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A CHARACTERIZATION OF THE GENERALIZED KKM MAPPINGS VIA THE MEASURE OF NONCOMPACTNESS IN COMPLETE GEODESIC SPACES

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Abstract. By using the measure of noncompactness, we characterize the class of generalized KKM mappings on complete geodesic spaces. Existence theorems of fixed points and the best approximation for set-valued mappings are established. Applications are also provided. The results presented in this paper improve and extend some recent results in the literature.

Keywords. $CAT(0)$ space; Generalized KKM mapping; Fixed point; Best proximity point; Measure of noncompactness.

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1. Introduction

The terminology of $CAT(\kappa)$ spaces was coined by Gromov in 1987 [5]. The initials are in honour of Cartan, Alexandrov and Toponogov. A geodesic metric space X is said to be a $CAT(\kappa)$ space if for any geodesic triangle of appropriate size is not fatter than its comparison triangle in the model space M^2_κ. A metric space X is said to be of curvature bounded above by κ if it is locally a $CAT(\kappa)$ space. It is well known that any complete, simply connected

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Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. The classical hyperbolic spaces, Euclidean buildings, and the complex Hilbert ball with a hyperbolic metric are examples of $CAT(0)$ spaces. A general reference for facts about $CAT(0)$ spaces used here is [5].

KKM theory which was proposed by Knaster, Kuratowski, and Mazurkiewicz, in 1929 [21] and later developed by Ky Fan in 1972 [9] has a lot of valuable consequences in nonlinear analysis, such as fixed point theorems. In 1961 Ky Fan showed that the *KKM* theorems provide the foundation for a lot of modern essential results in diverse areas of mathematical sciences and in 1961 [8] he generalized the *KKM* theorem in the infinite dimensional topological vector spaces. In 1996 Khamsi [12] established an analogue of the famous *KKM*-map principle based on Ky Fan's theory for hyperconvex metric spaces. This result was used by Park [23] to establish some fixed point theorems and by Kirk *et al.* [20] to establish some fixed point theorems and saddle point theorems in hyperconvex metric spaces. Such contents of the *KKM* theory have numerous applications on various fields, especially, on fixed point theory and equilibrium theory; for more details and references see [25]. Recently, Niculescu and Roventa [22] obtained an analogue to *KKM* principle in $CAT(0)$ spaces; see [2, 3, 11, 17, 18, 24–28, 30–33] and the references therein.

The present paper mainly aims at establishing the class of generalized *KKM* type mappings on $CAT(0)$ spaces to specify the characteristics of the family of subsets in finite intersection properties. These in turn are applied to obtain some existence theorem of fixed point and best approximation. Moreover the paper extends the finite intersection property theorems and its application to noncompact version of the *KKM* principle in $CAT(0)$ spaces.

2. Preliminaries

Let (X,d) be a metric space and $x,y \in X$ with $d(x,y) = l$. A geodesic path from x to y is an isometry $c : [0,l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$ and $c(l) = y$. The image of c is called a geodesic segment joining x and y. A geodesic segment joining x and y is not necessarily unique in general. When it is unique, this geodesic segment is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that $d(x,z) = (1-\alpha)d(x,y)$ and $d(y,z) = \alpha d(x,y)$. In this case, we write $z = \alpha x \oplus (1-\alpha)y$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be a uniquely geodesic if there is exactly one geodesic joining x to y for each $x,y \in X$. In a geodesic space (X,d), the metric $d : X \times X \to \mathbb{R}$ is convex if for any $x,y,z \in X$ and $\alpha \in [0,1]$, one has

$$d(x, \alpha y \oplus (1-\alpha)z) \leq \alpha d(x,y) + (1-\alpha)d(x,z).$$

A subset Y of X is said to be convex if Y includes every geodesic segment joining any couple of its points. A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for a geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i,\bar{y}_j) = d(x_i,y_j)$ for $i,j \in \{1,2,3\}$.

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom:
"Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x,y \in \triangle$ and all comparison points $\bar{x},\bar{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{E}^2}(\bar{x},\bar{y})."$$

Also we will use $z = \bigoplus_{i=1}^{n} \alpha_i z_i = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \alpha_n z_n$ where $\alpha_1, \cdots, \alpha_n \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$, and $z_1, \cdots, z_n \in X$, to denote the unique point $z = (1-\alpha_n)z' \oplus \alpha_n z_n$ where

$$z' = \frac{\alpha_1}{1-\alpha_n} z_1 \oplus \frac{\alpha_2}{1-\alpha_n} z_2 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1}$$

for $\alpha_n \neq 1$ and $z = z_n$ for $\alpha_n = 1$.

