J. Nonlinear Funct. Anal. 2017 (2017), Article ID 7

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

A NEW OPTIMALITY CONDITION FOR WEAKLY EFFICIENT SOLUTIONS OF CONVEX VECTOR EQUILIBRIUM PROBLEMS WITH CONSTRAINTS

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Abstract. The main aim of this paper is to present a necessary and sufficient optimality condition for weakly efficient solution of a constrained convex vector equilibrium problem (which is denoted by CVEPC) by means of contingent epiderivatives. As an application, we also obtain a necessary and sufficient condition for weakly efficient solution of CVEPC in terms of contingent hypoderivatives.

Keywords. Optimality condition; Contingent hypoderivative; Cone-convex function; Steady function; Weakly efficient solution.

2000 Mathematics Subject Classification 90C29, 90C46, 90C48, 49J52.

1. Introduction

Throughout this article, let X, Y and Z be real Banach spaces, C be a nonempty convex subset in X, $Q \subset Y$ and $S \subset Z$ be pointed closed convex cones with its interiors nonempty, in which Q defines a partial order on Y and S defines a partial order on Z. Given a vector bifunction $F : X \times X \longrightarrow Y$ and a constraints function $g : X \longrightarrow Z$ is S-convex on C, which means that

$$tg(x_1) + (1-t)g(x_2) \in g(tx_1 + (1-t)x_2) + S \ \forall x_1, x_2 \in C, \ \forall t \in [0,1].$$

We assume that for each $\bar{x} \in C$, $F(\bar{x}, .)$ is a Q-convex function from $C \subset X$ into Y and such that $F(\bar{x}, \bar{x}) = 0$. Our convex vector equilibrium problem with constraints is denoted by CVEPC:

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Received July 1, 2016; Accepted November 29, 2016.

finding a vector $\bar{x} \in K$ such that

$$F(\bar{x}, y) \notin -intQ \ (\forall y \in K),$$

where $K = \{x \in C : g(x) \in -S\}$ is a convex set and it is also called a feasible set of CVEPC. Moreover, vector $\bar{x}$ is called a weakly efficient solution to the CVEPC.

A number of papers have been devoted to the optimality condition of vector equilibrium problems, see, for example, Jahn and Rauh [2], Marín and Sama [3, 4], Jiménez *et al.* [5, 6], Gong [8], Su [9], M. Bianchi *et al.* [10], Qiu [11], and the references therein. Gong [8] provided a necessary and sufficient optimality condition for weak efficient solution to the convex vector equilibrium problem with constraints CVEPC by using the separation theorems of convex sets and the tools of nonlinear analysis. Jiménez, Novo, Sama [6], Marín and Sama [3, 4] derived first-order optimality conditions in vector optimization involving stable functions in terms of contingent epiderivatives. Jahn and Rauh [2] proposed the concep of the contingent epiderivative for a set-valued mapping and its application to the optimality conditions for set-valued optimization problems, etc.

Our study now is to establish a new necessary and sufficient optimality condition for weakly efficient solution to the convex vector equilibrium problem with constraints CVEPC in terms of contingent epiderivatives and hypoderivatives.

The remainder of this paper is organized as follows. After some preliminaries and definitions, a necessary and sufficient optimality condition for weakly efficient solution to the CVEPC in terms of contingent epiderivatives is a well-presented analysis in Section 3. Besides, a necessary and sufficient optimality condition for weakly efficient solution to the CVEPC in terms of contingent hypoderivatives is also derived. An example to illustrate is also provided as well.

