J. Nonlinear Funct. Anal. 2017 (2017), Article ID 9

MATHRES
Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

SOME FIXED POINT THEOREMS FOR G-CONTRACTIONS IN CONE b-METRIC SPACES OVER A BANACH ALGEBRA

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Abstract. In this paper, we introduce the notion of G-contractions on cone b-metric spaces over Banach algebras endowed with a graph G. Fixed point theorems for G-contractions are proved. Some examples are also provided to illustrate the main results presented in this paper, which extend and generalize several known results in cone b-metric spaces over Banach algebras.

Keywords. Cone metric space; Cone b-metric space; Solid cone; Banach algebra; Fixed point.

2010 Mathematics Subject Classification. 47H10; 54H25.

1. Introduction

In 2007, Huang and Zhang [1] introduced the notion of cone metric spaces by assigning a vector-value in a Banach space to the metric defined on a set. They started the study of fixed point theorems in such spaces called cone metric spaces. Afterwards, several authors published many papers on this interesting topic, see, [2]. In some recent studies, (see, e.g., [3, 4, 5, 6]) it was shown that the results obtained in cone metric spaces are not a genuine generalization of the corresponding versions of metric spaces. Indeed, in these papers it was shown that the most

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Received June 30, 2016; Accepted December 5, 2016.

of the fixed point results of cone metric spaces are the consequences of their corresponding metric versions. Recently, Li and Xu [7] introduced the notion of cone metric spaces over Banach algebra and defined generalized Lipschitz contraction with vector contractive coefficient instead of usual real constant. They proved the existence of fixed points with the assumption that the underlying cone is normal, furthermore, they explained by an example that the fixed point theorems in cone metric spaces over Banach algebra are not equivalent to those in metric spaces, and so, such generalizations are the genuine ones. Latter, Xu and Radenović [8] showed that the normality of the cone can be removed from the results of Li and Xu [7].

Bakhtin [9] introduced the notion of b-metric spaces as a generalization of metric spaces and obtained the contraction mapping principle in such spaces and generalized the famous Banach contraction principle in this new setting. Hussain and Shah [10] introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. Some fixed point results on cone b-metric spaces over Banach algebra can be found in [11].

On the other hand, Jachymski [13] extended the famous Banach contraction principle for the spaces endowed with a graph. He obtained some fixed point results for the mappings satisfying some conditions involving the graphical structure. In this paper, we introduce the G-contractions in cone b-metric spaces over Banach algebra and endowed with a graph and prove some fixed point results for such contractions. Our results extend several known results in cone b-metric spaces over Banach algebra. Some example are presented which illustrate the results proved herein.

2. Preliminaries

First, we recall some definitions and results about the Banach algebras and cone b-metric spaces.

Let A be a real Banach algebra, i.e., A is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all $x,y,z \in A, a \in \mathbb{R}$

(1) $x(yz) = (xy)z$;

(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(3) $a(xy) = (ax)y = x(ay)$;

(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that the Banach algebra A has a unit, i.e., a multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details we refer to [14].

The following proposition is well known [14].

Proposition 2.1. Let A be a real Banach algebra with a unit e and $x \in A$. If the spectral radius $\rho(x)$ of x is less than *one*, i.e.,

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if

(1) P is non-empty, closed and $\{\theta, e\} \subset P$, where θ is the zero vector of A;
(2) $a_1P + a_2P \subset P$ for all non-negative real numbers a_1, a_2;
(3) $P^2 = PP \subset P$
(4) $P \bigcap (-P) = \{\theta\}$.

For a given cone $P \subset A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^\circ$, where P° denotes the interior of P.

The cone P is called normal if there exists a number $K > 0$ such that for all $a, b \in A$,

$$a \preceq b \quad \text{implies} \quad \|a\| \leq K\|b\|.$$

The least positive value of K satisfying the above inequality is called the normal constant (see [1]). Note that, for any normal cone P we have $K \geq 1$ (see [15]). In the following we always assume that P is a cone in a real Banach algebra A with $P^\circ \neq \phi$ (i.e., the cone P is a solid cone) and $\preceq$ is the partial ordering with respect to P.

The following lemmas and remarks play an important role in this article.