Let X and Y be topological Hausdorff spaces, $A \subseteq Y$ and $T : X \to 2^Y$ be a multivalued map with nonempty values. Define

$$T^-(A) = \{x \in X : T(x) \cap A \neq \phi\}$$

and let $\text{int}(A)$, ∂A and $\mathscr{F}(A)$ denote the interior, boundary and the set of all nonempty finite subsets of A, respectively. We recall that a map $F : X \to 2^Y$ is said to be upper semi-continuous (lower semi-continuous) if for each closed (open) set $B \subseteq Y$, $T^-(B)$ is closed (open) in X.

Typically (cf. [22]) the convex hull $\mathrm{co(A)}$ of a subset A of a $CAT(0)$ space X is defined as follows:

$$\mathrm{co(A)} = \bigcup_{\mathrm{n=0}}^{\infty} \mathrm{A_n},$$

where $A_0 = A$ and for $n \geq 1$ the set A_n consist of all points in X which lie on geodesics which start and end in A_{n-1}.

Definition 2.1. Let X be a nonempty set, Y be a $CAT(0)$ space and $C \subseteq Y$.

i. A multivalued mapping $G : C \to 2^Y$ is said a KKM map if

$$\mathrm{co}(F) \subseteq \bigcup_{x \in F} G(x)$$

for every $F \in \mathscr{F}(C)$.

ii. A multivalued mapping $G : X \to 2^Y$ is said generalized KKM map if for each nonempty finite subset $A = \{x_1, \cdots, x_n\}$ of X there exist a nonempty subset $\{y_1, \cdots, y_n\}$ (not necessarily disjoint) of Y such that for each subset $\{y_{i_1}, \cdots, y_{i_j}\}$ of $\{y_1, \cdots, y_n\}$ we have

$$\mathrm{co}(\{y_{i_1}, \cdots, y_{i_j}\}) \subseteq \bigcup_{k=1}^{j} G(x_{i_k}).$$

The concept of generalized KKM maps is defined by Chang and Zhang in topological vector spaces [6] motivated by the works of Knaster-Kuratowski and Mazurkiewicz [21]. These notions also have been studied by Khan *et al.* in GFC-spaces [14], and more recently by Khamsi et al. in metric type spaces [13], and Park in generalized convex spaces [26]. We consider generalized KKM mappings in complete geodesic spaces.

It is clear that each KKM mapping is a generalized KKM map but there are some examples of generalized KKM mappings which are not KKM, see [15, 19].

Definition 2.2. [10] Let X be a set and $(X_i)_{i \in I}$ a family of subsets of X, then $(X_i)_{i \in I}$ has finite intersection property if for any finite, nonempty subset J of I, $\bigcap_{i \in J} X_i$ is not empty.

Motivated by the work of [4, 7, 31] we summarize some known results which we will use in the ensuing section.

For $n \geq 0$, Δ_n denotes the standard n-simplex of $\mathbb{R}^{n+1}$ with vertices $e_0, \cdots, e_n$ where e_i is the i-th unit vector in $\mathbb{R}^{n+1}$, that is

$$\Delta_n = \{(\alpha_0, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} \alpha_i = 1, \forall \alpha_i \geq 0\}.$$

The following lemmas can be easily proved.

Lemma 2.3. [7] *Let X be a $CAT(0)$ space and $x, y, z \in X$ such that $x \neq y$. Then*

i. *The mapping $f : [0,1] \to [x,y]$ defined by $f(t) = (1-t)x \oplus ty$, is continuous.*

ii. $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$, *for all* $t \in [0,1]$.

Lemma 2.4. *Suppose X is a $CAT(0)$ space and $A \subseteq X$. Then the convex hull of A is introduced via the formula*

$$\mathrm{co(A)} = \bigcup_{\mathrm{n}=0}^{\infty} \mathrm{A}'_{\mathrm{n}},$$

where $A'_0 = A$ and $A'_i = \{\bigoplus_{i=1}^{n} a_i x_i : \sum_{i=1}^{n} a_i = 1, \{x_1, \cdots, x_n\} \in \mathscr{F}(A'_{i-1})\}$ for $i \geq 1$.

Lemma 2.5. *Let X be a $CAT(0)$ space. Then the mapping $f : \Delta_n \to X$ which is defined by*

$$f(t_0, t_1, \cdots, t_n) = t_0 x_0 \oplus t_1 x_1 \oplus \cdots \oplus t_n x_n,$$

is continuous for each $x_0, \cdots, x_n \in X$ and $n \in \mathbb{N}$.