2. Preliminaries

Let us consider Problem CVEPC be given as in Sec. 1. Let $A \subset X$, as usual, we denote $intA$, clA and $cone(A)$ instead of the interior, closed and cone hull of A, respectively (shortly, resp.). $\mathbb{R}$ (resp., $\mathbb{N}$) instead of the set of all real (resp. natural) numbers. Let Y^* be a topological dual space of Y. The dual cone of Q is given as $Q^+ = \{\xi \in Y^* \mid \langle \xi, q \rangle \geq 0 \quad \forall\, q \in Q\}$, where $\langle .\,, . \rangle$

denotes the coupling between Y and Y^*. Let $F : X \longrightarrow 2^Y$ be a set-valued mapping from X into 2^Y, where 2^Y instead of the family of all the subsets of Y. The effective domain, graph, epigraph and hypograph of a set-valued mapping F are given respectively as

$$dom(F) = \{x \in X \,|\, F(x) \neq \emptyset\},$$

$$graph(F) = \{(x,y) \in X \times Y \,|\, y \in F(x)\},$$

$$epi(F) = \{(x,y) \in X \times Y \,:\, x \in dom(F),\ y \in F(x) + Q\}.$$

$$hypo(F) = \{(x,y) \in X \times Y \,:\, x \in dom(F),\ y \in F(x) - Q\}.$$

Let us next provide the definitions about the contingent cone, steady function, contingent derivative, hypoderivative and epiderivative, which will be used in this paper

Definition 2.1. [1, 2, 3, 4, 5, 6, 9] Let $M \subset Y$ and $\bar{z} \in clM$.

(i) The contingent cone $T(M,\bar{z})$ of the set M at $\bar{z}$ is given as

$$T(M,\bar{z}) = \{y \in Y : \exists t_n \longrightarrow 0^+, \exists y_n \longrightarrow y \text{ such that } \bar{z} + t_n y_n \in M \;\forall\, n \geq 1\}.$$

(ii) The adjacent cone $A(M,\bar{z})$ of the set M at $\bar{z}$ is given as

$$A(M,\bar{z}) = \{y \in Y : \forall t_n \longrightarrow 0^+, \exists y_n \longrightarrow y \text{ such that } \bar{z} + t_n y_n \in M \;\forall\, n \geq 1\}.$$

Definition 2.2. [1, 2, 3, 4, 5, 6, 9] Let $F : X \longrightarrow 2^Y$ and $(\bar{x},\bar{y}) \in graph(F)$.

(i) The contingent derivative of F at $(\bar{x},\bar{y})$ is the set-valued mapping $D_cF(\bar{x},\bar{y})$ from X into 2^Y defined as

$$graph(D_cF(\bar{x},\bar{y})) = T(graph(F),(\bar{x},\bar{y})).$$

(ii) The contingent epiderivative (resp., hypoderivative) of F at $(\bar{x},\bar{y})$ is the single-valued mapping $\underline{D}F(\bar{x},\bar{y})$ (resp., $\overline{D}F(\bar{x},\bar{y})$) from X into Y defined as

$$epi(\underline{D}F(\bar{x},\bar{y})) = T(epi(F),(\bar{x},\bar{y}))$$

$$\big(\text{resp., } hypo(\overline{D}F(\bar{x},\bar{y})) = T(hypo(F),(\bar{x},\bar{y}))\big),$$

where we denote $t_n \longrightarrow 0^+$ instead of a sequence of positive numbers with limit 0.

Remark 2.1. When $F = f : X \longrightarrow Y$, we write $D_c f(\overline{x})v$ instead of $D_c f(\overline{x}, f(\overline{x}))v$ for all $v \in X$. From here, it is not hard to see that if $f : X \longrightarrow Y$ and $\overline{x}, v \in X$ then

$$D_c f(\overline{x})(v) = \Big\{ y \in Y : \exists (t_n, v_n) \to (0^+, v) \text{ such that } \lim_{n \longrightarrow \infty} \frac{f(\overline{x} + t_n v_n) - f(\overline{x})}{t_n} = y \Big\}$$

(can see in [5]). So, $D_c f(\overline{x})(v)$ is a closed set and the set-valued map $v \mapsto Dcf(\overline{x})(v)$ is positively homogeneous.