Lemma 2.2. [16] *If E is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

Lemma 2.3. [17] *If E is a real Banach space with a solid cone P and if $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 2.4. [17] *If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n < n_0$.*

Remark 2.5. [8] If $\rho(x) < 1$, then $\|x^n\| \to 0$ as $n \to \infty$.

Definition 2.6. [7, 18, 1] Let X be a non-empty set. Suppose that the mapping $d\colon X \times X \to A$ satisfies:

(1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$.
(3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space over the Banach algebra A.

Definition 2.7. [11] Let X be a non-empty set and $s \geq 1$ be a constant and A be a Banach algebra. Suppose that the mapping $d\colon X \times X \to A$ satisfies for all $x,y,z \in X$:

(1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$.
(3) $d(x,y) \preceq s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$.

Then d is called a cone b-metric on X, and (X,d,s) is called a cone b-metric space over the Banach algebra A.

Definition 2.8. [11] Let (X,d,s) be a cone b-metric space over a Banach algebra A, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

(1) The sequence $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n,x) \ll c$ for all $n > n_0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2) The sequence $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n,x_m) \ll c$ for all $n,m > n_0$.

(3) (X,d,s) is a complete cone b-metric space if every Cauchy sequence is convergent in X.

It is obvious that the limit of a convergent sequence in a cone b-metric space is unique. A mapping $T\colon X\to X$ is called continuous at $x\in X$, if for every sequence $\{x_n\}$ in X such that $x_n\to x$ as $n\to\infty$, we have $Tx_n\to Tx$ as $n\to\infty$.

The following definition about the graphs can be found in [12, 13, 19].

Let X be a nonempty set and by $\mathscr{D}$ denote the diagonal of the cartesian product $X\times X$. Let $G=(V(G),E(G))$ be a directed graph such that the set of vertices $V(G)=X$ and the set of its edges $E(G)\supseteq\mathscr{D}$, i.e., $E(G)$ contains all loops, then the set X is said to be endowed with the graph $G=(V(G),E(G))$. The graph G is assumed to be free of parallel edges, and so, G can be identified by the pair $(V(G),E(G))$.

The conversion of graph G is denoted by G^{-1} and

$$V(G^{-1})=V(G)\text{ and }E(G^{-1})=\{(x,y)\in X\times X\colon (y,x)\in E(G)\}.$$

By $\widetilde{G}$, we denote the undirected graph obtained from G by including all the edges of G^{-1}. More precisely, we define

$$V(\widetilde{G})=V(G)\text{ and }E(\widetilde{G})=E(G)\cup E(G^{-1}).$$

It is obvious that $E(\widetilde{G})$ is a symmetric relation on X. If $x,y\in V(G)$, then a path in G from x to y of length $k\in\mathbb{N}$ is a sequence $\{x_n\}_{n=0}^{k}$ of $k+1$ vertices such that $x_0=x, x_k=y$ and $(x_{n-1},x_n)\in E(G)$ for $n=1,2,\dots,k$. A graph G is called connected if, there is a path between any two vertices of G. The graph G is weakly connected if $\widetilde{G}$ is connected.

Throughout this paper, we assume that the graphs under consideration are directed and are with nonempty sets of vertices and edges.

Definition 2.9. Let X be a nonempty set endowed with the graph G and $T\colon X\to X$ be a mapping. Then T is called G-edge preserving if

$$(x,y)\in E(G)\implies (Tx,Ty)\in E(G)\quad\text{for all}\quad x,y\in X.$$

The mapping T is called weak G-edge preserving if

$$(x,y)\in E(G)\implies (Tx,Ty)\in E(\widetilde{G})\quad\text{for all}\quad x,y\in X.$$

Remark 2.10. We consider the following four conditions:

(i) T is G-edge preserving;
(ii) T is G^{-1}-edge preserving;
(iii) T is $\widetilde{G}$-edge preserving;
(iv) T is weak G-edge preserving.

By the above definition it is obvious that the conditions (i) and (ii) are equivalent. Similarly, the conditions (iii) and (iv) are equivalent, while (i) (and so (ii)) implies the condition (iii) (and so (iv)). But the condition (iii) is actually weaker than the first two. Indeed, if $X = \mathbb{R}$, $E(G)$ is the usual order relation on X, i.e., $E(G) = \{(x,y) \in X \times X : x \leq y\}$ and $T : X \to X$ is define by $Tx = 1 - x$ for all $x \in X$, then it is easy to see that T is a weak G-edge preserving mapping but not G-edge preserving.

