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ON $p(x)$-KIRCHHOFF EQUATIONS WITH CONCAVE-CONVEX TERMS IN UNBOUNDED DOMAINS

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Abstract. In this paper, we study the existence of solutions for a class of $p(x)$-Kirchhoff problems in unbounded domains with concave-convex terms. The technical approach in this paper is mainly based on the mountain pass theorem and Ekeland's variational principle.

Keywords. $p(x)$-Laplacian; Kirchhoff equation; Concave-convex term; Ekeland's variational principle.

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1. INTRODUCTION

In this article, we are concerned with the following problem

$$\begin{cases} L(u) = \lambda f(x)|u|^{q(x)-2}u + g(x)|u|^{r(x)-2}u + h(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

where $L(u) := -M\left(\int_\Omega \frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\right)\left(\Delta_{p(x)}u - |u|^{p(x)-2}u\right)$, $\lambda > 0$, $\Omega = \mathbb{R}^N \backslash \overline{\Omega_b}$ with $\Omega_b \subset \mathbb{R}^N$ is a smooth bounded domain, M is a positive continuous function on $[0,+\infty)$, g is a nonnegative continuous function, f and h are continuous functions which may change sign on Ω and $p, q, r \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \{\sigma \in C(\overline{\Omega}) \cap L^\infty(\Omega) : \inf_{x\in\overline{\Omega}} \sigma(x) > 1\}.$$

Moreover p is Lipschitz continuous, $1 < p^- := \inf_{x\in\Omega} p(x) \le p^+ := \sup_{x\in\Omega} p(x) < N$ and $r(x) < p^*(x)$, $\forall x \in \overline{\Omega}$.

The study of various problems with nonstandard growth conditions has been received considerable attention because of their large applications in elastic mechanics electrorheological fluids, image restoration, dielectric breakdown, electrical resistivity and polycrystal plasticity and continuum mechanics; see

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[1, 2, 3, 4, 5, 6, 7] and the references therein. Elliptic problems of Kirchhoff-type involving operators like the $p(x)$-Laplacian or its generalizations have been the object of ever more attention in these latest years; see, for instance, [8, 9, 10, 11, 12, 13, 14, 15] and the references therein. However, to our knowledge, there are not many papers [16, 17, 18] which dealt with nonlocal p-Kirchhoff equations with concave-convex terms.

Motivated by the above mentioned works, we will use the Mountain Pass theorem and Ekeland's variational principle to prove the existence of two solutions for problem (P_λ) in unbounded domains with nonhomogeneous term $h(x)$.

The assumptions under problem (P_λ) will be considered are the following:

(M_1) There exist $m_1 \geq m_0 > 0$ and $\frac{q^+}{p^-} < \alpha \leq \beta < \frac{r^-}{p^+}$ such that

$$m_0 t^\alpha \leq \widehat{M}(t) := \int_0^t M(s)ds \leq m_1 t^\beta \text{ for all } t > 0;$$

(M_2) There exists $0 < \theta < \frac{r^-}{p^+}$ such that

$$\theta \widehat{M}(t) \geq M(t)t \text{ for all } t > T_0 > 0;$$

(H_0) $f \in L^\infty(\Omega) \cap L^{q_0(x)}(\Omega)$, with $q_0(x) = \frac{p^*(x)}{p^*(x)-q(x)}\ \forall x \in \overline{\Omega}$;

(H_1) $0 \leq g \in L^\infty(\Omega) \cap L^{r_0(x)}(\Omega)$, with $r_0(x) = \frac{p^*(x)}{p^*(x)-r(x)}\ \forall x \in \overline{\Omega}$;

(H_2) $h \in L^\infty(\Omega) \cap L^{s_0(x)}(\Omega)$, with $s_0(x) = \frac{p^*(x)}{p^*(x)-1}\ \forall x \in \overline{\Omega}$.