Definition 2.3. [1, 5, 9]

(i) A function $f : X \longrightarrow Y$ is said to be stable at $\overline{x}$ if there exist a neighborhood U of $\overline{x}$ and a real number $L_f > 0$ such that

$$\|f(x) - f(\overline{x})\| \leq L_f \|x - \overline{x}\| \quad \forall\, x \in U.$$

(ii) A function $f : X \longrightarrow Y$ is said to be steady at $\overline{x}$ in the direction $v \in X$ if

$$\lim_{(t,u) \longrightarrow (0^+, v)} \frac{f(\overline{x} + tu) - f(\overline{x} + tv)}{t} = 0.$$

It is said that f is steady at $\overline{x}$ if f is steady at $\overline{x}$ in all the directions.

Definition 2.4. [8] A function $f : X \longrightarrow Y$ is called Q-concave on $C \subset X$ if f is $(-Q)$-convex on C, meaning that for all $x_1, x_2 \in C$, $t \in [0,1]$, we get

$$t f(x_1) + (1-t) f(x_2) \in f(tx_1 + (1-t)x_2) - Q.$$

We next state some important results of Xun-Hua Gong[8] and J. Jahn and R. Rauh [2], which will be used in this paper.

Theorem 2.1. (see in Gong [8, Theorem 3.1, p. 1458]) *Let us assume that condition (DKA) holds, which means that there exists $x_0 \in C$ such that $g(x_0) \in -intS$. Then $x \in K$ is a weakly efficient solution to the CVEPC if and only if there exists $y^* \in Q^+ \setminus \{0\}$, $z^* \in S^+$ such that*

$$\langle z, g(x) \rangle = 0$$

and

$$\langle y^*, F(x,x) \rangle + \langle z^*, g(x) \rangle = \min_{y \in C} \big\{ \langle y^*, F(x,y) \rangle + \langle z^*, g(y) \rangle \big\}. \tag{2.1}$$

Proof. Argue similarly as in the proof of Theorem 3.1 [8] with K is replaced by $-S$, X_0 is replaced by C and the $K-$ concavexity of g on X_0 is replaced by the $S-$ convexity of g on C. □

Remark 2.2. If the condition (DKA) is satisfied, which means that $\exists x_0 \in C : g(x_0) \in -intS$, then $x_0 \in K$. This follows that the feasible set $K \neq \emptyset$.

Theorem 2.2. (see in Jahn and Rauh [2, Theorem 6, p. 202]) *Let $(X, \|.\|_X)$ be a real normed space, $F : X \longrightarrow \mathbb{R}$ be a single-valued function and let $\bar{x} \in X$ be given. If F is continuous at $\bar{x}$ and convex, then the contingent epiderivatives of F at $(\bar{x}, F(\bar{x}))$ exists.*

3. Optimality conditions

Using Theorem 2.2 we obtain existence results as follows.

Proposition 3.1. *Let a function $k : X \longrightarrow \mathbb{R}$ be steady at a point $\bar{x} \in C$, where $C \subset X$ be a nonempty convex subset. Then*

(i) *If k is $\mathbb{R}_+-$ convex on C then the contingent epiderivative of k at $(\bar{x}, k(\bar{x}))$ exists.*
(ii) *If k is $\mathbb{R}_+-$ concave on C then the contingent hypoderivative of k at $(\bar{x}, k(\bar{x}))$ exists.*

Proof. Case (i). From the fact that $k : X \longrightarrow Y$ is steady at $\bar{x} \in C$, which follows that $k : X \longrightarrow Y$ is steady at $(\bar{x}, 0)$. By taking into account B. Jiménez and V. Novo [5] yields that k is stable at $\bar{x}$. It is easily seen that k is continuous at $\bar{x}$ and convex on C. In view of Theorem 2.2, we deduce that $\underline{D}k(\bar{x}, k(\bar{x}))$ exists.