Definition 2.11. Let (X,d,s) be a complete cone b-metric space over a Banach algebra A and P be the underlying solid cone. Suppose, the space (X,d,s) be endowed with a graph G and $T : X \to X$ be a mapping. Then T is called a G-contraction with contractive vector k if there exists $k \in P$ such that $\rho(k) < \frac{1}{s}$ and

$$d(Tx,Ty) \preceq kd(x,y)$$

for all $x,y \in X$ with $(x,y) \in E(G)$.

The proof of the following lemma follows from the definition of $G^{-1}, \widetilde{G}$ and the symmetry of the cone b-metric.

Lemma 2.12. *Let (X,d,s) be a complete cone b-metric space over a Banach algebra A and P be the underlying solid cone. Suppose, the space (X,d,s) be endowed with a graph G and $T : X \to X$ be a mapping. Then the following three conditions are equivalent:*

(i) T is a G-contraction;
(ii) T is a G^{-1}-contraction;
(iii) T is a $\widetilde{G}$-contraction.

Definition 2.13. Let X be a nonempty set endowed with the graph G and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be a G-sequence if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. The sequence $\{x_n\}$ is said to be weak G-sequence if it is $\widetilde{G}$-sequence, that is, $(x_n, x_{n+1}) \in E(\widetilde{G})$ for all $n \in \mathbb{N}$.

Remark 2.14. Since $E(\widetilde{G}) \supseteq E(G), E(G^{-1})$ every G-sequence (as well as G^{-1}-sequence) is a weak G-sequence. But the converse is not true in general. Indeed, if $X = \mathbb{R}$, $E(G)$ is the usual order relation on X, i.e., $E(G) = \{(x,y) \in X \times X \colon x \leq y\}$ and $\{x_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$. Then $E(G^{-1}) = \{(x,y) \in X \times X \colon y \leq x\}$ and $E(\widetilde{G}) = X \times X$, and so, $\{x_n\}$ is a weak G-sequence but neither a G-sequence nor a G^{-1}-sequence.

3. Main results

First, we prove the following lemma which will be used in the sequel.

Lemma 3.1. *Let X be a nonempty set and G be the corresponding associated graph. Let $T \colon X \to X$ be a mapping and $x_0 \in X$ be such that $(x_0, Tx_0) \in E(\widetilde{G})$. If T is weak G-edge preserving, then the Picard sequence $\{T^n x_0\} = \{x_n\}$ is a weak G-sequence.*

Proof. Let $x_0 \in X$ such that $(x_0, Tx_0) \in E(\widetilde{G})$. We shall show that the Picard sequence $\{T^n x_0\}$ is a weak G-sequence. Since $(x_0, Tx_0) \in E(\widetilde{G})$, we consider the following two cases:

(i) If $(x_0, Tx_0) \in E(G)$, i.e., $(x_0, x_1) \in E(G)$, then as T is weak G-edge preserving we have $(Tx_0, Tx_1) = (x_1, x_2) \in E(\widetilde{G})$.

(ii) If $(Tx_0, x_0) \in E(G)$, i.e., $(x_1, x_0) \in E(G)$, then again we have $(Tx_1, Tx_0) = (x_2, x_1) \in E(\widetilde{G})$.

Since $E(\widetilde{G})$ is a symmetric relation, in both the cases we have $(x_1, x_2) \in E(\widetilde{G})$. Repeating the similar arguments, we obtain that $(x_n, x_{n+1}) \in E(\widetilde{G})$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is a weak G-sequence.

Now, we are in a position to prove our main results.

Theorem 3.2. *Let (X, d, s) be a complete cone b-metric space over a Banach algebra A and P be the underlying solid cone. Suppose, $T \colon X \to X$ be a G-contraction with contractive vector k and the following conditions are satisfied:*

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \in E(\widetilde{G})$;

(ii) T is weak G-edge preserving;

(iii) at least one of the following conditions is satisfied:

(a) T is continuous;

(b) if $\{x_n\}$ be a weak G-sequence in X and $x_n \to x$ then there exists $N \in \mathbb{N}$ such that $(x_n, x) \in E(\widetilde{G})$ for all $n > N$.

