$i(N_\lambda(\cdot,1), P_R, P) = i(N_\lambda(\cdot,0), P_R, P) = 1.$$

Then equation $N_\lambda(u,1) = u$ admits a solution $u_0 \in P_R \setminus \{0_E\}$ and $\lambda \in \Lambda_P^N$. For all $N \in Q_K^P(E)$, the constants λ_P^N and θ_P^N will play an important role in all this paper and they are defined by

$$\lambda_P^N = \inf \Lambda_P^N, \ \theta_P^N = \sup \Theta_P^N.$$

Lemma 3.4. *Let $N \in Q_K^P(E)$ and assume that $\lambda_P^N, \theta_P^N > 0$. Then for all $\gamma, R > 0$ we have*

$$i(\gamma N, P_R, P) = \begin{cases} 1, \ \textit{if } \gamma\theta_P^N < 1, \\ 0, \ \textit{if } \gamma\lambda_P^N > 1. \end{cases}$$

Proof. Let $R > 0$, $\gamma \in (0, 1/\theta_P^N)$ and $u \in \partial P_R$ such that $\gamma Nu \succeq u$. This implies that $1/\gamma \in \Theta_P^N$ and $1/\gamma \leq \theta_P^N$ which contradicts $\gamma \in (0, 1/\theta_P^N)$. So, the hypothesis of Lemma 2.8 hold and $i(\gamma N, P_R, P) = 1$. Let $R > 0$, $\gamma > 1/\lambda_P^N$ and $u \in \partial P_R$ such that $\gamma Nu \preceq u$. This implies that $1/\gamma \in \Lambda_P^N$ and $1/\gamma \geq \inf \Lambda_p^N = \lambda_P^N$ which contradicts $\gamma > 1/\lambda_P^N$. So, the hypothesis of Lemma 2.7 hold and $i(\gamma N, P_R, P) = 0$. This completes the proof.

Lemma 3.5. *For all $N \in Q_K^P(E)$ we have $\lambda_P^N \leq \theta_P^N$.*

Proof. Indeed, if $\lambda_P^N > \theta_P^N$ we have then from Lemma 3.4, for $\gamma \in \left(1/\lambda_P^N, 1/\theta_P^N\right)$, the contradiction

$$i(\gamma L, K_R, K) = \begin{cases} 1, \text{ since } \gamma\theta_P^N < 1, \\ 0, \text{ since } \gamma\lambda_P^N > 1. \end{cases}$$

Remark 3.6. We deduce from d) in Remark 3.1 and Lemma 3.5 that for all $N \in Q_K^P(E)$, $\sigma_K(N) \subset \left[\lambda_P^N, \theta_P^N\right]$.

Remark 3.7. Let $N \in Q_K^P(E)$, $v \in P \smallsetminus \{0_E\}$ and assume that the constant λ_P^N is positive. Then, equation $\lambda u - Nu = v$ has no solution in P for all $\lambda \in \left(0, \lambda_P^N\right)$.

The index jump property

Let $N \in Q_K^P(E)$ and $\gamma \in (0, +\infty) \smallsetminus \sigma_K(N)$. The integer $i(\gamma N, K_R, K)$ is defined for all $R > 0$ and the excision property of the fixed point index, make it independant of R. Moreover, if $\gamma < 1/\|N\|$, we have for all $u \succ 0_E$ and $\lambda \geq 1$, $\gamma Nu \neq \lambda u$ and Lemma 2.7 leads to $i(\gamma N, P_R, P) = 1$. This justifies the following definition.

Definition 3.8. An operator $N \in Q_K^P(E)$ is said to have the index-jump property (IJP for short) if

$$\nu_N = \sup\{\gamma > 0 : i(\gamma N, P_R, P) = 1\} < \infty$$

and in this case we say that N has the IJP at ν_N.

Theorem 3.9. *An operator $N \in Q_K^P(E)$ has the IJP if and only if $\sigma_K(N) \neq \emptyset$. Moreover, we have that $\nu_N = \sigma_N^+$.*

Proof. Let $N \in Q_K^P(E)$ having the IJP at ν_N and let γ be a positive real number. By the contrary, suppose that $\sigma_K(N) = \emptyset$. In this case, for all $\lambda \geq 1$ and $u \succ 0_E$, $\gamma Nu \neq \lambda u$. Hence, we have from Lemma 2.7 that for all $R > 0$, $i(\gamma N, P_R, P) = 1$ contradicting the IJP property of N. Now, suppose that for some $N \in Q_K^P(E)$, , $\sigma_K(N) \neq \emptyset$ and let $\gamma > 1/\sigma_N^+$. Consider for $t > 0$ the equation

$$u - \gamma N(u) = t\phi_1, \tag{3.2}$$

where $\phi_1 \succeq 0_E$ is such that $N(\phi_1) = \sigma_N^+ \phi_1$ and $t > 0$. We claim that (2.2) has no positive solution. Indeed, if there exists $x \succ 0_E$ satisfying (3.2), then, one has

$$x \succeq \gamma N(x) \tag{3.3}$$

and

$$x \succeq t\phi_1. \tag{3.4}$$

We obtain from (3.3) that

$$x \succeq (\gamma N)(x) \succeq (\gamma N)^2(x) \succeq (\gamma N)^3(x) \succeq \cdots \succeq (\gamma N)^k(x) \succeq \cdots$$

From (3.4), we have

$$x \succeq (\gamma N)^k(x) \succeq (\gamma N)^k(t\phi_1) = \gamma^k (\sigma_{N,K}^+)^k (t\phi_1),$$

which leads to

$$t\phi_1 \preceq x / \left(\gamma \sigma_N^+\right)^k. \tag{3.5}$$

Taking in account, $\gamma\sigma_N^+ > 1$, we obtain from (3.5) the contradiction

$$0_E \prec t\phi_1 \preceq x / \left(\gamma \sigma_N^+\right)^k \to 0_E \text{ as } k \to \infty.$$

Thus, we have from Lemma 2.10, $i(\gamma N, P_R, P) = 0$ whenever $1/\gamma \notin \sigma_K(N)$ and this shows that $\nu_N = \sup\{\gamma > 0 : i(\gamma N, P_R, P) = 1\} < \infty$. At the end, we have from Lemma 2.7 that for all $\gamma < 1/\sigma_N^+$, $i(\gamma N, P_R, P) = 1$. This shows that $\sigma_N^+ = \nu_N$ and ending the proof.

We present now a concept which is stronger that the IJP, which plays an important role in our main results.

Definition 3.11. An operator $N \in Q_K^P(E)$ is said to have the strongly index-jump property (SIJP for short) if $\lambda_P^N > 0$. Moreover, if $\lambda_P^N = \theta_P^N = \nu$ then we say that N has the SIJP at ν.

Proposition 3.12. *Let $N_1, N_2 \in N_K^P(E)$ and assume that $N_1 \preceq N_2$. Then $\lambda_P^{N_1} \leq \lambda_P^{N_2}$ and $\theta_P^{N_1} \leq \theta_P^{N_2}$. Moreover if $N_1, N_2 \in Q_K^P(E)$ and N_1 has the SIJP, then N_2 has the SIJP.*

Proof. Indeed, we have

$$\Theta_P^{N_1} \subset \Theta_P^{N_2} \text{ and } \Lambda_P^{N_2} \subset \Lambda_P^{N_1},$$

which leads to

$$\lambda_P^{N_1} \leq \lambda_P^{N_2} \text{ and } \theta_P^{N_1} \leq \theta_P^{N_2}.$$

Proposition 3.13. *If $N \in Q_K^P(E)$ is lower bounded on P, then N has the SIJP.*

Proof. By the contrary, suppose that $\lambda_P^N = 0$, in this case, there exists sequences $(\lambda_n) \subset (0, +\infty)$ and $(u_n) \subset \partial P_1$ such that $\lim_{n \longrightarrow +\infty} \lambda_n = 0$ and $Nu_n \preceq \lambda_n u_n$. Let (u_{n_k}) be a subsequence such that $\lim Nu_{n_k} = v \in P$. In one hand, we have that

$$v = \lim Nu_{n_k} \preceq \lim \lambda_{n_k} u_{n_k} = 0_E,$$

and in the other,

$$\|v\| = \lim \|Nu_{n_k}\| \geq N_P^- \|u_{n_k}\| = N_P^- > 0.$$

This ends the proof.

In the reminder of this subsection, we answer to the question: what represent the constants λ_P^N and θ_P^N for the operator $N \in Q_K^P(E)$?