Case (ii). By virtue of Definition 1.3 in Section 1, it follows that $(-k)$ is $\mathbb{R}_+-$ convex on C and continuous at $\bar{x}$. Using the preceding obtained result, it leads to $\underline{D}(-k)(\bar{x}, (-k)(\bar{x}))$ exists. Again making use of Definition 2.1 (iv), the following relation holds

$$\overline{D}k(\bar{x}, k(\bar{x})) = -\underline{D}(-k)(\bar{x}, (-k)(\bar{x})).$$

Therefore $\overline{D}k(\bar{x}, k(\bar{x}))$ exists, and the claim follows. □

Proposition 3.2. *Let $k, C, \bar{x}$ be given as in Proposition 3.1. Assume, in addition, that k is $\mathbb{R}_+$-convex on C. Then $\bar{x}$ is a solution of problem (P_k) if and only if*

$$\underline{D}k(\bar{x}, k(\bar{x}))(v) \geq 0, \quad \forall v \in A(C, \bar{x}). \tag{3.1}$$

Where

$$(P_k): \qquad min\{k(x) : x \in C\}$$

Proof. *Necessary condition:* Suppose, to the contrary, that the inequality of (3.1) does not hold, meaning that there exists at least a direction $v_0 \in A(C,\bar{x})$ such that

$$w_0 := \underline{D}k(\bar{x},k(\bar{x}))(v_0) < 0. \tag{3.2}$$

It is obvious that

$$(v_0,w_0) \in epi(\underline{D}k(\bar{x},k(\bar{x}))).$$

By the definition of contingent epiderivative of k at $(\bar{x},k(\bar{x}))$, there exist sequences $t_n \to 0^+$ and $(v_n,w_n) \to (v_0,w_0)$ such that

$$(\bar{x},k(\bar{x})) + t_n(v_n,w_n) \in epi(k), \; \forall n \in \mathbb{N},$$

which is equivalent to

$$w_n \geq \frac{k(\bar{x}+t_n v_n) - k(\bar{x})}{t_n}, \; \forall n \in \mathbb{N}. \tag{3.3}$$

By letting $n \to +\infty$, one obtains the following result

$$w_0 \geq lim_{n\to+\infty} \frac{k(\bar{x}+t_n w_n) - k(\bar{x})}{t_n}. \tag{3.4}$$

Because $v_0 \in A(C,\bar{x})$, hence for the preceding sequence $(t_n)_{n\geq 1}$, there exists sequence $w'_n \longrightarrow w_0$ such that $\bar{x} + t_n w'_n \in C$ for all $n \in \mathbb{N}$. As $\bar{x} \in C$ is a solution of (P_k), thus the following inequality holds

$$k(\bar{x}+t_n w'_n) - k(\bar{x}) \geq 0, \; \forall n \in \mathbb{N}. \tag{3.5}$$

On the other hand, we have a represention as follows

$$\frac{k(\bar{x}+t_n w_n) - k(\bar{x})}{t_n} = \frac{k(\bar{x}+t_n w'_n) - k(\bar{x})}{t_n} + \frac{k(\bar{x}+t_n w_n) - k(\bar{x}+t_n w'_n)}{t_n} \; \forall n \in \mathbb{N},$$

which yields that

$$\lim_{n\to+\infty} \frac{k(\bar{x}+t_n w_n) - k(\bar{x})}{t_n} = \lim_{n\to+\infty} \frac{k(\bar{x}+t_n w'_n) - k(\bar{x})}{t_n}, \tag{3.6}$$

because k is steady at $\bar{x}$, and moreover, by the definition 2.2 above, it is easy to see that

$$\lim_{n\to+\infty} \frac{k(\bar{x}+t_n w_n) - k(\bar{x}+t_n w'_n)}{t_n} = 0.$$

Combining (3.4)-(3.6), yields that

$$w_0 \geq 0,$$

which contradicting condition (3.2).