Then T has a fixed point $x^* \in X$. Moreover, the Picard sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \in E(\widetilde{G})$. We consider the Picard sequence $\{T^n x_0\} = \{x_n\}$. Then by Lemma 3.1 the sequence $\{x_n\}$ is a weak G-sequence. We shall show that $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\}$ is a weak G-sequence we have $(x_{n-1}, x_n) \in E(\widetilde{G})$ for all $n \in \mathbb{N}$, and so, by Lemma 2.12 we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq kd(x_n, x_{n-1}).$$

Repeating this process we obtain

$$d(x_{n+1}, x_n) \preceq kd(x_n, x_{n-1}) \preceq \cdots \preceq k^n d(x_0, x_1). \tag{1}$$

Since $\rho(k) < \frac{1}{s}$ we have $\rho(ks) < 1$, and so, by Proposition 2.1, $e - sk$ is invertible and

$$(e - sk)^{-1} = \sum_{i=0}^{\infty} (sk)^i.$$

Suppose, $n, m \in \mathbb{N}$ and $m > n$, then we have

$$\begin{aligned} d(x_n, x_m) &\preceq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\preceq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\preceq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n-1} d(x_{m-1}, x_m). \end{aligned}$$

Using (1) in the above inequality, we obtain

$$\begin{aligned} d(x_n, x_m) &\preceq sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + \cdots + s^{m-n-1} k^{m-1} d(x_0, x_1) \\ &= sk^n [e + sk + \cdots + s^{m-n-2} k^{m-n-1}] d(x_0, x_1) \\ &\preceq sk^n (e - sk)^{-1} d(x_0, x_1). \end{aligned}$$

Since $\rho(k) < \frac{1}{s} \leq 1$, we find from Remark 2.5 that $\|k^n\| \to 0$ as $n \to \infty$. Therefore, by Lemma 2.4 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \preceq sk^n(e - sk)^{-1}d(x_0, x_1) \ll c$$

for all $n > n_0$. It implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d, s) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We shall show that x^* is a fixed point of T. Suppose, the condition (a) holds, i.e., T is continuous. Since $x_n \to x^*$ as $n \to \infty$, by continuity of T we have $x_{n+1} = Tx_n \to Tx^*$. By uniqueness of limit of sequence in X we have $Tx^* = x^*$. Thus, x^* is a fixed point of T.

Now, suppose the condition (b) holds, then there exists $N \in \mathbb{N}$ such that $(x_n, x) \in E(\widetilde{G})$ for all $n > N$. Therefore, by Lemma 2.12 we have

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_n) + d(x_n, Tx^*)] \\ &= s[d(x^*, x_n) + d(Tx_{n-1}, Tx^*)] \\ &\preceq s[d(x^*, x_n) + kd(x_{n-1}, x^*)] \end{aligned}$$

for all $n > N$. Since $x_n \to x^*$ as $n \to \infty$, for every $c \in P$ with $\theta \ll c$ and for every $m \in \mathbb{N}$ there exists $n(m)$ such that $d(x_{n-1}, x^*) \ll \frac{c}{ms}$ for all $n > n(m)$. Therefore, it follows from the above inequality that

$$d(x^*, Tx^*) \preceq \frac{c}{m} + \frac{kc}{m} = \frac{c}{m}(e + k) \text{ for all } n > \max\{N, n(m)\}, m \in \mathbb{N}.$$

It implies that $\frac{c}{m}(e + k) - d(x^*, Tx^*) \in P$ for all $m \in \mathbb{N}$. Since P is closed, letting $m \to \infty$ we obtain $\theta - d(x^*, Tx^*) \in P$. By definition, we must have $d(x^*, Tx^*) = \theta$, i.e., $Tx^* = x^*$. Thus, x^* is a fixed point of T.

Next, we give some examples which illustrate the above result. The following example shows that the importance of G-edge preservity of mapping T in Theorem 3.2.