2. Preliminaries and main result

In this section, we recall some interesting properties of the variable exponent Lebesgue and Sobolev spaces that will be used to study problem (P_λ). Let $\Omega \subset \mathbb{R}^N$ an open domain. Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\text{measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

This space endowed with the Luxemburg norm,

$$||u||_{L^{p(x)}(\Omega)} = \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}$$

is a separable and reflexive Banach space. Denoting by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder type inequality

$$\int_\Omega |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) ||u||_{L^{p(x)}(\Omega)} ||v||_{L^{p'(x)}(\Omega)}.$$

Now, we introduce the modular of the Lebesgue-Sobolev space $L^{p(x)}(\Omega)$ as the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx,\ \forall u \in L^{p(x)}(\Omega).$$

The relation between modular and Luxemburg norm is clarified by the following proposition:

Proposition 2.1 ([19])**.** *If $u, u_n \in L^{p(x)}(\Omega)$, then following properties hold:*
(1) $||u||_{L^{p(x)}(\Omega)} \leq 1 \Rightarrow ||u||^{p^+}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq ||u||^{p^-}_{L^{p(x)}(\Omega)}$;
(2) $||u||_{L^{p(x)}(\Omega)} \geq 1 \Rightarrow ||u||^{p^-}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq ||u||^{p^+}_{L^{p(x)}(\Omega)}$;
(3) $\lim\limits_{n\to\infty} ||u_n - u||_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

endowed with the norm

$$||u|| = ||u||_{L^{p(x)}(\Omega)} + ||\nabla u||_{L^{p(x)}(\Omega)}.$$

$W^{1,p(x)}(\Omega)$ is a Banach space which is reflexive under condition $1 < p^- \leq p^+ < +\infty$. If $q \in C_+(\overline{\Omega})$ and $p(x) \leq q(x) \leq p^*(x) \forall x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous. This last embedding is compact when Ω is bounded and that $q(x) < p^*(x) \forall x \in \overline{\Omega}$. Setting

$$\rho_{1,p(x)}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx, \ \forall u \in W^{1,p(x)}(\Omega),$$

similarly to Proposition 2.1, it holds:

Proposition 2.2 ([19])**.** *If $u, u_n \in W^{1,p(x)}(\Omega)$, then following properties hold true:*
(1) $||u|| \leq 1 \Rightarrow ||u||^{p^+} \leq \rho_{1,p(x)}(u) \leq ||u||^{p^-}$;
(2) $||u|| \geq 1 \Rightarrow ||u||^{p^-} \leq \rho_{1,p(x)}(u) \leq ||u||^{p^+}$;
(3) $\lim\limits_{n\to\infty} ||u_n - u|| = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho_{1,p(x)}(u_n - u) = 0$.

Definition 2.3. We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (P_λ) if

$$M\left(\int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx \right) \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv \right) dx$$
$$- \lambda \int_\Omega f(x)|u|^{q(x)-2} uv dx - \int_\Omega g(x)|u|^{r(x)-2} uv dx - \int_\Omega h(x) v dx = 0,$$

for all $v \in W^{1,p(x)}(\Omega)$.

Our main result is the following theorem.

Theorem 2.4. *Assume that $(M_1) - (M_2)$ and $(H_0) - (H_2)$ hold. Then, there exist $\lambda_0, \mu > 0$ such that for all $\lambda \in (0, \lambda_0)$, problem ($P_\lambda$) has at least two nontrivial weak solutions provided that $||h||_{L^{s_0(x)}(\Omega)} \leq \mu$.*

3. Proof of main result

Since we will rely on the critical point theory, we define the energy functional corresponding to problem (P_λ) as $I_\lambda : W^{1,p(x)}(\Omega) \mapsto \mathbb{R}$,

$$I_\lambda(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx \right) - \lambda \int_\Omega f(x) \frac{|u|^{q(x)}}{q(x)} dx - \int_\Omega g(x) \frac{|u|^{r(x)}}{r(x)} dx - \int_\Omega h(x) u dx.$$

Tacking account that M is continue on $[0, +\infty)$, by assumptions $(H_0) - (H_2)$ and standard arguments, we see that $I_\lambda \in C^1\left(W^{1,p(x)}(\Omega), \mathbb{R}\right)$ (see for example, [20, 21, 22]. Moreover, the critical points of I_λ are weak solutions of problem (P_λ).