Sufficient condition: Let us given a vector $\bar{x} \in C$ and assuming that (3.1) holds. It is well-known that

$$A(C,\bar{x}) = T(C,\bar{x}) = cl\, cone(C - \bar{x}),$$

because C is convex. Consequently,

$$v_x := x - \bar{x} \in A(C,\bar{x}), \ \forall x \in C. \tag{3.7}$$

By invoke the result in Luc [7], we have

$$k(x) - k(\bar{x}) \in D_c k(\bar{x}, k(\bar{x}))(v_x), \ \forall x \in C,$$

which yields that

$$k(x) - k(\bar{x}) \in D_c(k + \mathbb{R}_+)(\bar{x}, k(\bar{x}))(v_x), \ \forall x \in C. \tag{3.8}$$

By the definition of contingent epiderivative of k at $(\bar{x}, k(\bar{x}))$, we have that

$$\underline{D}k(\bar{x}, k(\bar{x}))(v_x) + \mathbb{R}_+ = D_c(k + \mathbb{R}_+)(\bar{x}, k(\bar{x}))(v_x) \ \forall x \in C,$$

which combines with (3.8), yields that

$$k(x) - k(\bar{x}) \in \underline{D}k(\bar{x}, k(\bar{x}))(v_x) + \mathbb{R}_+ \ \forall x \in C,$$

which is equivalent to

$$k(x) \geq k(\bar{x}) + \underline{D}k(\bar{x}, k(\bar{x}))(v_x) \ \forall x \in C.$$

We combine this with (3.1), it leads to $\bar{x} \in C$ being a solution of (P_k).

As was to be shown. □

Proposition 3.3. *Let $k, C, \overline{x}$ be given as in Proposition 3.1. Assume, in addition, that k is $\mathbb{R}_+$-concave on C. Then $\overline{x}$ is a solution of problem (Q_k) if and only if*

$$\overline{D}k(\overline{x}, k(\overline{x}))(v) \leq 0, \ \ \forall v \in A(C, \overline{x}), \tag{3.9}$$

where

$$(Q_k): \qquad max\{k(x) : x \in C\}.$$

Proof. It is not difficult to verify that $\overline{x}$ is a solution of the problem (Q_k) if and only if $\overline{x}$ is a solution of the problem (P_{-k}). Furthermore, it can easily be seen that k is $\mathbb{R}_+$-concave on C if and only if $(-k)$ is $\mathbb{R}_+$-convex on C. Therefore, by virtue of Proposition 3.2, $\overline{x}$ is a solution of the problem (Q_k) if and only if

$$\underline{D}(-k)(\overline{x}, (-k)(\overline{x}))(v) \geq 0 \ \ \forall v \in A(C, \overline{x}). \tag{3.10}$$

Notice that condition (3.10) equivalents

$$-\overline{D}k(\overline{x}, k(\overline{x}))(v) \geq 0 \ \ \forall v \in A(C, \overline{x}),$$

which is equivalent to condition (3.9) holds. This completes the proof. □

Remark 3.1. The results obtained of Propositions 3.2 and 3.3 are still true when $A(C, \overline{x})$ is replaced by $T(C, \overline{x})$, because $A(C, \overline{x}) \subset T(C, \overline{x})$.

We next give a necessary and sufficient efficiency condition for weakly efficient solution of constrained convex vector equilibrium problem in terms of contingent epiderivatives.

Theorem 3.1. *Let us consider the problem CVEPC as in Section 1. Assume, furthermore, that $dim(Y)$ is finite and for each $\overline{x} \in K$, the following conditions are fulfilled*

(i) *$F(\overline{x}, .)$ and g are steady at $\overline{x}$;*

(ii) *There exists $v_0 \in T(C, \overline{x}) \cap dom(D_c g(\overline{x}, g(\overline{x})))$ satisfying*

$$D_c g(\overline{x}, g(\overline{x}))(v_0) + \{g(\overline{x})\} \subset -int\, S. \tag{3.11}$$

Then $\overline{x} \in K$ is a weakly efficient solution to the problem CVEPC if and only if there exists a pair $(\lambda, \eta) \in Y^* \times Z^*$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\begin{cases} \lambda \in Q^+ \setminus \{0\}, \\ \eta \in S^+ \text{ with } \langle \eta, g(\overline{x}) \rangle = 0, \\ \underline{D}(\lambda_0 F(\overline{x}, .) + \eta_0 g)(\overline{x}, \lambda_0 F(\overline{x}, \overline{x}) + \eta_0 g(\overline{x}))(v) \geq 0 \quad \forall v \in T(C, \overline{x}). \end{cases} \tag{3.12}$$