Example 3.3. *Let $A = C^1_{\mathbb{R}}[0, 1]$ be the Banach algebra with the norm defined by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for all $x \in A$, multiplication as pointwise multiplication and the unity $e = 1$. Let the non-normal cone be taken as $P = \{x \in A \colon x = x(t) \geq 0 \; \forall \, t \in [0, 1]\}$. Suppose $X = \{a, b, c\}$ and the* *mapping $d\colon X\times X\to A$ be defined by*

$$d(x,x)(t)=0, d(x,y)(t)=d(y,x)(t) \textit{ for all } x,y\in X;$$

$$d(a,b)(t)=e^t, d(b,c)(t)=2e^t, d(a,c)=4e^t.$$

Then, (X,d,s) is a complete cone b-metric space with $s=\dfrac{4}{3}$. Define the mapping $T\colon X\to X$ and the graph G by

$$T=\begin{pmatrix} a & b & c\\ c & c & b\end{pmatrix} \textit{ and } V(G)=X, E(G)=\{(a,a),(b,b),(c,c),(a,c)\}.$$

Then, it is easy to see that T is a G-contraction with contractive vector $k=k(t)\in P$, $k(t)=k_1t+k_2$ for all $t\in[0,1]$ such that $0\le k_1<\dfrac{3}{4}-k_2$ and $\dfrac{1}{2}\le k_2<1$. Now, since $(a,c)\in E(G)$, we have $(a,Ta)=(a,c)\in E(\widetilde{G})$. Therefore, the condition (i) of Theorem 3.2 is satisfied. Further, it is easy to see that the condition (b) of (iii) of Theorem 3.2 is also satisfied. Notice that, $(a,c)\in E(G)$, but $(Ta,Tc)\notin E(\widetilde{G})$. Thus, all the conditions of Theorem 3.2 (except (ii), i.e., T is G-weak edge preserving) are satisfied. Notice that, T is a fixed point free mapping.

The following example shows the importance of condition (i) of Theorem 3.2.

Example 3.4. *Let (X,d,s) be the cone b-metric space as defined in the Example 3.3. Define the mapping $T\colon X\to X$ and the graph G by*

$$T=\begin{pmatrix} a & b & c\\ b & c & b\end{pmatrix} \textit{ and } V(G)=X, E(G)=\{(a,a),(b,b),(c,c),(a,c)\}.$$

Then, it is easy to see that T is a G-contraction with contractive vector $k=k(t)\in P$ such that $\rho(k)\in\left[0,\dfrac{3}{4}\right)$. Now, since $E(G)\supseteq\mathscr{D}$, therefore by the definition of T we obtain that T is G-weak edge preserving. Therefore, the condition (ii) of Theorem 3.2 is satisfied. Further, it is easy to see that the condition (b) of (iii) of Theorem 3.2 is also satisfied. Notice that, there exists no $x_0\in X$ such that $(x_0,Tx_0)\in E(\widetilde{G})$. Thus, all the conditions of Theorem 3.2 (except (i)) are satisfied. Notice that, T is a fixed point free mapping.

Example 3.5. *Let $A=\mathbb{R}^2$ with norm $\|(u_1,u_2)\|=|u_1|+|u_2|$ and the multiplication is defined as $uv=(u_1,u_2)(v_1,v_2)=(u_1v_1,u_1v_2+u_2v_1)$. Let $P=\{u=(u_1,u_2)\in A\colon u_1,u_2\ge 0\}$. Then*

P is a cone in A and A is a Banach algebra with unit $e=(1,0)$. Let $X=\mathbb{R}\times\mathbb{R}$ and for all $x=(x_1,x_2), y=(y_1,y_2)\in X$,

$$d(x,y)=(|x_1-y_1|^2,|x_2-y_2|^2).$$

Then (X,d,s) is a complete cone b-metric space with $s=2$. Define a mapping $T: X\to X$ by

$$T(x_1,x_2)=\begin{cases}(ax_1,bx_2), & \text{if } x_1,x_2\in\mathbb{Q}\setminus\{0\};\\ (1,1), & \text{otherwise}\end{cases}$$

where a,b are two positive rationals such that $\max\{a^2,b^2\}<\frac{1}{2}$. Let the graph G be defined by

$$V(G)=X, E(G)=\{((x_1,x_2),(x_3,x_4)): x_1,x_2,x_3,x_4\in\mathbb{Q}\setminus\{0\}\}.$$