Proposition 3.14. *Let $N \in Q_K^P(E)$ and assume $\theta_P^N > 0$. Then $\theta_P^N = \sigma_N^+$.*

Proof. Let $(\theta_n) \subset \left(0, \theta_P^N\right) \subset \Theta_P^N$ be an increasing sequence converging to θ_P^N and consider for all integer $n \geq 1$, the cone

$$P_n = \{u \in P : Nu \geq \theta_n u\}$$

and note that K_n is not the trivial cone and $N(P_n) \subset P_n$. Consider also the sets

$$\Lambda_n^N = \{\lambda \geq 0 : \text{ there exist } u \in P_n \setminus \{0_E\} \text{ such that } Nu \preceq \lambda u\},$$

$$\Theta_n^N = \{\theta \geq 0 : \text{ there exist } u \in P_n \setminus \{0_E\} \text{ such that } Nu \succeq \theta u\}$$

and the constants

$$\lambda_n^N = \inf \Lambda_n^N \text{ and } \theta_n^N = \sup \Theta_n^N.$$

By simple computations, one obtains

$$0 < \theta_n \leq \lambda_n^N \leq \theta_n^N \leq \theta_P^N$$

and N admits for all $n \geq 1$ a positive eigenvalue μ_n associated with a normalized eigenvector $\phi_n \in P_n \setminus \{0_E\}$ with

$$0 < \theta_n \leq \lambda_n^N \leq \mu_n \leq \theta_n^N \leq \theta_P^N.$$

Clearly, we have $\lim \mu_n = \theta_P^N$. Thus, we have for all $n \geq 1$

$$N^2 \phi_n = \mu_n N \phi_n = \mu_n^2 \phi_n$$

and the compcteness of N leads to $\phi = \lim N\phi_n$ (up to a subsequence) satisfies $N\phi = \theta_P^N \phi$ and $\|\phi\| = \theta_P^N > 0$. At the end, Remark 3.6 leads to $\theta_P^N = \sigma_N^+$.

Proposition 3.15. *Let $N \in Q_K^P(E)$ be subadditive (in particular linear) and assume that $\lambda_P^N > 0$. Then $\lambda_P^N = \sigma_N^-$.*

Proof. Let $(\lambda_n) \subset \left(\lambda_P^N, +\infty\right) \subset \Lambda_P^N$ be a decreasing sequence converging to λ_P^N and $(\phi_n) \subset P \setminus \{0_E\}$ such that $N\phi_n \preceq \lambda_n \phi_n$. Consider for all integer $n \geq 1$, $P_n = \{u \in P : Nu \preceq \lambda_n u\}$ and note that the subadditivity of N makes of the set P_n convex and so, a cone in E which is diffrent from the trivial one, since $\phi_n \in P_n$. We have also, $N(P_n) \subset P_n$ and so, consider the sets

$$\Lambda_n^N = \{\lambda \geq 0 : \text{ there exist } u \in P_n \setminus \{0_E\} \text{ such that } Nu \preceq \lambda u\},$$

$$\Theta_n^N = \{\theta \geq 0 : \text{ there exist } u \in P_n \setminus \{0_E\} \text{ such that } Nu \succeq \theta u\}$$

and the constants

$$\lambda_n^N = \inf \Lambda_n^N \text{ and } \theta_n^N = \sup \Theta_n^N.$$

Clearly, we have $0 < \lambda_P^N \leq \lambda_n^N \leq \theta_n^N \leq \lambda_n$ and N admits for all $n \geq 1$ a positive eigenvalue μ_n associated with a normalized eigenvector $\psi_n \in P_n$ with

$$0 < \lambda_P^N \leq \lambda_n^N \leq \mu_n \leq \theta_n^N \leq \lambda_n.$$

Thus, we have $\lim \mu_n = \lambda_P^N$ and for all $n \geq 1$

$$N^2 \psi_n = \mu_n N \psi_n = \mu_n^2 \psi_n$$

and the compcteness of N leads to $\phi = \lim N\psi_n$ (up to a subsequence) satisfies $N\phi = \lambda_P^N \phi$ and $\|\phi\| = \lambda_P^N > 0$. At the end, Remark 3.6 leads to $\lambda_P^N = \sigma_N^-$. The proof is complete.

Theorem 3.16. *Let $N \in Q_K^P(E)$ be increasing and assume that the cone K is solid and $N(\partial K \setminus \{0_E\}) \subset int(K)$. Then N has a unique positive eigenvalue μ_N at which it has the SIJP.*

Proof. First let us prove that $\theta_K^N > 0$. Let $u \in \partial K \setminus \{0_E\}$, since $Nu \in int(K)$ we deduce from Lemma 3.7 in [20] that there exists $r_0 > 0$ small enough such that $Nu \succeq r_0 u$ and namely, $\theta_K^N \geq r_0 > 0$. Since Proposition 3.14 claims that $\mu_N = \theta_K^N$ is a positive eigenvalue of N and Lemma 3.5 states that $\lambda_K^N \leq \theta_K^N$, we have to show that $\lambda_K^N \geq \mu_N = \theta_K^N$. By the contrary, suppose that $\lambda_K^N < \mu_N = \theta_K^N$ and let $\lambda \in (\lambda_K^N, \mu_N)$, $\phi, \psi \succ 0_E$ be such that $N\phi = \mu_N \phi$ and $N\psi \preceq \lambda \psi$. Set

$$\tilde{\psi} = \begin{cases} \psi \text{ if } \psi \in int(K), \\ N(\psi) \text{ if } \psi \in \partial K \end{cases}$$

and observe that $\tilde{\psi} \in int(K)$, $N(\tilde{\psi}) \preceq \lambda \tilde{\psi}$ and from Lemma 3.7 in [20], there exists $s_0 > 0$ small enough such that $\tilde{\psi} \succeq s_0 \phi$.

Thus, we have

$$\phi = (\mu_N)^{-1} N\phi \preceq (s_0 \mu_N)^{-1} N\tilde{\psi} \preceq \lambda (s_0 \mu_N)^{-1} \tilde{\psi}.$$

Again we have,

$$\phi = (\mu_N)^{-1} N\phi \preceq (\mu_N)^{-1} N\left(\lambda (s_0 \mu_N)^{-1} \tilde{\psi}\right) \preceq (s_0)^{-1} (\lambda/\mu_N)^2 \tilde{\psi}.$$

By induction, we obatin that for all integer $n \geq 1$

$$\phi \preceq (s_0)^{-1} (\lambda/\mu_N)^n \tilde{\psi},$$

which leads to the contradiction $0_E \prec \psi \preceq \alpha (\lambda/\mu_N)^n \phi \to 0$ as $n \to \infty$. This ends the proof.

Corollary 3.17. *Assume that the cone K is solid and let $N \in Q_K^P(E)$ be a strongly positive and increasing. Then N has a unique positive eigenvalue μ_N at which it has the SIJP.*

Let $\Gamma(E)$ be the class of operators $N \in Q_K^P(E)$ such that there exists a sequence of cones (P^n) and an increasing sequence of operators N_n, such that $P^n \subset P$, $N_n(K) \subset P^n$ and N_n has the SIJP at μ_n and $N_n \longrightarrow N$ in operator norm.

Theorem 3.18. *Let $N \in \Gamma(E)$, then N has a unique positive eigenvalue μ_N at which it has the SIJP.*

Proof. Let $(P^n), (N_n)$ and (μ_n) be the sequences making of N an operator in the class $\Gamma(E)$ and let ϕ_n be the normalized eigenvector associated with λ_n.

First, we have that $\Theta_{P^n}^{N_n} = \Theta_P^{N_n}$. Indeed; it is obvious that $\Theta_{P^n}^{N_n} \subset \Theta_P^{N_n}$ and if $\theta > 0$, $u \in P \setminus \{0_E\}$ are such that $N_n u \succeq \theta u$ then $N_n u \in P^n \setminus \{0_E\}$ and $N_n(N_n u) \succeq \theta N_n u$. This shows that $\theta \in \Theta_{P^n}^{N_n}$ and $\Theta_P^{N_n} \subset \Theta_{P^n}^{N_n}$. By similar way, we obtain $\Lambda_{P^n}^{N_n} = \Lambda_P^{N_n}$ and we have so,

$$0 < \lambda_P^{N_n} = \lambda_{P^n}^{N_n} = \mu_n = \theta_{P^n}^{N_n} = \theta_P^{N_n}.$$

Since the sequence (N_n) is nondecreasing, we obtain by means of Proposition 3.12 that (μ_n) is nondecreasing and

$$0 < \lambda_P^{N_n} = \lambda_{P^n}^{N_n} = \mu_n = \theta_{P^n}^{N_n} = \theta_P^{N_n} \leq \lambda_P^N \leq \theta_P^N.$$

Thus, we have

$$0 < \mu_N = \lim \mu_n \leq \lambda_P^N \leq \theta_P^N$$

and it follows from Lemma 2.6 that μ_N is a positive eigenvalue of N and $\mu_N = \lambda_P^N \leq \theta_P^N$.