Proof. We first prove that the condition (DKA) is fulfilled. In fact, we fixed $\overline{x} \in K$. By the assumption (ii), one can find a direction $v_0 \in T(C, \overline{x}) \cap dom(D_c g(\overline{x}, g(\overline{x})))$ such that condition (3.11) is valid. We arbitrarily take $z \in D_c g(\overline{x}, g(\overline{x}))(v_0)$ such that

$$z + g(\overline{x}) \in -intS. \tag{3.13}$$

By invoke the definition of contingent derivative of g at $(\overline{x}, g(\overline{x}))$, one has

$$(v_0, z) \in T(graph(g), (\overline{x}, g(\overline{x}))),$$

which yields the existence of sequences $t_n \to 0^+$, $(v_n, z_n) \to (v_0, z)$ such that

$$(\overline{x}, g(\overline{x})) + t_n(v_n, z_n) \in graph(g) \ \forall n \in \mathbb{N},$$

which is equivalent to

$$z_n = \frac{g(\overline{x} + t_n v_n) - g(\overline{x})}{t_n} \ \forall n \in \mathbb{N}.$$

Because of $dim(Y) < +\infty$, and from the assumption (i) it leads to g being stable at $\overline{x}$ (see the proof of Proposition 3.1 for details), hence the following limit exists

$$\lim_{n \to +\infty} \frac{g(\overline{x} + t_n v_n) - g(\overline{x})}{t_n} := z.$$

In the same way as in the proof of Proposition 3.2, we also obtain the result as follows

$$z = \lim_{n \to +\infty} \frac{g(\overline{x} + t_n v_n') - g(\overline{x})}{t_n},$$

where $v_n' \longrightarrow v_0$ and $\overline{x} + t_n v_n' \in C$ for all $n \in \mathbb{N}$. In other words, it follows from (3.13) that

$$\lim_{n \to +\infty} \Big(\frac{g(\overline{x} + t_n v_n) - g(\overline{x}) + t_n g(\overline{x})}{t_n} \Big) \in -intS.$$

Therefore, for n big enough,

$$\frac{g(\bar{x}+t_nv_n)-g(\bar{x})+t_ng(\bar{x})}{t_n} \in -intS,$$

which yields that

$$g(\bar{x}+t_nv_n)-g(\bar{x})+t_ng(\bar{x}) \in -intS$$

for sufficiently large n.

Consequently

$$g(\bar{x}+t_nv_n') \in -intS \text{ for sufficiently large n such that } t_n \in (0,1).$$

From there we conclude that condition (DKA) is fulfilled. Making use of Theorem 2.1, we get $\bar{x}$ is a weakly efficient solution to the CVEPC if and only if there exists $\lambda \in Q^+ \setminus \{0\}$ and $\eta \in S^+$ such that $\langle \eta, g(\bar{x}) \rangle = 0$ and condition (2.1) is satisfied. We set

$$F = \lambda_0 F(\bar{x}, .) + \eta_0 g : X \longrightarrow \mathbb{R}.$$

Then it is evident that F is $\mathbb{R}_+$-convex on C and continuous at $\bar{x}$ with $F(\bar{x}) = 0$. Moreover, $\bar{x}$ is a solution of (P_F). By directly using Propositions 3.1 and 3.2, $\underline{D}F(\bar{x}, F(\bar{x}))$ exists and satisfying

$$\underline{D}F(\bar{x}, F(\bar{x}))(v) \geq 0 \;\; \forall v \in T(C, \bar{x}),$$

therefore condition (3.12) is true. This completes the proof. □

Remark 3.2. If the condition of (3.11) is replaced by $D_cg(\bar{x})(v_0) - \{g(\bar{x})\} \subset -intS$, it is easy to see that the statements of Theorem 3.1 are still holds.