Then, it is easy to see that T is a G-contraction with contractive vector $k=(k_1,k_2)\in P$ such that $k_1=\max\{a^2,b^2\}$ and k_2 is arbitrary. Note that, for any $(x_0,y_0)\in\mathbb{Q}\setminus\{0\}$ we have

$$((x_0,y_0),T(x_0,y_0))=((x_0,y_0),(ax_0,by_0))\in E(\widetilde{G}).$$

Therefore, the condition (i) of Theorem 3.2 is satisfied. Since a,b positive rationals, the condition (ii) of Theorem 3.2 is also satisfied. Notice that:

(a) the mapping T is not continuous;

(b) for any $x_1,x_2\in\mathbb{Q}\setminus\{0\}$ the sequence (x_n^1,x_n^2), where $x_n^1=\frac{x_1}{2^n}, x_n^2=\frac{x_2}{2^n}$, is a G-weak sequence in X and $(x_n^1,x_n^2)\to(0,0)$ as $n\to\infty$, but $((x_n^1,x_n^2),(0,0))\notin E(\widetilde{G})$ for all $n\in\mathbb{N}$.

Therefore, the condition (iii) of Theorem 3.2 is not satisfied. Thus, all the conditions of Theorem 3.2, except (iii), are satisfied and again, the mapping T is a fixed point free mapping.

The next example shows that Theorem 3.2 ensures only the existence (but, not the uniqueness) of the fixed point of T.

Example 3.6. *Let (X,d,s) be the cone b-metric space as defined in the Example 3.3. Define the mapping $T: X\to X$ and the graph G by*

$$T=\begin{pmatrix} a & b & c\\ a & b & b\end{pmatrix} \text{ and } V(G)=X, E(G)=\{(a,a),(b,b),(c,c),(c,b)\}.$$

Then, it is easy to see that T is a G-contraction with contractive vector $k = k(t) \in P$ such that $\rho(k) \in \left[0, \frac{3}{4}\right)$. Now, since $(a,Ta),(b,Tb) \in E(G)$, the condition (i) of Theorem 3.2 is satisfied. Again, since $E(G) \supseteq \mathscr{D}$, therefore by the definition of T we obtain that T is G-weak edge preserving. Therefore, the condition (ii) of Theorem 3.2 is satisfied. Further, it is easy to see that the condition (b) of (iii) of Theorem 3.2 is also satisfied. Thus, all the conditions of Theorem 3.2 are satisfied, and so, by Theorem 3.2 T has at least one fixed point in X. Indeed, $a,b \in X$ are two fixed points of T.

In view of the above example, we give the following sufficient condition for uniqueness of fixed point of T.

Theorem 3.7. *Let all the conditions of Theorem 3.1 are satisfied. In addition, suppose the graph G is weakly connected, then T has a unique fixed point.*

Proof. First, note that the existence of fixed point of T follows from Theorem 3.2. Suppose G is weakly connected, then we shall show the uniqueness of fixed point. On contrary, suppose there exist two distinct fixed points $x^*, y^* \in X$, i.e., $Tx^* = x^* \neq Ty^* = y^*$. Since G is weakly connected there exists a path $\{x_0\}_{i=0}^{m}$ such that $x_0 = x^*, x_k = y^*$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$ for $i = 1, \ldots, m$. Since $(x_{i-1}, x_i) \in E(\widetilde{G})$ for $i = 1, \ldots, m$, by weak G-edge preservity of mapping T we have $(Tx_{i-1}, Tx_i) \in E(\widetilde{G})$ for $i = 1, \ldots, m$. Similarly, we can show that

$$(T^n x_{i-1}, T^n x_i) \in E(\widetilde{G}) \text{ for } i = 1, \ldots, m \text{ for all } n \in \mathbb{N}.$$