Lemma 3.1. *Assume that $(M_1)-(M_2)$ and $(H_0)-(H_2)$ hold. Then there exist $\lambda_0,\mu,\rho,\kappa>0$ such that for $\lambda\in(0,\lambda_0)$ and $||h||_{L^{s_0(x)}(\Omega)}\le\mu$,*

$$I_\lambda(u)\ge\kappa \text{ for all } ||u||=\rho.$$

Moreover, there exists $e\in W^{1,p(x)}(\Omega)$ with $||e||>\rho$, such that $I_\lambda(e)<0$.

Proof. From (H_0), Hölder's inequality and Proposition 2.1, we have

$$\begin{aligned}\int_\Omega |f(x)||u|^{q(x)}dx &\le c_0||f||_{L^{q_0(x)}(\Omega)}||\,|u|^{q(x)}||_{L^{\frac{p^*(x)}{q(x)}}(\Omega)}\\ &\le c_0||f||_{L^{q_0(x)}(\Omega)}\max\left(||u||^{q^+}_{L^{p^*(x)}(\Omega)},||u||^{q^-}_{L^{p^*(x)}(\Omega)}\right)\\ &\le c_1||f||_{L^{q_0(x)}(\Omega)}\max\left(||u||^{q^+},||u||^{q^-}\right). \quad (3.1)\end{aligned}$$

Similarly, one has

$$\int_\Omega g(x)|u|^{r(x)}dx\le c_2||g||_{L^{r_0(x)}(\Omega)}\max\left(||u||^{r^+},||u||^{r^-}\right) \quad (3.2)$$

and

$$\int_\Omega |h(x)||u|dx\le c_3||h||_{L^{s_0(x)}(\Omega)}||u||. \quad (3.3)$$

On the other hand, using Young's inequality, for given $\varepsilon>0$, there exists C_ε such that

$$||h||_{L^{s_0(x)}(\Omega)}||u||\le\varepsilon||u||^{\alpha p^+}+C_\varepsilon||h||^{\frac{\alpha p^+}{\alpha p^+-1}}_{L^{s_0(x)}(\Omega)}. \quad (3.4)$$

By (M_1), (3.1)-(3.4) and Proposition 2.2, for $||u||<1$, we have

$$\begin{aligned}I_\lambda(u)\ge& m_0\left(\int_\Omega\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\right)^\alpha-\frac{\lambda}{q^-}\int_\Omega|f(x)||u|^{q(x)}dx\\ &-\frac{1}{r^-}\int_\Omega g(x)|u|^{r(x)}dx-\int_\Omega|h(x)||u|dx\\ \ge&\frac{m_0}{(p^+)^\alpha}||u||^{\alpha p^+}-\frac{\lambda c_1}{q^-}||f||_{L^{q_0(x)}(\Omega)}||u||^{q^-}-\frac{c_2}{r^-}||g||_{L^{r_0(x)}(\Omega)}||u||^{r^-}\\ &-c_3\varepsilon||u||^{\alpha p^+}-c_3C_\varepsilon||h||^{\frac{\alpha p^+}{\alpha p^+-1}}_{L^{s_0(x)}(\Omega)}.\end{aligned}$$

Choosing $\varepsilon=\frac{m_0}{2c_3(p^+)^\alpha}$, we obtain

$$\begin{aligned}I_\lambda(u)\ge&\frac{m_0}{2(p^+)^\alpha}||u||^{\alpha p^+}-\frac{\lambda c_1}{q^-}||f||_{L^{q_0(x)}(\Omega)}||u||^{q^-}-\frac{c_2}{r^-}||g||_{L^{r_0(x)}(\Omega)}||u||^{r^-}-c_3C_\varepsilon||h||^{\frac{\alpha p^+}{\alpha p^+-1}}_{L^{s_0(x)}(\Omega)}\\ =&||u||^{\alpha p^+}\left(\frac{m_0}{2(p^+)^\alpha}-\frac{\lambda c_1}{q^-}||f||_{L^{q_0(x)}(\Omega)}||u||^{q^--\alpha p^+}-\frac{c_2}{r^-}||g||_{L^{r_0(x)}(\Omega)}||u||^{r^--\alpha p^+}\right)\\ &-c_3C_\varepsilon||h||^{\frac{\alpha p^+}{\alpha p^+-1}}_{L^{s_0(x)}(\Omega)}. \quad (3.5)\end{aligned}$$