It remains to show that $\theta_P^N = \lambda_P^N = \mu_N$. Let $\theta \in (0, +\infty) \setminus \sigma_K(N)$ and $R > 0$, if $\theta > \mu_N$, we have then for all integer $n \geq 1$, $\mu_n < \theta$ and $i\left(\theta^{-1} N_n, P_R, P\right) = 1$. Letting $n \to \infty$, we get $i\left(\theta^{-1} N, P_R, P\right) = 1$. If $\theta < \mu_N$, then there exists an integer n_0 such that $\theta < \mu_n$ and $i\left(\theta^{-1} N_n, P_R, P\right) = 0$ for all $n \geq n_0$. Letting $n \to \infty$, we get $i\left(\theta^{-1} N, P_R, P\right) = 0$. Hence, we have proved that N has the IJP at μ_N and it follows from Proposition 3.14 and Theorem 3.9 that $\mu_N = \sigma_N^+ = \theta_P^N$. This ends the proof.

Remark 3.19. The problem of existence of a positive eigenvalue for increasing, 1-homogeneous and completely continuous operators has been the subject of several recent and old works (see [5], [6], [15], [16], [17] and [18]). Note that Proposition 3.13 and Theorems 3.16, 3.18 present classes of such operators having at least one positive eigenvalue.

Fixed point theorems for positive maps

Let $T : K \to K$ be a completely continuous mapping, the main goal of this section is to prove fixed point theorems for the mapping T.

Theorem 3.20. *Assume that the cone K is normal and there exists two operators $N_1, N_2 \in Q_K^P(E)$, $\gamma > 0$, three functions $G_1, G_2, G_3 : K \to K$ such that $\theta_P^{N_1} < 1 < \lambda_P^{N_2}$ and for all $u \in K$*

$$Tu \preceq N_1 u + G_1 u,$$

$$N_2 u - G_2 u \preceq Tu \preceq \gamma N_2 u + G_3 u. \tag{3.6}$$

If either

$$G_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } G_i u = \circ(\|u\|) \text{ as } u \to \infty,\ i = 2, 3, \tag{3.7}$$

or

$$G_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } G_i u = \circ(\|u\|) \text{ as } u \to 0,\ i = 2, 3, \tag{3.8}$$

then T has at least one nontrivial fixed point.

Proof. We present the proof in the case where (3.7) holds, the other case is checked similarly. We have to prove existence of $0 < r < R$ such that

$$i(T, K_r, K) = 1 \text{ and } i(T, K_R, K) = 0.$$

In such a situation, additivity and solution properties of the fixed point index imply that

$$i(T, K_R \setminus \overline{K_r}, K) = i(T, K_R, K) - i(T, K_r, K) = -1$$

and T has a positive fixed point u with $r < \|u\| < R$.

Consider the function $H_1 : [0,1] \times K \to K$ defined by $H_1(t,u) = (1-t)Tu + tN_1 u$ and let us prove existence of $r > 0$ small enough, such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂K_r. By the contrary suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial K_{1/n}$ such that $u_n = (1-t_n)Tu_n + t_n N_1 u_n$. Note that $v_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$v_n = (1-t_n)\left(Tu_n / \|u_n\|\right) + t_n N_1 v_n. \tag{3.9}$$

Thus, the inequality

$$Tu_n / \|u_n\| \preceq N_1(v_n) + (F_1 u_n / \|u_n\|) \tag{3.10}$$

combines with the normality of the cone K and the fact that $F_1(u_n) = \circ(\|u_n\|)$ as $n \to \infty$ implying that $(Tu_n / \|u_n\|)$ is bounded. Because of the compactness of N_1 there exists a subsequence (v_{n_k}) such that $\lim N_1 v_{n_k} = v \in P$. In fact, we have that $v \succ 0_E$. Indeed, if $\lim N_1 v_{n_k} = 0_E$, then inequality (3.10), the normality of the cone K and the fact that $F_1(u_n) = \circ(\|u_n\|)$ as $n \to \infty$ imply $\lim (Tu_n / \|u_n\|) = 0_E$. This together with (3.9)leads to $\lim v_{n_k} = 0_E$ contradicting $\|v_{n_k}\| = 1$.

Therefore, letting $k \to \infty$ in

$$\begin{aligned} N_1 v_{n_k} &= N_1 \left((1-t_{n_k})(Tu_{n_k}/\|u_{n_k}\|) + t_{n_k} N_1 v_{n_k}\right) \\ &\preceq N_1 \left(N_2 v_{n_k} + (1-t_{n_k})(F_1 u_{n_k}/\|u_{n_k}\|)\right), \end{aligned}$$

we have $v \preceq N_2 v$ and $1 \leq \theta_P^{N_1}$, contradicting the hypothesis $\theta_P^{N_1} < 1$ in Theorem 3.20 and proves existence of $r > 0$ small enough such that for all $t \in [0,1]$ equation $H_2(t,u) = u$ has no solution in ∂K_r. For a such radius $r > 0$, homotopy and permanence properties of the fixed point index and Lemma 3.4 lead to

$$i(T, K_r, K) = i(H_2(0,\cdot), K_r, K) = i(H_2(1,\cdot), K_r, K) = i(N_1, K_r, K) = i(N_1, P_r, P) = 1.$$

In similar way, consider the function $H_2 : [0,1] \times K \to K$ defined by $H_1(t,u) = (1-t)Tu + tN_2 u$ and let us prove existence of $R > 0$ large enough, such that for all $t \in [0,1]$ equation $H_2(t,u) = u$ has no solution in ∂K_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial K_n$ such that

$$u_n = (1-t_n) Tu_n + t_n N_2 u_n.$$

Note that $w_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$w_n = (1-t_n)(Tu_n / \|u_n\|) + t_n N_2 w_n. \tag{3.11}$$

Thus, the inequality

$$Tu_n / \|u_n\| \preceq \gamma N_2 w_n + (F_3 u_n / \|u_n\|) \tag{3.12}$$

combined with the normality of the cone K and the fact that $F_3(u_n) = \circ(\|u_n\|)$ as $n \to \infty$, implies that the sequence $(Tu_n / \|u_n\|)$ is bounded. Because of the compactness of N_2 there exists a subsequence (w_{n_k}) such that $\lim N_2 w_{n_k} = w \in P$. In fact, we have that $w \succ 0_E$; Indeed, if $\lim N_2 v_{n_k} = 0_E$, then inequality (3.12), the normality of the cone K and the fact that $F_3(u_n) = \circ(\|u_n\|)$ as $n \to \infty$ imply $\lim (Tu_n / \|u_n\|) = 0_E$. This together with (3.11) leads to $\lim w_{n_k} = 0_E$

contradicting $\|w_{n_k}\| = 1$. Therefore, letting $k \to \infty$ in

$$\begin{aligned} N_2 w_{n_k} &= N_2\left((1-t_{n_k})\left(Tu_{n_k}/\|u_{n_k}\|\right) + t_{n_k} N_2 w_{n_k}\right) \\ &\succeq N_2\left(N_2 w_{n_k} - (1-t_{n_k})\left(F_2 u_{n_k}/\|u_{n_k}\|\right)\right), \end{aligned}$$

we have $w \succeq N_2 w$ and $1 \geq \lambda_P^{N_2}$, contradicting the hypothesis $\lambda_P^{N_2} > 1$ in Theorem 3.20 and proves existence of $R > 0$ large enough such that for all $t \in [0,1]$ equation $H_2(t,u) = u$ has no solution in ∂K_R. For such a radius $R > 0$, homotopy and permanence properties of the fixed point index and Lemma 3.4 lead to

$$i(T,K_R,K) = i(H_1(0,\cdot),K_R,K) = i(H_1(1,\cdot),K_R,K) = i(N_2,K_R,K) = i(N_2,P_R,P) = 0.$$

This completes the proof.

Theorem 3.21. *Assume that $T(P) \subset P$, there exists two operators $N_1, N_2 \in Q_K^P(E)$ and two functions $G_1, G_2 : K \to K$ such that N_1, N_2 are lower bounded on P, $\theta_P^{N_1} < 1 < \lambda_P^{N_2}$ and for all $u \in K$*

$$N_2 u - G_2 u \preceq Tu \preceq N_1 u + G_1 u.$$

If either

$$G_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } G_2 u = \circ(\|u\|) \text{ as } u \to 0 \tag{3.13}$$

or

$$G_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } G_2 u = \circ(\|u\|) \text{ as } u \to \infty, \tag{3.14}$$

then T has at least one nontrivial fixed point.