The following example is provided to explain for preceding Theorem 3.1.

Example 3.1. *Let $X = \mathbb{R}, Y = Z = \mathbb{R}^2$, $C = [-1,1]$, $Q = S = \mathbb{R}^2_+$ and let $\bar{x} \in [0,1)$ be arbitrary. The functions $F(\bar{x}, .), g : \mathbb{R} \longrightarrow \mathbb{R}^2$ are given respectively by*

$$F(\bar{x}, x) = (x - \bar{x}, x^2 - \bar{x}^2), \;\; \forall x \in \mathbb{R};$$

$$g(x) = (x, x), \;\; \forall x \in \mathbb{R}.$$

Then, we have the following assertions:

The feasible set of CVEPC has form

$$K = \{x \in C : g(x) \in -S\} = [-1, 0].$$

The objective functions $F(\bar{x},.)$ and g are steady at $\bar{x}\in K$ and moreover they are $\mathbb{R}^2_+$-convex on a convex set C, with $F(\bar{x},\bar{x})=0$.

We pick $x_0=-\frac{1}{2}$, by directly calculating one obtains as follows

$$g(x_0)\in -intS,$$

$$T(C,\bar{x})\cap dom\big(D_c g(\bar{x},g(\bar{x}))\big)=\mathbb{R}$$

$$D_c g(\bar{x},g(\bar{x}))(v)=\{(v,v)\}\ \forall v\in\mathbb{R}.$$

Therefore there exists at least a direction $v_0\in T(C,\bar{x})\cap dom\big(D_c g(\bar{x},g(\bar{x}))\big)$ (where we can pick $v_0<-\bar{x}$) such that

$$D_c g(\bar{x},g(\bar{x}))(v_0)+\{g(\bar{x})\}\subset -int\mathbb{R}^2_+.$$

Thus, it is evident that all the assumptions from (i) to (iii) in Theorem 3.1 are fulfilled. Let us next consider two cases can occur as follows:

Case 1. $\bar{x}=0$. *We pick $\lambda=(0,1)\in Q^+\setminus\{(0,0)\}=\mathbb{R}^2_+\setminus\{(0,0)\}$ and $\eta=0\in S^+=\mathbb{R}^2_+$ satisfying $\langle\eta,g(\bar{x})\rangle=0$. Then*

$$F(x)=\lambda_0 F(0,x)=x^2\ \textit{for all}\ x\in\mathbb{R}.$$

It is plain that the real-valued function F is steady at 0 and $\mathbb{R}^2_+$- convex on C. By a direct computation, it follows that

$$\underline{D}F(\bar{x},F(\bar{x}))(v)=0,\ \forall v\in T(C,\bar{x})=\mathbb{R}.$$

Case 2. $\bar{x}\in(0,1)$. *We pick $\lambda=(1,-\frac{1}{2\bar{x}})\in Q^+\setminus\{(0,0)\}=\mathbb{R}^2_+\setminus\{(0,0)\}$, $\eta=(0,0)\in S^+=\mathbb{R}^2_+$ satisfying $\langle\eta,g(\bar{x})\rangle=0$. Then the function*

$$F(.)=\lambda_0 F(\bar{x},.)+\eta_0 g(.):\mathbb{R}\longrightarrow\mathbb{R}$$

is defined by

$$F(x)=x-\bar{x}-\frac{1}{2\bar{x}}(x^2-\bar{x}^2),\ \forall x\in\mathbb{R}.$$

It is clear that F is steady at $\bar{x}$ and $\mathbb{R}^2_+$-convex on C. We have that

$$\underline{D}F(\bar{x},F(\bar{x}))(v)=0,\ \forall v\in T(C,\bar{x})=\mathbb{R}.$$

Theorem 3.1 leads to $\bar{x}$ being a weakly efficient solution to the CVEPC.

In the same way as in the statement of Theorem 3.1, we also obtain a necessary and sufficient efficiency condition for weakly efficient solution of constrained convex vector equilibrium problem via contingent hypoderivatives.