Let

$$\gamma(\tau)=a_0\lambda\tau^{q^--\alpha p^+}+a_1\tau^{r^--\alpha p^+},\ 0<\tau<1,$$

where $a_0=\frac{c_1}{q^-}||f||_{L^{q_0(x)}(\Omega)}$, $a_1=\frac{c_2}{r^-}||g||_{L^{r_0(x)}(\Omega)}$. Since $q^-<\alpha p^+$, we see that $\gamma(\tau)\to+\infty$ as $\tau\to0^+$. Then γ has a minimum at

$$\tau_0:=\left(\frac{a_0\lambda(\alpha p^+-q^-)}{a_1(r^--\alpha p^+)}\right)^{1/(r^--q^-)}.$$

Moreover

$$\gamma(\tau_0) = \frac{a_0(r^- - q^-)}{r^- - \alpha p^+}\left(\frac{a_0(\alpha p^+ - q^-)}{a_1(r^- - \alpha p^+)}\right)^{(q^- - \alpha p^+)/(r^- - q^-)} \lambda^{(r^- - \alpha p^+)/(r^- - q^-)} \underset{\lambda\to 0^+}{\longrightarrow} 0.$$

Thus, we can find $\lambda_0 > 0$ such that $\gamma(\tau_0) < \frac{m_0}{2(p^+)^\alpha}$ $\forall \lambda \in (0,\lambda_0)$. From (3.5), for $||u|| = \tau_0$, we have

$$I_\lambda(u) \geq \tau_0^{\alpha p^+}\left(\frac{m_0}{2(p^+)^\alpha} - \gamma(\tau_0)\right) - c_3 C_\varepsilon ||h||_{L^{s_0(x)}(\Omega)}^{\frac{\alpha p^+}{\alpha p^+ - 1}}. \tag{3.6}$$

Set

$$\kappa := \tau_0^{\alpha p^+}\left(\frac{m_0}{2(p^+)^\alpha} - \gamma(\tau_0)\right) - c_3 C_\varepsilon ||h||_{L^{s_0(x)}(\Omega)}^{\frac{\alpha p^+}{\alpha p^+ - 1}},$$

and

$$\mu := \left[\frac{\tau_0^{\alpha p^+}}{c_3 C_\varepsilon}\left(\frac{m_0}{2(p^+)^\alpha} - \gamma(\tau_0)\right)\right]^{\frac{\alpha p^+ - 1}{\alpha p^+}}.$$

It follows from (3.6) that for each $\lambda \in (0,\lambda_0)$ and $||h||_{L^{s_0(x)}(\Omega)} < \mu$,

$$I_\lambda(u) \geq \kappa > 0 \text{ for } ||u|| = \rho = \tau_0.$$

Now, let $\varphi_0 \in C_0^\infty(\Omega_0)$, where $\Omega_0 \subset \{x \in \Omega : g(x) > 0\}$. In view of (M_1), for t large enough

$$\begin{aligned} I_\lambda(t\varphi_0) =& \widehat{M}\left(\int_\Omega \frac{|\nabla t\varphi_0|^{p(x)} + |t\varphi_0|^{p(x)}}{p(x)}dx\right) - \lambda\int_\Omega f(x)\frac{|t\varphi_0|^{q(x)}}{q(x)}dx \\ &- \int_\Omega g(x)\frac{|t\varphi_0|^{r(x)}}{r(x)}dx - t\int_\Omega h(x)\varphi_0 dx \\ \leq& \frac{m_1 t^{\beta p^+}}{(p^-)^\beta}\left(\int_\Omega\left(|\nabla\varphi_0|^{p(x)} + |\varphi_0|^{p(x)}\right)dx\right)^\beta + \frac{\lambda t^{q^+}}{q^-}\int_\Omega |f(x)||\varphi_0|^{q(x)}dx \\ &- \frac{t^{r^-}}{r^+}\int_{\Omega_0} g(x)|\varphi_0|^{r(x)}dx - t\int_\Omega h(x)\varphi_0 dx. \end{aligned}$$