Proof. We present the proof in the case where (3.13) holds, the other case is checked similarly. As in proof of Theorem 3.20, we have to prove existence of $0 < r < R$ such that

$$i(T,P_r,P) = 1 \text{ and } i(T,P_R,P) = 0.$$

Consider the function $H_1 : [0,1] \times P \to P$ defined by $H_1(t,u) = (1-t)Tu + tN_1 u$ and let us prove existence of $r > 0$ small enough, such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂P_r. By the contrary suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial P_{1/n}$ such that

$$u_n = (1-t_n)Tu_n + t_n N_1 u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial P_1$ and satisfies

$$v_n = (1-t_n)\left(Tu_n/\|u_n\|\right) + t_n N_1 v_n$$

then

$$N_1 v_n = N_1\left((1-t_n)\left(Tu_n/\|u_n\|\right) + t_n N_1 v_n\right).$$

Because of the compacteness and the lower boundeness of N_1, there exists a subsequence (v_{n_k}) such that $\lim N_1 v_{n_k} = v$ and $\|v\| = \lim \|N_1 v_{n_k}\| \geq N_{1,P}^{-} > 0$. Thus, letting $k \to \infty$ in

$$\begin{aligned} N_1 v_{n_k} &= N_1\left((1-t_{n_k})\left(Tu_{n_k}/\|u_{n_k}\|\right) + t_{n_k} N_1 v_{n_k}\right) \\ &\preceq N_1\left(N_1 v_{n_k} + (1-t_{n_k})\left(G_1 u_{n_k}/\|u_{n_k}\|\right)\right) \end{aligned}$$

we obtain $v \preceq N_1 v$ and $1 \leq \theta_P^{N_1}$ contradicting the hypothesis $\theta_P^{N_1} < 1$ in Theorem 3.21 and proves existence of $r > 0$ small enough such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂P_r. For such a radius $r > 0$, homotopy property of the fixed point index and Lemma 3.4 lead to

$$i(T, P_r, P) = i(H_2(0,\cdot), P_r, P) = i(H_2(1,\cdot), P_r, P) = i(N_1, P_r, P) = 1.$$

In similar way, consider the function $H_2 : [0,1] \times P \to P$ defined by $H_2(t,u) = (1-t)Tu + tN_2 u$ and let us prove existence of $R > 0$ large enough, such that for all $t \in [0,1]$, equation $H_2(t,u) = u$ has no solution in ∂P_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial P_n$ such that

$$u_n = (1-t_n)Tu_n + t_n N_2 u_n.$$

Note that $w_n = u_n / \|u_n\| \in \partial P_1$ satisfies

$$w_n = (1-t_n)\left(Tu_n/\|u_n\|\right) + t_n N_2 w_n.$$

It follows that

$$N_2 w_n = N_2\left((1-t_n)\left(Tu_n/\|u_n\|\right) + t_n N_2 w_n\right). \tag{3.15}$$

Because of the compacteness and the lower boundeness of N_2, there exists a subsequence (w_{n_k}) such that $\lim N_2 w_{n_k} = w$ and $\|w\| = \lim \|N_2 w_{n_k}\| \geq N^-_{2,P} > 0$. Thus, letting $k \to \infty$ in

$$\begin{aligned} N_2 w_{n_k} &= N_2\left((1-t_{n_k})\left(Tu_{n_k}/\left\|u_{n_k}\right\|\right) + t_{n_k} N_2 w_{n_k}\right) \\ &\succeq N_2\left(N_2 w_{n_k} - (1-t_{n_k})\left(G_2 u_{n_k}/\left\|u_{n_k}\right\|\right)\right), \end{aligned}$$

we obtain $w \succeq N_2 w$ and $1 \geq \lambda_P^{N_2}$ contradicting the hypothesis $\lambda_P^{N_2} > 1$ in Theorem 3.21 and proves existence of $R > 0$ large enough such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂K_R. For such a radius $R > 0$, homotopy and permanence property of the fixed point index and Lemma 3.4 lead to

$$i(T, P_R, P) = i(H_1(0,\cdot), P_R, P) = i(H_1(1,\cdot), P_R, P) = i(N_2, P_R, P) = 0.$$

This completes the proof.

Theorem 3.22. *Assume that K is a normal cone in E and there exist $N_1, N_2 \in Q_K^P(E)$, $\alpha > 0$ and three functions $G_1, G_2, G_3 : K \to K$ such that $\theta_P^{N_1} < 1 < \lambda_P^{N_2}$, N_1, N_2 are uniformely continuous on $\overline{B}(0_E, 2)$ and for all $u \in K$,*

$$Tu \preceq N_1\left(u + G_1(u)\right),$$

$$N_2\left(u - G_2(u)\right) \preceq T(u) \preceq \alpha N_2\left(u + G_3(u)\right).$$

If either

$$G_1(u) = \circ\left(\|u\|\right) \text{ near } 0 \text{ and } G_i(u) = \circ\left(\|u\|\right) \text{ near } \infty \text{ for } i = 2,3, \tag{3.16}$$

or

$$G_1(u) = \circ\left(\|u\|\right) \text{ near } \infty \text{ and } G_i(u) = \circ\left(\|u\|\right) \text{ near } 0 \text{ for } i = 2,3, \tag{3.17}$$

then T admits at least one nontrivial fixed point.

Proof. We present the proof in the case where (3.16) holds, the other case is checked similarly. As in proof of Theorem 3.20, we have to prove existence of $0 < r < R$ such that

$$i(T, K_r, K) = 1 \quad \text{and} \quad i(T, K_R, K) = 0.$$

Consider the function $H_1 : [0,1] \times K \to K$ defined by $H_1(t,u) = tTu + (1-t)N_1 u$ and let us prove existence of $r > 0$ small enough such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂K_r. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial K_{1/n}$ such that $u_n = t_n T u_n + (1-t_n) N_1(u_n)$. Note that $v_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$v_n = t_n \left(T u_n / \|u_n\| \right) + (1-t_n) N_1(v_n). \tag{3.18}$$

Then $N_1(v_n) = N_1 \left(t_n \left(T u_n / \|u_n\| \right) + (1-t_n) N_1(v_n) \right)$. Because of the compacteness and the uniform continuity of N_1, there is a subsequence of integers (n_l), $\widehat{t} \in [0,1]$ and $v \in P$ such that

$$\lim t_{n_l} = \widehat{t}, \; \lim N_1(v_{n_l}) = \lim N_1(v_{n_l} + (G_1 u_{n_l} / \|u_{n_l}\|) = v.$$

We claim that $v \succ 0_E$; Indeed, if $\lim N_1(v_n) = 0_E$, then the normality of the cone K and the estimate

$$\left(T \left(u_{n_l} \right) / \|u_{n_l}\| \right) \preceq N_1 \left(v_{n_l} + \left(G_1 u_{n_l} / \|u_{n_l}\| \right) \right),$$

lead to $\lim \left(T \left(u_{n_l} \right) / \|u_{n_l}\| \right) = 0_E$ and this combined with (3.18) implies $\lim v_{n_l} = 0_E$, contradicting $\|v_{n_l}\| = 1$. At this stage, letting $l \to \infty$ in

$$\begin{aligned} N_1(v_{n_l}) &= N_1(t_{n_l} \left(T \left(u_{n_l} \right) / \|u_{n_l}\| \right) + (1-t_{n_l}) N_1(v_{n_l})) \\ &\preceq N_1(t_{n_l} \left(N_2 \left(v_{n_l} + \left(G_1 / \|u_{n_l}\| \right) \right) \right) + (1-t_{n_l}) N_1(v_{n_l})), \end{aligned}$$

we obtain $v \preceq N_1 v$ and $1 \in \Theta_P^{N_1}$. This contradicts the hypothesis $1 > \theta_P^{N_1}$ in Theorem 3.22 and proves existence of $r > 0$ small enough such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂K_r. For a such $r > 0$ we deduce from homotopy and permanance properties of the fixed point index and Lemma 3.4 that

$$i(T, K_r, K) = i(H_1(1,\cdot), K_r, K) = i(H_1(0,\cdot), K_r, K) = i(N_1, K_r, K) = i(N_1, P_r, P) = 1.$$

In similar way, consider the function $H_2 : [0,1] \times K \to K$ defined by $H_2(t,u) = tTu + (1-t) N_2(u)$ and let us prove existence of $R > 0$ large enough such that for all $t \in [0,1]$ equation $H_2(t,u) = u$ has no solution in ∂K_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial K_n$ such that $u_n = t_n T(u_n) + (1-t_n) N_2(u_n)$. Note that $w_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$w_n = t_n \left(T \left(u_n \right) / \|u_n\| \right) + (1-t_n) N_2(w_n). \tag{3.19}$$

Then $N_2(w_n) = N_2(t_n (T(u_n)/\|u_n\|) + (1-t_n)N_2(w_n))$. Because of the compacteness and the uniform continuity of N_2, there is a subsequence of integers (n_l), $\widehat{t} \in [0,1]$ and $w \in P$ such that

$$\lim t_{n_l} = \widehat{t},\ \lim N_2(w_{n_l}) = \lim N_2(w_{n_l} - (G_2 u_{n_l}/\|u_{n_l}\|)) = \lim N_2(w_{n_l} + (G_3 u_{n_l}/\|u_{n_l}\|) = w.$$