Proof. Our proof is by induction on n. For case $n = 1$, the result is obvious by Lemma 2.3Now assume that for case $n = k-1$ the result is true. To prove the result for $n = k$, let $\{\alpha_m\}_{m=1}^{\infty}$ be a sequence in $\triangle_k$ such that $\alpha_m \to \alpha$, and $x_0, \cdots, x_n \in X$. Our goal is to show that

$$\alpha_0^m x_0 \oplus \alpha_1^m x_1 \oplus \cdots \oplus \alpha_k^m x_k \to \alpha_0 x_0 \oplus \alpha_1 x_1 \oplus \cdots \oplus \alpha_k x_k,$$

where $\alpha_m = (\alpha_0^m, \cdots, \alpha_k^m)$ for each $m \in \mathbb{N}$ and $\alpha = (\alpha_0, \cdots, \alpha_k)$. Assume that $\alpha_k^m \to 1$ and we can also assume that $\alpha_k^m \neq 1$. Define

$$z_{k-1}^m = \frac{\alpha_0^m}{1-\alpha_k^m} x_0 \oplus \cdots \oplus \frac{\alpha_{k-1}^m}{1-\alpha_k^m} x_{k-1}.$$

By the induction hypothesis, we have

$$z_{k-1}^m \to \frac{\alpha_0}{1-\alpha_k} x_0 \oplus \cdots \oplus \frac{\alpha_{k-1}}{1-\alpha_k} x_{k-1},$$

as $m \to \infty$. Set

$$L = \lim_{m\to\infty} \alpha_0^m x_0 \oplus \alpha_1^m x_1 \oplus \cdots \oplus \alpha_k^m x_k.$$

Then we have

$$\begin{aligned} L &= (\lim_{m\to\infty}(1-\alpha_k^m)) \lim_{m\to\infty} z_{k-1}^m \oplus \lim_{m\to\infty} \alpha_k^m x_k \\ &= (1-\alpha_k)\left(\frac{\alpha_0}{1-\alpha_k} x_0 \oplus \cdots \oplus \frac{\alpha_{k-1}}{1-\alpha_k} x_n\right) \oplus \alpha_k x_k \\ &= \alpha_0 x_0 \oplus \cdots \oplus \alpha_{k-1} x_{k-1} \oplus \alpha_k x_k. \end{aligned}$$

Thus we have proved that

$$\alpha_1^m x_0 \oplus \alpha_1^m x_1 \oplus \alpha_k^m x_k \to \alpha_1 x_0 \oplus \cdots \oplus \alpha_k x_k,$$

when $\alpha_k^m \nrightarrow 1$ as $m \to \infty$. If $\alpha_k^m \to 1$, then $\alpha_i^m \to 0$ as $m \to \infty$ for each $i = 0, \cdots, k-1$. This condition is satisfied due to $\sum_{i=0}^k \alpha_i^m = 1$.

Thus in order to finish our proof we have to show that

$$\alpha_0^m x_0 \oplus \alpha_2^m x_2 \oplus \cdots \oplus \alpha_k^m x_k \to x_k,$$

as $m \to \infty$. Consider $\{\beta_m\}$ be a subsequence of $\{\alpha_m\}$ such that $\beta_k^m \neq 1$ and $\{\gamma_m\}$ be a subsequence of $\{\alpha_m\}$ such that $\gamma_k^m = 1$, where $\beta_m = (\beta_0^m, \cdots, \beta_k^m)$, and $\gamma_m = (\gamma_0^m, \cdots, \gamma_k^m)$. Since

$$\gamma_0^m x_0 \oplus \gamma_1^m x_1 \oplus \cdots \oplus \gamma_k^m x_k \to x_k,$$

as $m \to \infty$. So it is enough to check that

$$\beta_0^m x_0 \oplus \beta_1^m x_1 \oplus \cdots \oplus \beta_k^m x_k \to x_k,$$

as $n\to\infty$. In order to prove, put $d=d(\beta_0^m x_0\oplus\beta_1^m x_1\oplus\cdots\oplus\beta_k^m x_k, x_k)$. By Lemma 2.3, we have

$$\begin{aligned} d &= d\left((1-\beta_k^m)\left(\frac{\beta_0^m}{1-\beta_k^m}x_0\oplus\cdots\oplus\frac{\beta_{k-1}^m}{1-\beta_k^m}x_{k-1}\right)\oplus\beta_k x_k, x_k\right)\\ &\le (1-\beta_k^m)d(\frac{\beta_0^m}{1-\beta_k^m}x_0\oplus\cdots\oplus\frac{\beta_{k-1}^m}{1-\beta_k^m}x_{k-1}, x_k)\\ &\le (1-\beta_k^m-\beta_{k-1}^m)d(y_{k-2},x_k)+\beta_{k-1}^m d(x_{k-1},x_k)\\ &\vdots\\ &\le \sum_{i=0}^{n}\beta_i^m d(x_i,x_k)\to 0, \end{aligned}$$

as $m\to\infty$, where

$$y_{k-2}=\frac{\beta_0^m}{1-\beta_k^m-\beta_{k-1}^m}x_0\oplus\cdots\oplus\frac{\beta_{k-2}^m}{1-\beta_k^m-\beta_{k-1}^m}x_{k-2}.$$