Therefore $I_\lambda(t\varphi_0) \to -\infty$ as $t \to +\infty$ since $1 < q^+ < \beta p^+ < r^-$. So, for some t_0 large enough, $I_\lambda(t_0\varphi_0) < 0$. This ends the proof of Lemma 3.1. □

Lemma 3.2. *Assume $(M_1)-(M_2)$ and $(H_0)-(H_2)$. Then, I_λ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ such that

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0 \text{ in } \left(W^{1,p(x)}(\Omega)\right)^*.$$

We claim that $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Suppose by contradiction up to subsequence $||u_n|| \to +\infty$. Then, for n large enough, it follows from $(M_1)-(M_2)$ and (H_1) that

$$\begin{aligned}
&c+1+||u_n|| \\
\geq & I_\lambda(u_n) - \frac{1}{r^-}\langle I'_\lambda(u_n),u_n\rangle \\
\geq & \widehat{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx\right) \\
&-\frac{1}{r^-}M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega\left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)dx \\
&+\lambda\int_\Omega\left(\frac{1}{r^-}-\frac{1}{q(x)}\right)f(x)|u_n|^{q(x)}dx+\int_\Omega\left(\frac{1}{r^-}-\frac{1}{r(x)}\right)g(x)|u_n|^{r(x)}dx+\left(\frac{1}{r^-}-1\right)\int_\Omega h(x)u_n dx \\
\geq & \left(1-\frac{\theta p^+}{r^-}\right)\widehat{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx\right) \\
&-\lambda\left(\frac{1}{r^-}+\frac{1}{q^-}\right)\int_\Omega |f(x)||u_n|^{q(x)}dx+\left(\frac{1}{r^-}-1\right)\int_\Omega h(x)u_n dx \\
\geq & \frac{m_0}{(p^+)^\alpha}\left(1-\frac{\theta p^+}{r^-}\right)||u_n||^{\alpha p^-}-\lambda\left(\frac{1}{r^-}+\frac{1}{q^-}\right)c_1||f||_{L^{q_0(x)}(\Omega)}||u_n||^{q^+} \\
&-c_3\left(1-\frac{1}{r^-}\right)||h||_{L^{s_0(x)}(\Omega)}||u_n||.
\end{aligned}$$

Since $1<q^+<\alpha p^-$ and $\theta p^+<r^-$, this last inequality is an absurd and hence $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Then, up to subsequence $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$ and

$$\int_\Omega \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx \to l_0, \text{ as } n\to+\infty. \tag{3.7}$$

If $l_0=0$, then $||u_n||\to 0$. The proof is complete. Suppose that $l_0>0$ and prove that $||u_n-u||\to 0$. Indeed, we first note that

$$\begin{aligned}
&M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx\right) \\
&\times\left[\int_\Omega\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\nabla(u_n-u)dx+\int_\Omega\left(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right)(u_n-u)dx\right] \\
&=\langle I'_\lambda(u_n),u_n-u\rangle - M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega\left(|\nabla u|^{p(x)-2}\nabla u\nabla(u_n-u)+|u|^{p(x)-2}u(u_n-u)\right)dx \\
&+\lambda\int_\Omega f(x)|u_n|^{q(x)-2}u_n(u_n-u)dx+\int_\Omega g(x)|u_n|^{r(x)-2}u_n(u_n-u)dx+\int_\Omega h(x)(u_n-u)dx.
\end{aligned} \tag{3.8}$$

Since $u_n \rightharpoonup u$, one has

$$\begin{cases} \langle I'_\lambda(u_n),u_n-u\rangle \to 0, \\ \int_\Omega\left(|\nabla u|^{p(x)-2}\nabla u\nabla(u_n-u)+|u|^{p(x)-2}u(u_n-u)\right)dx \to 0. \end{cases} \tag{3.9}$$