We claim that $w \succ 0_E$; Indeed, if $\lim N_2(w_{n_l}) = 0_E$, then the normality of the cone K and the estimate $(T(u_{n_l})/\|u_{n_l}\|) \preceq \alpha N_2 (w_{n_l} + (G_3 u_{n_l}/\|u_{n_l}\|))$, lead to $\lim (T(u_{n_l})/\|u_{n_l}\|) = 0_E$ and this combined with (3.19) implies $\lim w_{n_l} = 0_E$, contradicting $\|w_{n_l}\| = 1$. At this stage, letting $l \to \infty$ in

$$\begin{aligned} N_2(w_{n_l}) &= N_2(t_{n_l}(T(u_{n_l})/\|u_{n_l}\|) + (1-t_{n_l})N_2(w_{n_l})) \\ &\succeq N_2(t_{n_l}(N_2(w_{n_l} - (G_2/\|u_{n_l}\|))) + (1-t_{n_l})N_2(w_{n_l})) \end{aligned}$$

we obtain $w \succeq N_2 w$ and $1 \in \Lambda_P^{N_2}$. This contradicts the hypothesis $1 < \lambda_P^{N_2}$ in Theorem 3.22 and proves existence of $R > 0$ large enough such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂K_R. For a such $R > 0$, we deduce from homotopy and permanance properties of the fixed point index and Lemma 3.4 that

$$i(T, K_R, K) = i(H_2(1,\cdot), K_R, K) = i(H_2(0,\cdot), K_R, K) = i(N_2, K_R, K) = i(N_2, P_R, P) = 0.$$

This completes the proof.

Theorem 3.23. *Assume that $T(P) \subset P$, there exist $N_1, N_2 \in Q_K^P(E)$ and functions $G_1, G_2 : K \to K$ such that N_1,N_2 are lower bounded on P and uniformely continuous on $\overline{B}(0_E, 2)$, $\theta_P^{N_1} < 1 < \lambda_P^{N_2}$ and for all $u \in K$,*

$$N_2(u - G_1(u)) \preceq T(u) \preceq N_1(u + G_2(u)).$$

If either

$$G_1(u) = \circ(\|u\|) \text{ near } 0 \text{ and } G_2(u) = \circ(\|u\|) \text{ near } \infty \tag{3.20}$$

or

$$G_1(u) = \circ(\|u\|) \text{ near } \infty \text{ and } G_2(u) = \circ(\|u\|) \text{ near } 0, \tag{3.21}$$

then T admits at least one nontrivial fixed point.

Proof. We present the proof in the case where (3.20) holds, the other case is checked similarly. As in the proof of Theorem 3.20, we have to prove existence of $0 < r < R$ such that

$$i(T,P_r,P) = 1 \text{ and } i(T,P_R,P) = 0.$$

Consider the function $H_1 : [0,1] \times P \to P$ defined by $H_1(t,u) = (1-t)Tu + tN_1(u)$ and let us prove existence of $r > 0$ small enough such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂P_r. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial P_n$ such that $u_n = H_1(t_n,u_n) = (1-t_n)Tu_n + t_nN_1(u_n)$. Note that $v_n = u_n/\|u_n\| \in \partial P_1$ and satisfies $v_n = (1-t_n)(T(u_n)/\|u_n\|) + t_nN_1(v_n)$. Then $N_1(v_n) = N_1((1-t_n)(T(u_n)/\|u_n\|) + t_nN_1(v_n))$. Because of the compacteness and the uniform continuity of N_1, there is a subsequence of integers (n_l), $\widehat{t} \in [0,1]$ and $v \in P$ such that

$$\lim t_{n_l} = \widehat{t},\ \lim N_1(v_{n_l}) = \lim N_1(v_{n_l} + (G_1(u_{n_l})/\|u_{n_l}\|) = v.$$

Moreover, we have that $\|v\| = \lim \|N_1(v_{n_l})\| \geq N_{1,P}^- > 0$. Thus, letting $l \to \infty$ in

$$\begin{aligned} N_1(v_{n_l}) &= N_1((1-t_{n_l})(T(u_{n_l})/\|u_{n_l}\|) + t_{n_l}N_1(v_{n_l})) \\ &\preceq N_1((1-t_{n_l})(N_2(v_{n_l} + (G_1(u_{n_l})/\|u_{n_l}\|))) + t_{n_l}N_1(w_{n_l})) \end{aligned}$$

we obtain $v \preceq N_1 v$ and $1 \in \Theta_P^{N_1}$. This contradicts the hypothesis $1 > \theta_P^{N_1}$ in Theorem 3.23 and proves existence of $r > 0$ small enough such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂P_r. For a such $r > 0$, we deduce from homotopy property of the fixed point index and Lemma 3.4 that

$$i(T,P_r,P) = i(H_1(1,\cdot),P_r,P) = i(H_1(0,\cdot),P_r,P) = i(N_1,P_r,P) = 1.$$

Consider the function $H_2 : [0,1] \times P \to P$ defined by $H_2(t,u) = (1-t)Tu + tN_2(u)$ and let us prove existence of $R > 0$ large enough such that for all $t \in [0,1]$ equation $H_2(t,u) = u$ has no solution in ∂P_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial P_n$ such that $u_n = H_2(t_n,u_n) = (1-t_n)Tu_n + t_nN_2(u_n)$. Note that $w_n = u_n/\|u_n\| \in \partial K_1$ and satisfies $w_n = (1-t_n)(T(u_n)/\|u_n\|) + t_nN_2(w_n)$. Then

$$N_2(w_n) = N_2((1-t_n)(T(u_n)/\|u_n\|) + t_nN_2(w_n)). \tag{3.22}$$

Because of the compacteness and the uniform continuity of N_2, there is a subsequence of integers (n_l), $\widehat{t} \in [0,1]$ and $w \in P$ such that

$$\lim t_{n_l} = \widehat{t},\ \lim N_2(w_{n_l}) = \lim N_2(w_{n_l} - (G_2(u_{n_l}) / \|u_{n_l}\|) = w.$$

Moreover, we have that $\|w\| = \lim \|N_2(w_{n_l})\| \geq N_{2,P}^{-} > 0$. Thus, letting $l \to \infty$ in

$$\begin{aligned} N_2(w_{n_l}) &= N_2((1-t_{n_l})(T(u_{n_l}) / \|u_{n_l}\|) + t_{n_l} N_2(w_{n_l})) \\ &\succeq N_2((1-t_{n_l})(N_2(w_{n_l} - (G_2(u_{n_l}) / \|u_{n_l}\|))) + t_{n_l} N_2(w_{n_l})) \end{aligned}$$

we obtain $w \succeq N_2 w$ and $1 \in \Lambda_P^{N_2}$. This contradicts the hypothesis $1 < \lambda_P^{N_2}$ in Theorem 3.23 and proves existence of $R > 0$ large enough such that for all $t \in [0,1]$ equation $H_2(t,u) = u$ has no solution in ∂P_R. For a such $R > 0$ we deduce from homotopy and permanance properties of the fixed point index and Lemma 3.4 that

$$i(T, P_R, P) = i(H_2(1,\cdot), P_R, P) = i(H_2(0,\cdot), P_R, P) = i(N_2, P_R, P) = 0.$$

This completes the proof.

We consider now, the particular case $T = NF$ where $N \in Q_K^P(E)$, $F : K \to K$ is a continuous and bounded map. We deduce from the above theorems, existence results for positive solution to the abstract Hammerstein equation (1.1).

Corollary 3.24. *Assume that the cone K is normal, N is uniformely continuous on $B(0_E, 2)$ and there exist three nonnegative real numbers α, β, γ and three functions $G_1, G_2, G_3 : K \to K$ such that $\alpha \theta_P^N < 1 < \beta \lambda_P^N$ and for all $u \in K$*

$$Fu \preceq \alpha u + G_1(u),$$
$$\beta u - G_2(u) \preceq F(u) \preceq \gamma u + G_3(u).$$

If either

$$G_1(u) = \circ(\|u\|) \text{ at } 0 \text{ and } G_i(u) = \circ(\|u\|) \text{ at } \infty \text{ for } i = 2,3 \tag{3.23}$$

or

$$G_1(u) = \circ(\|u\|) \text{ at } \infty \text{ and } G_i(u) = \circ(\|u\|) \text{ at } 0 \text{ for } i = 2,3 \tag{3.24}$$

then Equation (1.1) admits at least one positive solution.