Theorem 3.2. *Assume that $dim(Y)$ is finite and for each $\bar{x} \in K$, the following conditions are fulfilled*

(i) *$F(\bar{x},.)$ and g are steady at $\bar{x}$;*
(ii) *$F(\bar{x},.)$ and g are Q-concave and S-concave on C, respectively;*
(iii) *There exists $v_0 \in T(C,\bar{x}) \cap dom(D_c g(\bar{x}, g(\bar{x})))$ satisfying*

$$D_c g(\bar{x}, g(\bar{x}))(v_0) + \{g(\bar{x})\} \subset int\, S. \tag{3.14}$$

Then $\bar{x} \in K$ is a weakly efficient solution to the problem CVEPC if and only if there exists a pair $(\lambda, \eta) \in Y^* \times Z^*$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\begin{cases} \lambda \in Q^+ \setminus \{0\}, \\ \eta \in S^+ \text{ with } \langle \eta, g(\bar{x}) \rangle = 0, \\ \overline{D}(\lambda_0 F(\bar{x},.) + \eta_0 g)(\bar{x}, \lambda_0 F(\bar{x},\bar{x}) + \eta_0 g(\bar{x}))(v) \leq 0 \quad \forall v \in T(C,\bar{x}). \end{cases} \tag{3.15}$$

Proof. We fixed $\bar{x} \in K$ and $v_0 \in T(C,\bar{x}) \cap dom(D_c g(\bar{x}, g(\bar{x})))$ such that all the assumptions from (i) to (iii) of Theorem 3.2 are fulfilled. We set $F'(\bar{x},.) = -F(\bar{x},.)$ and $g' = -g$. Then the objective functions $F'(\bar{x},.)$ and g' are steady at $\bar{x}$ and they are $Q-$ convex and $S-$ convex on C, respectively. It is not hard to check that the condition of (3.14) is equivalent to

$$D_c g'(\bar{x}, g'(\bar{x}))(v_0) + \{g'(\bar{x})\} \subset -int\, S.$$

Where we note that

$$D_c g'(\bar{x}, g'(\bar{x}))(v_0) = -D_c g(\bar{x}, g(\bar{x}))(v_0).$$

Making use of Theorem 3.1 to deduce that $\bar{x} \in K$ is a weakly efficient solution to the CVEPC if and only if there exists the pair $(\lambda, \eta) \in Y^* \times Z^*$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\begin{cases} \lambda \in Q^+ \setminus \{0\}, \\ \eta \in S^+ \text{ with } \langle \eta, g(\bar{x}) \rangle = 0, \\ \underline{D}(\lambda_0 F'(\bar{x}, .) + \eta_0 g')(\bar{x}, \lambda_0' F(\bar{x}, \bar{x}) + \eta_0 g'(\bar{x}))(v) \geq 0 \quad \forall v \in T(C, \bar{x}). \end{cases} \tag{3.16}$$

In other words, we always have

$$\begin{aligned} \overline{D}(\lambda_0 F(\bar{x}, .) + \eta_0 g)(\bar{x}, \lambda_0 F(\bar{x}, \bar{x}) + \eta_0 g(\bar{x}))(v) = -\underline{D}(\lambda_0 F'(\bar{x}, .) \\ + \eta_0 g')(\bar{x}, \lambda_0 F'(\bar{x}, \bar{x}) + \eta_0 g'(\bar{x}))(v) \\ \forall v \in T(C, \bar{x}), \end{aligned} \tag{3.17}$$

which yields that condition (3.15) holds and the conclusion follows. This completes the proof. □

Remark 3.3. If we replace the condition of (3.14) by $D_c g(\bar{x}, g(\bar{x}))(v_0) - \{g(\bar{x})\} \subset int\, S$, then it is not hard to see that the statements of Theorem 3.2 are still true.

Acknowledgements

The author is grateful to the reviewers for useful suggestions which improve the contents of this article. This research was supported by National Foundation for Science and Technology Development of Vietnam (NAPOSTED) under Grant 101.01-2014.61.

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