Using Lemma 2.12 and the above inclusion, we obtain

$$d(T^n x_{i-1}, T^n x_i) = d(TT^{n-1} x_{i-1}, TT^{n-1} x_i) \preceq kd(T^{n-1} x_{i-1}, T^{n-1} x_i)$$

for $i = 1, \ldots, m$. Repeating this process we get:

$$d(T^n x_{i-1}, T^n x_i) \preceq k^n d(x_{i-1}, x_i) \text{ for } i = 1, \ldots, m \text{ for all } n \in \mathbb{N}. \tag{2}$$

Therefore, for any $n \in \mathbb{N}$ we obtain

$$
\begin{aligned}
d(x^*,y^*) &= d(T^n x^*, T^n y^*) = d(T^n x_0, T^n x_m) \preceq s[d(T^n x_0, T^n x_1) + d(T^n x_1, T^n x_m)] \\
&\preceq sd(T^n x_0, T^n x_1) + s^2 d(T^n x_1, T^n x_2) + s^2 d(T^n x_2, T^n x_m) \\
&\vdots \\
&\preceq sd(T^n x_0, T^n x_1) + s^2 d(T^n x_1, T^n x_2) + \cdots + s^{m-1} d(T^n x_{m-2}, T^n x_{m-1}) \\
&\quad + s^{m-1} d(T^n x_{m-1}, T^n x_m) \\
&= \sum_{i=1}^{m-1} s^i d(T^n x_{i-1}, T^n x_i) + s^{m-1} d(T^n x_{m-1}, T^n x_m).
\end{aligned}
$$

Using (2) in the above inequality we obtain

$$
\begin{aligned}
d(x^*,y^*) &\preceq \sum_{i=1}^{m-1} s^i k^n d(x_{i-1}, x_i) + s^{m-1} k^n d(x_{m-1}, x_m) \\
&= k^n \left[\sum_{i=1}^{m} s^i d(x_{i-1}, x_i) + s^{m-1} d(x_{m-1}, x_m) \right].
\end{aligned}
$$

Since $\rho(k) < \frac{1}{s} \leq 1$, we find from Remark 2.5 $\|k^n\| \to 0$ as $n \to \infty$. Therefore, by Lemma 2.4 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$
d(x^*,y^*) \preceq k^n \left[\sum_{i=1}^{m} s^i d(x_{i-1}, x_i) + s^{m-1} d(x_{m-1}, x_m) \right] \ll \frac{c}{l}
$$

for all $n > n_0$ and for all $l \in \mathbb{N}$. Since P is closed, letting $l \to \infty$ we obtain $\theta - d(x^*,y^*) \in P$. By definition, we must have $d(x^*,y^*) = \theta$, i.e., $x^* = y^*$. This contradiction proves the uniqueness of fixed point.

Next, we give some consequences of our main results.

Theorem 3.8. *Let (X,d,s) be a complete cone b-metric space over a Banach algebra A and P be the underlying solid cone. Suppose the mapping $T : X \to X$ satisfies generalized Lipschitz condition:*

$$
d(Tx,Ty) \preceq kd(x,y) \quad \textit{for all} \quad x,y \in X
$$

where $k \in P$ with $\rho(k) < \dfrac{1}{s}$. Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point of X.

Proof. Define the graph G by $V(G) = X$ and $E(G) = X \times X$. Then, all the conditions of Theorem 3.7 are satisfied, and so, the mapping T has a unique fixed point in X.

Next, we derive the ordered and cyclic versions of Banach contraction principle in cone b-metric spaces. In the next theorems, we generalize and unify the results of Ran and Reurings [20], Liu and Xu [7] and Nieto, Rodríguez-López [21] and Kirk *et al.* [22].

Theorem 3.9. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X,d,s) be a complete cone b-metric space over a Banach algebra A with P the underlying solid cone. Let $T: X \to X$ be a continuous nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assumptions hold:*

(i) there exists $k \in P$ such that $\rho(k) < \frac{1}{s}$ and $d(Tx,Ty) \preceq kd(x,y)$ for all $x,y \in X$ with $x \sqsubseteq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$.

Then, T has a fixed point in X.

Proof. Define the graph G_r by $V(G_r) = X$ and $E(G_r) = \{(x,y) \in X \times X : x \sqsubseteq y\}$. Note that, the condition (i) implies that the mapping T a G-contraction with Lipschitz vector k. Since T is nondecreasing it is a weak G-edge preserving mapping. The condition (ii) implies that, there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(\widetilde{G}_r)$. Therefore, all the conditions of Theorem 3.2 are satisfied, and so, the mapping T has a fixed point in X.