Corollary 3.25. *Assume that N is lower bounded on the cone P and there exists three nonnegative real numbers α, β and two functions $G_1, G_2 : K \to K$ such that $\alpha\theta_P^N < 1 < \beta\lambda_P^N$ and for all $u \in P \setminus \{0_E\}$*

$$\alpha u - G_1(u) \preceq F(u) \preceq \beta u + G_2(u).$$

If either

$$G_1(u) = \circ(\|u\|) \text{ at } 0 \text{ and } G_2(u) = \circ(\|u\|) \text{ at } \infty, \tag{3.25}$$

or

$$G_1(u) = \circ(\|u\|) \text{ at } \infty \text{ and } G_2(u) = \circ(\|u\|) \text{ at } 0 \tag{3.26}$$

then Equation (1.1) admits at least one positive solution.

Remark 3.26. Conditions $1 < \lambda_P^{N_2}$ in Theorems 3.20, 3.21, 3.22, 3.23 and $1 < \lambda_P^N$ in Corollaries 3.24, 3.25 mean that operators N_2 and N have the SIJP.

4. Application to p-Laplacian BVPs

We discuss in this section existence of at least one positive solution to the boundary value problem (bvp for short)

$$\begin{cases} -(a\phi_p(u'))'(t) = b(t) f(t, u(t)) \ a.e. \ t \in (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{4.1}$$

where $p > 1$, $\phi_p(x) = |x|^{p-2}x$, $a, b : (0,1) \to [0, +\infty)$ are measurable functions.

In all this section, we assume that $a(t) > 0$ a. e. $t \in [0,1]$, $mes\{t \in (0,1) : b(t) > 0\} > 0$, $\psi_p(1/a)$ is integrable on any compact subset of $(0,1]$ where ψ_p is the inverse function of ϕ_p, b is integrable on any compact subset of $[0,1)$ and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

In all what follows we let for $\nu = 0$ or $+\infty$

$$f_\nu = \liminf_{u \to \nu} \left(\min_{t \in [0,1]} \psi_p(f(t,u))/u \right), \quad f^\nu = \limsup_{u \to \nu} \left(\max_{t \in [0,1]} \psi_p(f(t,u))/u \right).$$

In all this section, we let E be the Banach space of all continuous functions defined on $[0,1]$ equipped with its sup-norm (for $u \in E$, $\|u\| = \sup\{|u(t)| : t \in [0,1]\}$) and K be the normal cone of nonnegative functions in E.

The regular case

We assume here that

$$\psi_p(1/a),\ b \in L^1[0,1] \tag{4.2}$$

and

$$\lim_{t\to 0}\frac{1}{a(t)}\int_0^t b(s)ds = 0. \tag{4.3}$$

Because of Hypothesis (4.2), the operator $N_p : E \to E$ given for $u \in E$ by

$$N_p u(x) = \int_x^1 \psi_p\left(\frac{1}{a(t)}\int_0^t b(s)\phi_p(u(s))ds\right)dt, \tag{4.4}$$

is well defined. Let $F_p : K \to K$, the Nemitski operator defined for $u \in K$ by $F_p u(x) = \psi_p(f(x,u(x)))$, and $T_p = N_p F_p$. It is easy to see that, N_p is completely continuous (by Ascoli-Arzela theorem), F_p is bounded. Moreover if Hypothesis (4.3) holds, then all fixed points of T_p are positive solutions to bvp (4.1).

Let P be the cone in E defined by

$$P = \{u \in C : u(x) \geq \rho_p(x)\|u\| \text{ in } [0,1]\} \tag{4.5}$$

where

$$\rho(x) = \frac{1}{\overline{\rho}}\int_x^1 \frac{dt}{\psi_p(a(t))},\ \overline{\rho} = \int_0^1 \frac{dt}{\psi_p(a(t))}.$$

Lemma 4.1. *Assume that Hypothesis (4.2) holds, then we have that $N_p \in Q_K^P(E)$ and N_p is lower bounded on the cone P and has a unique positive eigenvalue at which it has the SIJP.*

Proof. First, let us prove that $N_p(K) \subset P$. Let $u \in K$, $v = N_p u$ and $w = v - \rho\|v\|$. Assume that for some $t_* \in (0,1)$, $w(t_*) < 0$ and let $t_0 \in (0,1)$ be such that

$$w(t_0) = \min_{t\in[0,1]} w(t), \quad w'(t_0) = 0.$$

In this case and since $w(0) = w(1) = 0$, there exists $\tau_1, \tau_2 \in (0,1)$ such that

$$\tau_1 < t_0 < \tau_2 \text{ and } w'(\tau_1) < w'(t_0) = 0 < w'(\tau_2),$$

or

$$v'(\tau_1) - \rho'(\tau_1)\|v\| < 0 < v'(\tau_2) - \rho'(\tau_2)\|v\|.$$

Since for all x, y, with $x \neq y$, $(\phi_p(x) - \phi_p(y))(x - y) > 0$, we obtain

$$a(\tau_1)(\phi_p(v'(\tau_1)) - \phi_p(\rho'(\tau_1)\|v\|)) < 0 < a(\tau_2)(\phi_p(v'(\tau_2)) - \phi_p(\rho'(\tau_2))\|v\|),$$

contradicting $(a(\phi_p(v') - \phi_p(\rho')\|v\|))'(t) = -b(t)\phi_p(u(t)) \leq 0$. This shows that $N_p u \in P$ and $N_p(K) \subset P$. Consider the space

$$X = \left\{u \in C^1([0,1]),\ u(1) = 0\right\}$$

equipped with the C^1-norm denoted $\|\cdot\|_1$ defined for $u \in X$, by $\|u\|_1 = \sup_{t\in[0,1]} |u'(t)|$ and let O be the subset of X defined as follows,

$$O = \left\{u \in K_X :\ u'(1) < 0 \text{ and } u(x) > 0\ \forall x \in [0,1)\right\}$$

where $K_X = K \cap X$ is a cone in X. Arguing as in the proof of Lemma 4.5 in [4], we obtain that O is an open set in X and if $N_{p,X} : X \to X$ is the restriction of N_p to X, then $N_X(K_X \setminus \{0_X\}) \subset O$. Therefore, we conclude from Corollary 3.17 that $N_{p,X}$ admits a unique positive eigenvalue μ_p at which it has the SIJP.

Let $u \in P \setminus \{0_E\}$ and $\theta \geq 0$ be such that $N_p u \succeq \theta u$, because of $N_p^{-1}(0_E) = \{0_E\}$ and $N_p(E) \subset X$, we have $N_p^2 u \succeq \theta N_p u$. Hence, we have proved that $\Theta_P^N = \Theta_K^{N_X}$ and $\theta_P^N = \theta_K^{N_X} = \mu_p$. Similarly, we have $\Lambda_P^N = \Lambda_K^{N_X}$ and $\lambda_P^N = \lambda_K^{N_X} = \mu_p = \theta_P^N = \theta_K^{N_X}$. This shows that N_p has as a unique positive eigenvalue at which it has the SIJP.

Let $u \in P$, we have

$$\begin{aligned} \left\|N_p u\right\| &= N_p u(0) \geq \int_0^1 \psi_p\left(\frac{1}{a(t)}\int_0^t b(s)\phi_p(\rho(s)\|u\|)ds\right)dt \\ &\geq \|u\| \int_0^1 \psi_p\left(\frac{1}{a(t)}\int_0^t b(s)\phi_p(\rho(s))ds\right)dt \end{aligned}$$

and shows that N_p is lower bounded on the cone P. This ends the proof.

Theorem 4.2. *Assume that Hypotheses (4.2) and (4.3) hold and let μ_p be the unique positive eigenvalue of N_p, then bvp (4.1) admits at least one positive solution whenever one of the following conditions*

$$f^0 < 1/\mu_p < f_\infty \tag{4.6}$$

and

$$f^{\infty} < 1/\mu_p < f_0 \tag{4.7}$$

holds true.

Proof. We present the proof in the case where Hypothesis (4.6) holds, the other case is checked similarly.

Let $\varepsilon > 0$ be such that $f^0 + \varepsilon < \mu_p < f_\infty$, It follows from definitions of f^0 and f_∞ that there exits a positive constant C such that for all $t \in [0,1]$ and $u \geq 0$,

$$(f_\infty - \varepsilon)\, u - C \leq f(t,u) \leq f^0 u + g_2(u)$$

where $g_2(u) = \max\left(0, f(t,u) - f^0 u\right)$. Therefore, we have

$$(f_\infty - \varepsilon)\, u - G_1(u) \leq Fu \leq \left(f^0 + \varepsilon\right) u + G_2(u) \text{ for all } u \in P,$$

where for all $u \in K$, $G_1 u(t) = C, G_2 u(t) = g_2(u(t))$ and Hypothesis (3.6) holds. At the end Corollary 3.25 guaranties existence of a positive solution to bvp (4.1).

The singular case

We assume in this subsection that

$$\int_0^1 \psi_p \left(\frac{1}{a(t)} \int_0^t b(s) ds \right) dt < \infty. \tag{4.8}$$

As in the regular case, because of Hypothesis (4.8), the operator N_p is well defined and if Hypothesis (4.3) holds, then all fixed points of T_p are positive solutions to bvp (4.1).