It follows that

$$\beta_0^m x_0\oplus\beta_1^m x_1\oplus\cdots\oplus\beta_k^m x_k\to x_k,$$

as $m\to\infty$. This completes the proof.

Lemma 2.6. [4] *Suppose $F_0, F_1, \cdots, F_n$ are closed subset of standard n-simplex Δ_n in $\mathbb{R}^{n+1}$. If for any nonempty subset I of $\{0,1,\cdots,n\}$,*

$$\mathrm{co}(\{\mathrm{e_i} : \mathrm{i}\in \mathrm{I}\})\subseteq\bigcup_{\mathrm{i}\in\mathrm{I}}\mathrm{F_i},$$

then $\bigcap_{i=0}^n F_i\neq\phi$.

Now suppose that $\{x_n\}$ is a sequence in a $CAT(0)$ space X such that $x_n\to x\in X$ as $n\to\infty$, and $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\alpha_n\to\alpha$. Then we have

$$z_n=\alpha_n x_n\oplus(1-\alpha_n)y\to z=\alpha x\oplus(1-\alpha)y,$$

as $n\to\infty$, where $y\in X$. Indeed, by ([16], Lemma 3.2) and triangular inequality, we deduce

$$d(z_n,z)\le d(z_n,\alpha_n x\oplus(1-\alpha_n)y)+d(\alpha_n x\oplus(1-\alpha_n)y,z)\to 0,$$

as $n\to\infty$.

3. Main results

Now, we give a characterization of the generalized KKM mappings in $CAT(0)$ spaces.

Theorem 3.1. *Let X be a nonempty set, Y be a $CAT(0)$ space, and $F : X \to 2^Y$ be a multivalued mapping with closed values. Then the family*

$$\{F(x) : x \in X\},$$

has the finite intersection property if and only if the mapping F is a generalized KKM mapping.

Proof. Let $\{x_0, \cdots, x_n\}$ be a subset of X, and $F : X \to 2^Y$ be a generalized KKM mapping. Therefore, there exist corresponding points $y_0, \cdots, y_n$ of Y such that for each subsequence $y_{i_0}, \cdots, y_{i_k}$ we deduce

$$\mathrm{co}(\{y_{i_0}, \cdots, y_{i_k}\}) \subset \bigcup_{j=0}^{k} F(x_{i_j}).$$

Let $C = \mathrm{co}(\{y_0, y_1, \cdots, y_n\})$ and define

$$F_i = F(x_i) \cap C$$

for every $i = 0, \cdots, n$. Since for every $i = 0, \cdots, n$; F_i is a closed subset of C. Define $\phi : \Delta_n \to C$ by

$$\phi(a) = \bigoplus_{i=0}^{n} a_i y_i,$$

where $a = (a_0, a_1, \cdots, a_n)$. By Lemma 2.5, ϕ is continuous, and we deduce $\phi^{-1}(F_i)$ is closed in Δ_n for each $i = 0, \cdots, n$.

On the other hand, we have

$$\phi(\mathrm{co}(\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}) \subseteq \mathrm{co}(\{y_{i_0}, y_{i_1}, \cdots, y_{i_k}\}) \subseteq \bigcup_{j=0}^{k} F(x_{i_j})$$

for each subsequence $e_{i_0}, \cdots, e_{i_k}$ of $\{e_0, \cdots, e_n\}$. It implies

$$\mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\} \subseteq \bigcup_{j=0}^{k} \phi^{-1}(F(x_{i_j}))$$

for each subsequence $e_{i_0}, \cdots, e_{i_k}$. Therefore, by Lemma 2.6, $\bigcap_{i=0}^{n} \phi^{-1}(F_i) \neq \phi$. It implies there exists $a \in \Delta_n$ such that

$$a \in \bigcap_{i=0}^{n} \phi^{-1}(F(x_i) \cap C).$$

Then $\phi(a) \in \cap_{i=0}^{n} F(x_i) \cap C$, Finally, we have $\phi(a) \in \bigcap_{i=0}^{n} F(x_i)$, as we wanted.