The following theorem is the cone b-metric version of the result of Nieto, Rodríguez-López [21] when the space is endowed with a Banach algebra.

Theorem 3.10. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X,d,s) be a complete cone b-metric space over a Banach algebra A with P the underlying solid cone. Let $T: X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assumptions hold:*

(i) there exists $k \in P$ such that $\rho(k) < \frac{1}{s}$ and $d(Tx,Ty) \preceq kd(x,y)$ for all $x,y \in X$ with $x \sqsubseteq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;

(iii) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.

Then, T has a fixed point in X.

Proof. Define the graph G_r similar to that as in the proof of Theorem 3.9. Now, the proof follows from the Theorem 3.7.

Next, we define the cyclic contractions (see [22]) in cone b-metric spaces.

Let X be a nonempty set, $T: X \to X$ a mapping and $A_1, A_2, \dots, A_m$ be subsets of X. Then $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T if

(1) A_i, $i = 1, 2, \dots, m$ are nonempty sets;

(2) $T(A_1) \subset A_2, \dots, T(A_{m-1}) \subset T(A_m), T(A_m) \subset T(A_1)$.

Remark 3.11. [22] If $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T, then $\mathrm{Fix}(T) \subset \bigcap_{i=1}^{m} A_i$.

Definition 3.12. Let (X, d, s) be a complete cone b-metric space over a Banach algebra A and P be the underlying solid cone. Suppose, $A_1, A_2, \dots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. A mapping $T: \ Y \to Y$ is called a generalized cyclic Lipschitz contraction with Lipschitz vector k if following conditions hold:

(1) $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;

(2) there exists $k \in P$ such that $\rho(k) < \frac{1}{s}$ and

$$d(Tx, Ty) \preceq k d(x, y) \tag{3}$$

for any $x \in A_i, y \in A_{i+1}$ $(i = 1, 2, \dots, m$ where $A_{m+1} = A_1)$.

The following theorem is the cone b-metric version of Kirk *et al.* [22].

Theorem 3.13. *Let (X, d, s) be a complete cone b-metric space over a Banach algebra A with P the underlying solid cone. Suppose, $A_1, A_2, \dots, A_m$ be subsets of X, $Y = \bigcup_{i=1}^{m} A_i$ is complete as a subspace of X and $T: Y \to Y$ be a a generalized cyclic Lipschitz contraction with Lipschitz vector k. Then, T has a unique fixed point in Y.*

Proof. Define the graph G_c by $V(G_c) = X$ and

$$E(G_c) = \{(x, y) \in X \times X : \ (x, y) \in A_i \times A_{i+1}, i = 1, 2, \dots, m, \text{ where } A_{m+1} = A_1\}.$$

First, by definition of the graph G_c and the cyclic representation we have $(x_0, Tx_0) \in E(G_c)$ for all $x_0 \in Y$. Also, T is weak G_c-edge preserving. Again, by definition of the graph G_c, T is a generalized cyclic Lipschitz contraction with Lipschitz vector k. Suppose, for a sequence $\{x_n\}$ we have $(x_n, x_{n+1}) \in E(G_c)$ for all n and $x_n \to x \in X$ as $n \to \infty$. Then, as $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation with respect to T, we must have $x \in \bigcap_{i=1}^{m} A_i$. Therefore, $(x_n, x) \in E(G_c)$ for all $n \in \mathbb{N}$. Now, by Theorem 3.2 the Picard sequence $\{T^n x_0\}$ converges to the fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ of T. For uniqueness, we shall show that the graph G_c is weakly connected. Suppose, $x, y \in Y = \bigcup_{i=1}^{m} A_i$, then $x \in A_i, y \in A_j$ for some $1 \leq i, j \leq m$. Since $x^* \in \bigcap_{i=1}^{m} A_i$, we have $(x, x^*), (x^*, y) \in E(G_c)$, and so, the graph G_c is weakly connected. Therfore, the uniqueness of fixed points follows from Theorem 3.7.

Acknowledgement

The authors are grateful to the reviewers for useful suggestions which improve the contents of this paper.

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