Let (μ_n) and (ν_n) be two sequences in $(0,1)$ such that $\lim \mu_n = 0$, $\lim \nu_n = 1$, $a(\mu_n) < \infty$ and $b(\nu_n) < \infty$, and let for all integer $n \geq 1$, a_n and b_n be defined by

$$a_n(t) = \begin{cases} a(t), \text{ if } t \in (\mu_n, 1), \\ \sup(a(t), a(\mu_n)) \text{ if } t \in (0, \mu_n), \end{cases}$$

$$b_n(t) = \begin{cases} b(t), \text{ if } t \in (0, \nu_n), \\ \inf(b(t), b(\nu_n)) \text{ if } t \in (\nu_n, 1). \end{cases}$$

Since all integer $n \geq 1$, $\psi_p(1/a_n)$, $b_n \in L^1[0,1]$, the operator $N_{p,n} : E \longrightarrow E$ given by

$$N_{p,n}u(x) = \int_x^1 \psi_p \left(\frac{1}{a_n(t)} \int_0^t b(s) \phi_p(u(s)) ds \right) dt$$

is well defined. Let for all integer $n \geq 1$, $P^n = \{u \in K : u(x) \geq \rho_n(x)\|u\| \text{ in } [0,1]\}$, where

$$\rho_n(x) = \frac{1}{\overline{\rho}} \int_x^1 \frac{dt}{\psi_p(a_n(t))} \text{ and } \overline{\rho} = \int_0^1 \frac{dt}{\psi_p(a_n(t))}.$$

Lemma 4.3. *Assume that Hypothesis (4.8) holds, then the operator $N_p \in \Gamma(E)$ and so it has a unique positive eigenvalue μ_p at which it has the SIJP. Moreover the operator N_p is subadditive on the cone K.*

Proof. We have from Lemma 4.1 that for all integer $n \geq 1$, $N_{p,n} \in Q_K^P(E)$ and $N_{p,n}$ is lower bounded on the cone P^n and has a unique positive eigenvalue at which it has the SIJP. Note also that the sequence $\left(N_p^n\right)$ is increasing, therefore, we have to prove that $N_p^n \to N_p$ in operator norm. Let $u \in E$ with $\|u\| = 1$, taking in account definitions of (μ_n), (ν_n), (a_n) and (b_n), we obtain by straightforward computations

$$\begin{aligned} &\left|N_p u(x) - N_{p,n}u(x)\right| \\ &\leq \int_0^1 \left|\psi_p\left(\tfrac{1}{a(t)}\int_0^t b(s)\phi_p(|u(s)|)ds\right) - \psi_p\left(\frac{1}{a_n(t)}\int_0^t b_n(s)\phi_p(|u(s)|)ds\right)\right| dt \\ &\leq \int_0^{\mu_n} \psi_p\left(\frac{1}{a(t)}\int_0^t b(s)ds\right) dt + \int_{\nu_n}^1 \psi_p\left(\frac{1}{a(t)}\int_0^t b(s)ds\right) dt. \end{aligned}$$

Therefore, we have

$$\left\|N_p - N_{p,n}\right\| \leq \int_0^{\mu_n} \psi_p\left(\frac{1}{a(t)}\int_0^t b(s)ds\right) dt + \int_{\nu_n}^1 \psi_p\left(\frac{1}{a(t)}\int_0^t b(s)ds\right) dt \to 0 \text{ as } n \to \infty.$$

This ends the proof.

Arguing as in the proof of Theorem 4.2, we obtain from Corollary 3.24 the following result.

Theorem 4.4. *Assume that Hypotheses (4.3) and (4.8) hold, then bvp (4.1) admits at least one positive solution whenever one of the following conditions*

$$f^0 < 1/\mu_p < f_\infty \leq f^\infty < \infty \tag{4.9}$$

and

$$f^\infty < 1/\mu_p < f_0 \leq f^0 < \infty \tag{4.10}$$

holds true.

5. Application to Urysohn type integral equations

We consider in this section the integral equation of Urysohn type

$$u(t) = f(t,u(t)) + g\left(t, \int_0^1 G(t,s)\varphi_\alpha(u(s))ds\right), \tag{5.1}$$

where α is a positive real number, $\varphi_\alpha(x) = |x|^{\alpha-1}x$, $G : [0,1]\times[0,1] \longrightarrow [0,+\infty)$ is continuous and does not vanish identicaly and $f,g : [0,1]\times[0,+\infty) \to [0,+\infty)$ are continuous functions. This type of integral equations has been discussed in [7], [8], [9], [11] and [12]. The main goal of this section is to derive existence results for positive solutions to Equation (5.1) from Theorems 3.20 and 3.21. We assume in all this section that f is a contraction, i.e.

$$\begin{cases} \text{there exists } k \in [0,1) \text{ such that for all } t \in [0,1] \text{ and } x,y \geq 0, \\ \qquad |f(t,x) - f(t,y)| \leq k|x-y|. \end{cases} \tag{5.2}$$

We set

$$\widetilde{f}(t,u) = \begin{cases} f(t,u) \text{ if } u \geq 0, \\ f(t,0) \text{ if } u \leq 0. \end{cases}$$

Clearly

$$|\widetilde{f}(t,x) - \widetilde{f}(t,y)| \leq k|x-y| \;.$$

Note that because of (5.2), we have that the function $\left(I_{\mathbb{R}} - \widetilde{f}(t,\cdot)\right)$ is an homeomorphism for all $t \in [0,1]$. Moreover $h : [0,1]\times\mathbb{R} \to \mathbb{R}$ defined by $h(t,x) = \left(I_{\mathbb{R}} - \widetilde{f}(t,\cdot)\right)^{-1}(x)$, is continuous. Set for $v = 0, +\infty$

$$h_v = \liminf_{u\to v}\left(\min_{t\in[0,1]} \frac{h(t,g(t,u))}{\psi_\alpha(u)}\right) \quad h^v = \limsup_{u\to v}\left(\max_{t\in[0,1]} \frac{h(t,g(t,u))}{\psi_\alpha(u)}\right),$$

where ψ_α be the inverse function of φ_α. Let E and K be respectively the Banach space and the normal cone introduced in Section 4. Let $N : E \to E$ be the operator defined for $u \in E$ by $Nu(t) = \psi_\alpha(\int_0^1 G(t,s)\varphi_\alpha(u(s))ds)$. Clearly, N is positive, positively one homogeneous and completely continuous.

Lemma 5.1. *Assume that*

$$\text{there exists } [\xi,\eta] \subset [0,1] \text{ such that } G(t,s) > 0 \text{ for all } t,s \in [\xi,\eta]. \tag{5.3}$$

Then $\theta_K^N > 0$.

Proof. Let $u_0 : [0,1] \to [0,+\infty)$ be the function defined by

$$u_0(t) = \begin{cases} 0 \text{ if } t \in [0,\xi], \\ \frac{4}{\beta-\alpha}(t-\xi) \text{ if } t \in \left[\xi, \frac{3\xi+\eta}{4}\right], \\ 1 \text{ if } t \in \left[\frac{3\xi+\eta}{4}, \frac{\xi+3\eta}{4}\right], \\ \frac{4}{\beta-\alpha}(\eta-t) \text{ if } t \in \left[\frac{\xi+3\eta}{4}, \eta\right], \\ 0 \text{ if } t \in [\eta,1] \end{cases}$$

and $\theta_0 = \psi_\alpha(G_0(\eta-\xi)/2)$ where $G_0 = \min\{G(t,s) : t,s \in [\xi,\eta]\}$. We have for all $t \in [0,1]$,

$$Nu_0(t) = \psi_\alpha\left(\int_0^1 G(t,s)\varphi_\alpha(u_0(s))\,ds\right) \geq \psi_\alpha\left(\int_{(3\xi+\eta)/4}^{(\xi 3+\eta)/4} G(t,s)ds\right) \geq \theta_0 u_0(t).$$

This shows that $\theta_0 \in \Theta_K^N$ and $\theta_K^N \geq \theta_0 > 0$.

Lemma 5.2. *Assume that*

$$\begin{cases} \textit{there exists } [\xi,\eta] \subset [0,1] \textit{ such that} \\ G(t,s) > 0 \textit{ for all } t \in [\xi,\eta] \textit{ and } s \in [0,1]. \end{cases} \tag{5.4}$$

Then $\lambda_K^N > 0$.