In order to finish the proof we have to show that if the family

$$\{F(x) : x \in X\}$$

has the finite intersection property, then F is a generalized KKM mapping. Suppose $\{x_0, \cdots, x_n\}$ is a subset of X. We know that $\bigcap_{i=0}^{n} F(x_i) \neq \emptyset$, so choose $x^* \in \bigcap_{i=0}^{n} F(x_i)$. Set $y_i = x^*$ for $i = 0, \cdots, n$. Then for any $0 \leq k \leq n$ and any subsequence $y_{i_0}, \cdots, y_{i_k}$ it follows that

$$\operatorname{co}(\{y_{i_j} : j = 0, \cdots, k\}) = \operatorname{co}(\{x^*\}) = \{x^*\} \subseteq \bigcup_{i=0}^{k} F(x_i).$$

This proves that F is a generalized KKM mapping.

Corollary 3.2. *Let X be a nonempty set Y be a $CAT(0)$ space and $G : X \to 2^Y$ be a multivalued map with closed values. If the mapping G is a KKM mapping, then the family*

$$\{G(x) : x \in X\},$$

has the finite intersection property. In the sequel, we are going to obtain a characterization of a generalized KKM mapping $G : X \to 2^Y$ with transfer closed values. i.e., for each $x \in X$ and $y \notin G(x)$, there exist $x' \in X$ such that $y \notin \overline{(G(x'))}$, where X and Y are two $CAT(0)$ spaces.

Remark 3.3. It is obvious that every multivalued mapping with closed values is transfer closed valued. However, you can find some examples in [31] which show the converse is not true. Also it is easy to check that the mapping $G : X \to 2^Y$ is transfer closed valued if and only if $\bigcap_{x \in X} G(x) = \bigcap_{x \in X} \overline{(G(x))}$.

Motivated by the study of Kirk, Sims and Yuan [18] we now establish noncompactness version of both generalized KKM and the fixed point theorems for set-valued mappings in $CAT(0)$ spaces. These results are noncompact generalization of the corresponding results given in above.

Let M be a $CAT(0)$ space and let μ denote the usual Kuratowksi measure of noncompactness on M i.e. for each nonempty bounded $A \subseteq M$:

$$\mu(A) = \inf\{\varepsilon > 0\text{: there exists } n \text{ with } A \subset \textstyle\bigcup_{i=1}^{n} A_i, \text{ where } diam(A_i) \ < \varepsilon\}.$$

Note that $\mathrm{diam}(A_i) = \sup_{x,y \in A_i} d(x,y)$ for $i = 1, \cdots, n$. We also need the following result which is from Horvath [10].

Lemma 3.4. *Let Y be a complete metric space and let $\{F_i : i \in I\}$ be a family of nonempty closed subset of Y having the finite intersection property. If $\inf\limits_{i\in I} \mu(F_i) = 0$ then $\bigcap\limits_{i\in I} F_i$ is nonempty and compact.*

Theorem 3.5. *Let X be a nonempty set and Y be a complete $CAT(0)$ space. Suppose $G : X \to 2^Y \backslash \{\phi\}$ is a closed valued mapping and $\inf\limits_{x\in X} \mu(\overline{G(x)}) = 0$. Then $\bigcap_{x\in X} G(x)$ is nonempty if and only if the mapping $\overline{G}$ is a generalized KKM mapping.*

Proof. If $\overline{G(x)}$ is a generalized KKM mapping, it follows by Theorem 3.1 that the family $\{\overline{G(x)} : x \in X\}$ has the finite intersection property. Since

$$\inf_{x\in X} \mu(\overline{G(x)}) = 0,$$

according to Lemma 3.4 we imply that $\bigcap_{x\in X} \overline{G(x)} \neq \phi$, according to Remark 3.3 G is transfer closed valued so

$$\bigcap_{x\in X} G(x) \neq \phi.$$

Now suppose $\bigcap\limits_{x\in X} G(x) \neq \phi$ since G is transfer closed valued, so $\bigcap_{x\in X} \overline{G(x)} \neq \phi$ and the family $\{\overline{G(x)} : x \in X\}$ has the finite intersection property, so according to Theorem 3.1, $\overline{G(x)}$ is a generalized KKM mapping.