On the other hand, taking account the fact that $f \in L^{q_0(x)}(\Omega)$, for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ large enough such that $\Omega_b \subset B_{R_\varepsilon}$ and

$$||f||_{L^{q_0(x)}(\Omega \backslash \Omega_{R_\varepsilon})} < \varepsilon, \tag{3.10}$$

where $\Omega_{R_\varepsilon} = B_{R_\varepsilon} \backslash \Omega_b$. Since f is bounded in Ω_{R_ε}, Hölder's inequality and (3.10) imply

$$\begin{aligned}
\int_\Omega |f(x)||u_n - u|^{q(x)} dx \leq & ||f||_{L^\infty(\Omega_{R_\varepsilon})} \int_{\Omega_{R_\varepsilon}} |u_n - u|^{q(x)} dx + \int_{\Omega \backslash \Omega_{R_\varepsilon}} |f(x)||u_n - u|^{q(x)} dx \\
\leq & ||f||_{L^\infty(\Omega_{R_\varepsilon})} \int_{\Omega_{R_\varepsilon}} |u_n - u|^{q(x)} dx \\
& + c_0 ||f||_{L^{q_0}(\Omega \backslash \Omega_{R_\varepsilon})} \max \left(||u_n - u||^{q^+}_{L^{p^*(x)}(\Omega \backslash \Omega_{R_\varepsilon})}, ||u_n - u||^{q^-}_{L^{p^*(x)}(\Omega \backslash \Omega_{R_\varepsilon})} \right) \\
\leq & ||f||_{L^\infty(\Omega_{R_\varepsilon})} \int_{\Omega_{R_\varepsilon}} |u_n - u|^{q(x)} dx + c_0 \varepsilon \max \left(||u_n - u||^{q^+}_{L^{p^*(x)}(\Omega)}, ||u_n - u||^{q^-}_{L^{p^*(x)}(\Omega)} \right).
\end{aligned} \tag{3.11}$$

According to the Sobolev compact embedding theorem in the bounded domain Ω_{R_ε}, we see that $u_n \to u$ in $L^{q(x)}(\Omega_{R_\varepsilon})$. Having in mind that $\{u_n\}$ is bounded in $L^{p^*(x)}(\Omega)$, we deduce from (3.11) that

$$\int_\Omega |f(x)||u_n - u|^{q(x)} dx \to 0, \text{ as } n \to +\infty. \tag{3.12}$$

In the similar way, we obtain

$$\int_\Omega g(x)|u_n - u|^{r(x)} dx \to 0, \tag{3.13}$$

$$\int_\Omega h(x)(u_n - u) dx \to 0. \tag{3.14}$$

Using again Hölder's inequality and (3.12)-(3.13), we could easily establish that

$$\int_\Omega f(x)|u_n|^{q(x)-2} u_n (u_n - u) dx \to 0, \ \int_\Omega g(x)|u_n|^{r(x)-2} u_n (u_n - u) dx \to 0. \tag{3.15}$$

By the virtue of the continuity of the function M, it follows from (3.7)-(3.9), (3.14) and (3.15) that

$$\begin{aligned}
& M \left(\int_\Omega \frac{|\nabla u_n|^{p(x)} + |u_n|^{p(x)}}{p(x)} dx \right) \\
& \times \left[\int_\Omega \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) \nabla (u_n - u) dx + \int_\Omega \left(|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right) (u_n - u) dx \right] \\
& \to 0.
\end{aligned}$$

Hence

$$\begin{aligned}
& \lim_{n \to +\infty} \left[\int_\Omega \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) \nabla (u_n - u) dx + \int_\Omega \left(|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right) (u_n - u) dx \right] \\
& = 0.
\end{aligned} \tag{3.16}$$

Using the well known inequality in $\mathbb{R}^N$ given by

$$\left[(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta)(\xi - \eta) \right]^{\frac{p}{2}} (|\xi|^p + |\eta|^p)^{\frac{2-p}{2}} \geq (p-1)|\xi - \eta|^p \text{ if } 1 < p < 2,$$

and

$$(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta)(\xi - \eta) \geq 2^{-p} |\xi - \eta|^p \text{ if } p \geq 2,$$

we deduce from (3.16) that $||u_n - u|| \to 0$. This completes the proof of Lemma 3.2. □