Proof. Let $\lambda > 0$ and $u \in K \setminus \{0_E\}$ such that $Nu(t) \leq \lambda u(t)$ for all $t \in [0,1]$. It follows from Hypothesis (5.4) that $u(t) \geq \lambda^{-1}\psi_\alpha(\int_0^1 k(t,s)\varphi_\alpha(u(s))ds) > 0$ for all $t \in [\xi,\eta]$ then

$$\lambda \min_{t\in[\xi,\eta]} u(t) \geq \psi_\alpha(\min_{t\in[\xi,\eta]} \int_\xi^\eta G(t,s)\varphi_\alpha(u(s))ds) \geq \psi_\alpha(\min_{t\in[\xi,\eta]} \int_\xi^\eta G(t,s)ds) \min_{s\in[\xi,\eta]} u(s).$$

Leading to $\lambda \geq \psi_\alpha(\min_{t\in[\xi,\eta]} \int_\xi^\eta G(t,s)ds) > 0$ and $\lambda_K^N \geq \psi_\alpha(\min_{t\in[\xi,\eta]} \int_\xi^\eta G(t,s)ds) > 0$.

Lemma 5.3. *Assume that*

$$\begin{cases} \textit{there exists } [\xi,\eta] \subset [0,1],\ \rho > 0 \textit{ such that} \\ G(t,s) \geq \rho \max_{\tau\in[0,1]} G(\tau,s) \textit{ for all } t \in [\xi,\eta] \textit{ and } s \in [0,1] \end{cases} \tag{5.5}$$

and let P be the cone in E defined by

$$P = \{u \in K : u(t) \geq \psi_\alpha(\rho)\|u\| \textit{ for all } t \in [\xi,\eta]\}.$$

Then $N(K) \subset P$ *and N is lower bounded on P.*

Proof. We have for $u \in K$ and $t \in [\xi, \eta]$,

$$Nu(t) = \psi_\alpha(\int_0^1 G(t,s)\varphi_\alpha(u(s))ds) \geq \psi_\alpha(\int_0^1 \rho(\max_{\tau\in[0,1]} G(\tau,s))\varphi_\alpha(u(s))ds) \geq \psi_\alpha(\rho)\|Nu\|,$$

which proves that $Nu \in P$ and $N(K) \subset P$. Now, we have for all $u \in P$

$$\|Nu\| \geq Nu(t) \geq \psi_\alpha(\int_\xi^\eta k(t,s)\varphi_\alpha(u(s))ds) \geq \rho\,\psi_\alpha(\max_{t\in[0,1]} \int_\xi^\eta K(t,s)ds)\|u\|,$$

which proves that N is lower bounded on the cone P.

Remark 5.4. Note that if (5.5) is satisfied then (5.4) is satisfied and obviously (5.3) is satisfied.

Let $T_1 : E \to E,\ T_2 : K \to K$ and $T : K \to E$ be the operators defined by

$$T_1u(t) = \widetilde{f}(t,u(t)),\ T_2u(t) = g(t,Nu(t)) \text{ and } T = (I - T_1)^{-1}T_2.$$

It is easy to prove the following lemma.

Lemma 5.5. *Assume that*

$$h(t,x) \geq 0 \text{ for all } t \in [0,1] \text{ and } x \geq 0 \tag{5.6}$$

then $T(K) \subset K$ and K is completely continuous. Moreover, $u \in K$ is a solution to Equation (5.1) if and only if u is a fixed point of T.

Our first existence result for Equation (5.1) is obtained by means of Theorem 3.20.

Theorem 5.6. *Assume that Hypotheses (5.2), (5.4) and (5.6) hold. If either*

$$h^0 < 1/\theta_K^N \leq 1/\lambda_K^N < h_\infty \leq h^\infty < \infty \tag{5.7}$$

or

$$h^\infty < 1/\theta_K^N \leq 1/\lambda_K^N < h_0 \leq h^0 < \infty, \tag{5.8}$$

then Equation (5.1) admits at least one positive.

Proof. We present the proof in the case where Hypothesis (5.7) hold, the other case is checked similarly. Let $\varepsilon > 0$ be such that $(h^0 + \varepsilon)\theta_K^N < 1 < (h_\infty - \varepsilon)\lambda_K^N$, then there exists two positive constants C_1, C_2 such that for all $x \geq 0$ and $t \in [0,1]$

$$h(t,g(t,x)) \leq \left(h^0 + \varepsilon\right)\psi_\alpha(x) + f_1(t,x),$$

$$(h_\infty - \varepsilon)\,\psi_\alpha(x) - C_1 \le h(t,g(t,x)) \le (h^\infty + \varepsilon)\,\psi_\alpha(x) + C_2,$$

where $f_1(t,x) = \max\sup\{h(t,g(t,x)) - (h^0+\varepsilon)x, 0\}$. These inequalities lead to

$$Tu(t) \le N_1 u(t) + G_1 u(t),$$

$$N_2 u(t) - G_2 u(t) \le Tu(t) \le \gamma N_2 u(t) + G_3 u(t),$$

where

$$N_1 u(t) = \left(h^0+\varepsilon\right) Nu(t),\ N_2 u(t) = (h_\infty - \varepsilon) Nu(t),\ \gamma = (h^\infty+\varepsilon)/\left(h^0+\varepsilon\right),$$

$$G_1 u(t) = f_1(t, Nu(t)),\ G_2 u(t) = C_1 \text{ and } G_3 u(t) = C_2.$$

Clearly, we have

$$\theta_P^{N_1} = \left(h^0+\varepsilon\right)\theta_K^N < 1 < \lambda_K^{N_2} = (h_\infty - \varepsilon)\,\lambda_K^N,$$

$$G_1 u = \circ(\|u\|) \text{ near } 0 \text{ and } G_i u = \circ(\|u\|) \text{ near } \infty \text{ for } i = 2,3.$$

Therefore we conclude from Theorem 3.20 that T admits a nontrivial fixed point and then from Lemma 5.5 a positive solution to Equation (5.1).

Arguing as above we obtain by means of Theorem 3.21, the following existence result.

Theorem 5.7. *Assume that Hypotheses (5.2), (5.5), (5.6) hold and $T(K) \subset P$. If either*

$$h^0\theta_P^N < 1 < h_\infty \lambda_P^N, \tag{5.9}$$

or

$$h^\infty\theta_P^N < 1 < h_0 \lambda_P^N, \qquad ((5.10)$$

then Equation (5.1) admits at least one positive solution.

Example 5.8. Let $G(t,s) = s\phi(t)$ with

$$\phi(t) = \begin{cases} 0 \text{ if } t \in [0,1/2], \\ t-(1/2) \text{ if } t \in [1/2,1]. \end{cases}$$

It is easy to see that G satisfies (5.5) with $[\xi,\eta] = [3/4,1]$ and $\rho = 1/2$; in this case we have (as in the proof of Lemma 5.2) $\lambda_K^N = \lambda_P^N \ge \rho \max_{t\in[0,1]} \int_{3/4}^1 K(t,s)ds = 7/128$, where

$$P = \{u \in K : u(t) \ge 1/2\|u\| \text{ for all } t \in [3/4,1]\}.$$

We have also, $\theta_K^N = \theta_P^N \leq \max_{t\in[0,1]} \int_0^1 K(t,s)ds = 1/4$.

Case A. Let $\alpha = 1$ and $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = -x/(x+c)$, $c > 1$ and $g(x) = ax^2/\left(x^2+1\right)$. In this case we have

$$h(x) = \begin{cases} \frac{1}{2}\left(-(c+1-x) + \sqrt{(c+1-x)^2+4cx}\right) \text{ if } x \geq 0, \\ x \text{ if } x \leq 0. \end{cases}$$

Simple computations lead to $h_0 = h^0 = 0$ and $h_\infty = h^\infty = a$. Thus, we obtain from Theorem 5.6 that Equation (5.1) admits a positive solution whenever $a > 128/7$.

Case B. Let $\alpha = 2$ and $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = -x/2$ and $g(x) = a\sqrt{x} + bx$, $a, b > 0$. In this case we have

$$h(x) = \begin{cases} \dfrac{2x}{3} \text{ if } x \geq 0, \\ x \text{ if } x \leq 0, \end{cases}$$

and $h_0 = h^0 = 2a/3$ and $h_\infty = h^\infty = +\infty$. We have also $T(P) \subset P$, indeed, if $u(t) \geq (1/2)\|u\|$ for all $t \in [34, 1]$ then

$$g(u(t)) \geq a\sqrt{(1/2)\|u\|} + b(1/2)\|u\| \geq (1/2)g(\|u\|) = (1/2)\|g(u)\|.$$

Thus, we obtain from Theorem 5.7 that Equation (5.1) admits a positive solution whenever $a < 3/2$.

Case C. Let $\alpha = 1/2$ and $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = -x/2$ and $g(x) = a\sqrt{x} + bx$, $a, b > 0$. In this case we have $h_0 = h^0 = +\infty$ and $h_\infty = h^\infty = 0$.

Thus, we obtain from Theorem 5.7 that Equation (5.1) admits a positive solution whenever $a < 3/2$.

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