Corollary 3.6. *Let X be a nonempty set, Y be a complete $CAT(0)$ space, and $G : X \to 2^Y \backslash \{\phi\}$ be a closed valued mapping, and $\inf\limits_{x\in X} \mu(\overline{G(x)}) = 0$. Moreover, suppose for any finite nonempty subset J of I, $\bigcap_{i\in J} \overline{G(x_i)}$ is nonempty. Then*

$$\bigcap_{x\in X} G(x) = \bigcap_{x\in X} \overline{(G(x))} \neq \emptyset.$$

Theorem 3.7. *Let X be a nonempty set, Y be a complete $CAT(0)$ space, and $F : X \to 2^Y$ be a multivalued mapping with closed values and $\inf_{x\in X} \mu(F(x)) = 0$. Then $\bigcap_{x\in X} F(x) \neq \emptyset$ if and only if F is a generalized KKM map.*

Proof. Suppose $F : X \to 2^Y$ be a generalized KKM mapping, so according to Theorem the family $\{F(x) : x \in X\}$ has the finite intersection property. By assumption $\inf_{x\in X} \mu(F(x)) = 0$, so according to Lemma 3.4, $\bigcap_{x\in X} F(x) \neq \emptyset$.

Now suppose $\bigcap_{x\in X} F(x) \neq \emptyset$. So the family $\{F(x) : x \in X\}$ has the finite intersection property and according to Theorem 3.1, F is a generalized KKM mapping.

Theorem 3.8. *Let X be a nonempty set, Y be a complete $CAT(0)$ space and $G : X \to 2^Y$ be a multivalued mapping with closed values. Suppose there exists $x_0 \in X$ such that $\inf\limits_{x\in X} \mu(G(x_0)\bigcap G(x)) = 0$. Then the $\bigcap_{x\in X} G(x) \neq \emptyset$ if and only if the mapping G is a generalized KKM mapping.*

Proof. Since $\bigcap_{x\in X} G(x) \neq \phi$, the family $\{G(x) : x \in X\}$ has the finite intersection property. Since $G(x)$ is closed for each $x \in X$, thus by Theorem 3.1, G is a generalized KKM mapping.

Now suppose G is a generalized KKM mapping. Theorem 3.1 implies that the family $\{G(x) : x \in X\}$ has the finite intersection property. Therefore

$$\{G(x) \cap G(x_0) : x \in X\},$$

has the finite intersection property. Since $\inf\limits_{x\in X} \mu(G(x_0)\bigcap G(x)) = 0$, we find from Lemma 3.4 that

$$\emptyset \neq \bigcap_{x\in X} G(x) \bigcap G(x_0) = \bigcap_{x\in X} G(x),$$

which completes the proof.

4. Applications

Here are some applications of our results. At first we obtain some best approximation theorems and then we prove some fixed point theorems.

Theorem 4.1. *Suppose X is a compact subset of a complete $CAT(0)$ space Y and $F, G : X \to 2^Y$ are upper semi continuous maps with nonempty, compact and convex values. If $G^{-}(C)$ is convex for each convex subset C of Y, then $H : X \to 2^X$ is generalized KKM mapping, where*

$$H(y) = \{x \in X : d(G(x), F(x)) \leqslant d(G(y), F(x))\}$$

and $\inf_{y\in X}\mu(H(y))=0$*, also there exists* $x_0\in X$ *such that*

$$d(G(x_0),F(x_0))=\inf_{x\in X}d(G(x),F(x_0)).$$

Proof. Since $H:X\to 2^X$ is defined by

$$H(y)=\{x\in X: d(G(x),F(x))\leq d(G(y),F(x))\},$$

for each $y\in X$, $y\in H(y)$, so $H(y)\neq\emptyset$. We claim that $H(y)$ is closed for each $y\in X$. In order to show that, suppose $\{y_n\}$ is a sequence in $H(y)$ such that $y_n\to y^*$. We show that $y^*\in H(y)$. Let $\varepsilon>0$ be arbitrary. Since F is upper semicontinuous with compact values, there exists N_1 such that for each $n\geq N_1$, we have

$$F(y_n)\subseteq\overline{B}(F(y^*),\varepsilon).$$

Similarly, we can prove there exists N_2 such that for each $n\geq N_2$, we have

$$G(y_n)\subseteq\overline{B}(G(y^*),\varepsilon).$$

Let $N=\max\{N_1,N_2\}$. Then we have

$$\begin{aligned}
d(G(y^*),F(y^*)) &\leq d(G(y^*),G(y_n))+d(G(y_n),F(y_n))+d(F(y_n),F(y^*))\\
&\leq 2\varepsilon+d(G(y_n),F(y_n)) \qquad y_n\in H(y)\\
&\leq 2\varepsilon+d(G(y),F(y_n))\\
&\leq 2\varepsilon+d(G(y),F(y^*))+d(F(y^*),F(y_n))\\
&\leq 3\varepsilon+d(G(y),F(y^*)).
\end{aligned}$$

Since ε is arbitrary, $d(G(y^*),F(y^*))\leq d(G(y),F(y^*))$, and this proves our claim.