Proof of Theorem 2.4. In view of Lemmas 3.1, 3.2 and Mountain Pass theorem [23], there exists a weak solution u_1 of problem (P_λ) with $I_\lambda(u_1) > 0$. We now prove that there exists a second weak solution $u_2 \neq u_1$. Choosing $\varphi_1 \in C_0^\infty(\Omega)$ such that $\int_\Omega h(x)\varphi_1 dx > 0$. Then, by (M_1), for small $t > 0$,

$$I_\lambda(t\varphi_1) \leq \frac{m_1 t^{\beta p^-}}{(p^-)^\beta}\left(\int_\Omega \left(|\nabla\varphi_1|^{p(x)} + |\varphi_1|^{p(x)}\right)dx\right)^\beta + \frac{\lambda t^{q^-}}{q^-}\int_\Omega |f(x)||\varphi_1|^{q(x)}dx$$
$$+\frac{t^{r^-}}{r^-}\int_\Omega g(x)|\varphi_1|^{r(x)}dx - t\int_\Omega h(x)\varphi_1 dx < 0.$$

Thus for $\rho > 0$ given in Lemma 3.1,

$$-\infty < c_\rho := \inf_{u\in\overline{B_\rho}} I_\lambda(u) < 0, \qquad \inf_{u\in\partial B_\rho} I_\lambda(u) > 0,$$

where B_ρ is the ball centered at 0 and of radius ρ. Let us choose $0 < \varepsilon < \inf_{u\in\partial B_\rho} I_\lambda(u) - \inf_{u\in B_\rho} I_\lambda(u)$. Using the above information, $I_\lambda : \overline{B_\rho} \to \mathbb{R}$ is lower bounded on $\overline{B_\rho}$ and $I_\lambda \in C^1(\overline{B_\rho}, \mathbb{R})$. By Ekeland's variational principle [24], there exists $u_\varepsilon \in \overline{B_\rho}$ such that

$$\begin{cases} c_\rho \leq I_\lambda(u_\varepsilon) \leq c_\rho + \varepsilon, \\ I_\lambda(u_\varepsilon) < I_\lambda(u) + \varepsilon||u - u_\varepsilon||, \quad u \neq u_\varepsilon. \end{cases}$$

Since

$$I_\lambda(u_\varepsilon) \leq \inf_{u\in\overline{B_\rho}} I_\lambda(u) + \varepsilon \leq \inf_{u\in B_\rho} I_\lambda(u) + \varepsilon < \inf_{u\in\partial B_\rho} I_\lambda(u),$$

we can infer that $u_\varepsilon \in B_\rho$.

Now, we define $J_\lambda : \overline{B_\rho} \to \mathbb{R}$ by

$$J_\lambda(u) = I_\lambda(u) + \varepsilon||u - u_\varepsilon||.$$

It is easy to see that u_ε is a minimum point of J_λ. It follows that

$$\frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} \geq 0, \text{ for } t > 0 \text{ small enough and } v \in B_1.$$

It yields that

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} + \varepsilon||v|| \geq 0.$$

Letting $t \to 0$, we get

$$\langle I'_\lambda(u_\varepsilon), v\rangle + \varepsilon||v|| \geq 0,$$

which implies $||I'_\lambda(u_\varepsilon)||_{W^{1,p(x)}(\Omega)^*} \leq \varepsilon$. Therefore, we deduce that there exists a sequence $\{u_n\} \subset B_\rho$ such that

$$I_\lambda(u_n) \to c_\rho, \quad I'_\lambda(u_n) \to 0 \text{ in } W^{1,p(x)}(\Omega)^*.$$

Thanks to Lemma 3.2, up to subsequence $u_n \to u_2$ strongly in $W^{1,p(x)}(\Omega)$ with $I_\lambda(u_2) < 0$. Therefore, we conclude that problem (P_λ) admits at least two nontrivial solutions. This competes the proof of Theorem 2.4.

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