Now we show that for each $A\in\mathscr{F}(X)$, $\mathrm{co}(A)\subseteq H(A)$. On the contrary suppose $\mathrm{co}(A)\not\subseteq H(A)$ for some $A\in\mathscr{F}(X)$. Then there exists $y\in\mathrm{co}(A)$ such that $y\notin H(a)$ for every $a\in A$. Therefore

$$d(G(a),F(y))<d(G(y),F(y))$$

for all $a \in A$. So we have

$$G(a) \bigcap \left(\bigcup_{y' \in F(y)} B\left(y', \max_{a \in A} d(G(a), F(y))\right)\right) \neq \emptyset$$

for each $a \in A$. Since $F(y)$ is convex, we have

$$\bigcup_{y' \in F(y)} B\left(y', \max_{b \in A} d(G(b), F(y))\right)$$

is convex. It follows that

$$G(y) \bigcap \left(\bigcup_{y' \in F(y)} B\left(y', \max_{b \in A} d(G(b), F(y))\right)\right) \neq \emptyset.$$

Therefore, we have

$$d(G(y), F(y)) \leq \max_{b \in A} d(G(b), F(y)) < d(G(y), F(y)),$$

which is a contradiction. Now, Corollary 3.2 implies that $H(y)$ has finite intersection property. Since $\inf_{y \in X} \mu(H(y)) = 0$, we find from Lemma 3.4 that $\bigcap_{y \in X} H(y) \neq \emptyset$. So there exists $x_0 \in X$ such that

$$x_0 \in \bigcap_{y \in X} H(y).$$

Hence

$$d(G(x_0), F(x_0)) = \inf_{x \in X} d(G(x), F(x_0)),$$

which completes the proof.

Corollary 4.2. *Let Y be a complete $CAT(0)$ space and X be a nonempty subset of Y. Suppose $F : X \to 2^Y$ is a set-valued continuous mapping such that*

$$\inf_{x \in X} \mu(\{y \in X : d(y, F(y)) \leq d(x, F(y))\}) = 0.$$

Then there exists $x_0 \in X$ such that

$$d(x_0, F(x_0)) = \inf_{x \in X} d(x, F(x_0)).$$

Corollary 4.3. *Let X be a nonempty subset of $CAT(0)$ space Y and $X \in \mathscr{F}(Y)$. Suppose $F : X \to 2^X$ is a set-valued continuous mapping with nonempty closed values such that*

i. $\inf_{x\in X}\mu\{y\in X : d(y,F(y)) \leq d(x,F(y))\} = 0$

ii. *for each $x\in X$ with $x\notin F(x)$ there exist $z\in X$ such that*

$$d(z,F(x)) < d(x,F(x)),$$

then F has a fixed point in X.

Proof. By Corollary 4.2, we see that there exist $x_0 \in X$ such that

$$d(x_0,F(x_0)) = \inf_{x\in X} d(x,F(x_0)).$$

We claim x_0 is a fixed point of F. Indeed, assume this were not true, i.e., $x_0 \notin F(x_0)$. Then it follows $d(x_0,F(x_0)) > 0$. Then by assumption $z_0 \in X$ such that

$$d(z_0,F(x_0)) < d(x_0,F(x_0)).$$

On the other hand, note that $d(z_0,F(x_0)) \geq d(x_0,F(x_0)) > 0$. This implies $0 < d(z_0,F(x_0)) < d(z_0,F(x_0))$ which is a contradiction so x_0 is a fixed point of F.

We can state the following result which is an analogue of Fan's best approximation in $CAT(0)$ spaces.

Corollary 4.4. *Suppose X is a compact subset of a $CAT(0)$ space Y, and $F : X \to Y$ be continuous. Then there exists $x_0 \in X$ such that*

$$d(x_0,F(x_0)) = \inf_{x\in X} d(x,F(x_0)).$$

In general, the result obtained in Theorem 4.1 is not true if we change the compactness of X by completeness. In fact, define $f : [0,\infty) \to [0,\infty)$ by $f(x) = x+1$, It is not difficult to check that Theorem 4.1 does not hold. However, Kirk [16] proved an analogue of Theorem 4.1 for complete $\mathbb{R}$-trees. As a consequence a fixed point theorem for continuous mappings in complete $\mathbb{R}$-trees was found in [16], but it does not hold in $CAT(0)$ space [29]. Corollary 4.4 is similar to Theorem 2.13 of Amini *et al.* [1] in $\mathcal{N}\mathcal{R}$-metric spaces.

Theorem 4.5. *Let X be a $CAT(0)$ space and $F : X \to 2^X$ be a convex valued mapping. Then F has a fixed point if and only if the map $x \mapsto X\backslash F^{-}(x)$ is not a KKM map.*

Proof. Define $G : X \to 2^X$ by

$$G(x) = X \backslash F^{-}(x).$$

Suppose G is not a KKM mapping. It follows that there exist a subset $\{x_1, \cdots, x_n\}$ of X and $x_0 \in \mathrm{co}(\{x_1, \cdots, x_n\})$ such that $x_0 \notin G(x_i)$ for each $i = 1, \cdots, n$. It implies that $x_i \in F(x_0)$ for each $i = 1, \cdots, n$. Since $F(x_0)$ is convex, so it follows that

$$\mathrm{co}(\{x_1, \cdots, x_n\}) \subseteq F(x_0),$$

i.e. $x_0 \in F(x_0)$. Now suppose F has a fixed point, i.e., $x_0 \in F(x_0)$ for some $x_0 \in X$. Then $x_0 \notin G(x_0)$. So G is not a KKM mapping and the proof is complete.

Now by Theorem 4.5 and Corollary 3.6, we obtain the following form of Fan-Browder fixed point theorem in $CAT(0)$ spaces.

Corollary 4.6. *Let X be a convex and compact subset of a $CAT(0)$ space Y. Moreover suppose $F : X \to 2^Y$ satisfies:*

i. *for each $x \in X$, $X \backslash F^{-}(x)$ is transfer closed valued,*
ii. *for each $x \in X$, $F(x)$ is nonempty and convex.*

Then F has a fixed point.

Now we deduce an analogue of Fan's lemma in $CAT(0)$ spaces.

Corollary 4.7. *Let X be a convex and compact subset of a $CAT(0)$ space Y. Moreover suppose $C \subset X \times X$ satisfies:*

i. *for each $x \in X$, $(x,x) \in C$,*
ii. *for each $x \in X$, $\{y \in X : (x,y) \notin C\}$ is either nonempty and convex or empty,*
iii. *for each $y \in X$, $\{x \in X : (x,y) \notin C\}$ is closed.*

Then there exists $x_0 \in C$ such that $\{x_0\} \times C \subseteq C$.

Proof. Define $F : X \to 2^X$ by

$$F(x) = \{y \in X : (x,y) \notin C\}.$$

Then by Corollary 3.4, there exists $x_0 \in C$ such that $\{x_0\} \times C \subseteq C$.

Theorem 4.8. *Let $X_1, \cdots, X_n$ be $CAT(0)$ spaces and $X = \prod_{i=1}^n X_i$. Suppose for each $i = 1, \cdots, n$, $T_i : X \to 2^{X_i}$, is a generalized KKM mapping with nonempty closed values and $\inf_{x\in X} \mu(T_i(x)) = 0$ for each $i = 1, \cdots, n$. Then $T = \prod_{i=1}^n T_i : X \to 2^X$ has a fixed point.*

Proof. Suppose $\{x_1, \cdots, x_n\}$ be a nonempty finite subset of X. Since $T_j : X \to 2^{X_j}$, is a generalized KKM mapping for each $j = 1, \cdots, n$, so there exists $\{y_1^j, \cdots, y_m^j\}$ in X_j such that for each subset $\{y_{i_1}^j, \cdots, y_{i_k}^j\}$ of $\{y_1^j, \cdots, y_m^j\}$ we have

$$\mathrm{co}(\{y_{i_1}^j, \cdots, y_{i_k}^j\}) \subseteq \bigcup_{l=1}^k T_j(x_{i_l}).$$

Now let $y_j = (y_j^1, \cdots, y_j^n)$ for $j = 1, \cdots, m$. Therefore

$$\mathrm{co}(\{y_{i_1}, \cdots, y_{i_k}\}) \subseteq \bigcup_{j=1}^k T(y_{i_j}),$$

and T is a generalized KKM mapping so by Theorem 3.1, the family $\{T(x) : x \in X\}$ has the finite intersection property. On the other hand by assumption we have $\inf_{x\in X} \mu(T_i(x)) = 0$ for each $i = 1, \cdots, n$ since $T = \prod_{i=1}^n T_i$ so $\inf_{x\in X} \mu(T(x)) = 0$. Now by Lemma 3.4, $\bigcap_{x\in X} T(x) \neq \emptyset$. So there exists $x_0 \in X$ such that $x_0 \in \bigcap_{x\in X} T(x)$ then $x_0 \in T(x_0)$ and proof is complete.

Corollary 4.9. *Suppose X be a $CAT(0)$ space and $F : X \to 2^X$ be a generalized KKM map with nonempty, closed values and $\inf_{x\in X} \mu(F(x)) = 0$. Then F has a fixed point.*

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