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ON A COUPLED SYSTEM OF HILFER AND HILFER-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, by using the technique that relies on the concept of the measure of noncompactness and the fixed point theory, we prove some existence results for some coupled systems of Hilfer and Hilfer-Hadamard fractional differential equations.

Keywords. Coupled system; Fixed point; Fractional differential equation; Hilfer-Hadamard fractional derivative; Measure of noncompactness.

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1. INTRODUCTION

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [1, 2]. For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs of Abbas, Benchohra and N'Guérékata [3, 4], Samko, Kilbas and Marichev [5], Kilbas, Srivastava and Trujillo [6] and Zhou [7], the papers by Abbas *et al.* [8, 9, 10], and the references therein. Recently, considerable attention have been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [1, 11, 12, 13, 14, 15, 16, 17] and the references therein and a coupled system of Hadamard type sequential fractional differential equations was considered in [18]. In [19, 20, 21], the measure of noncompactness was applied to some classes of functional Riemann-Liouville or Caputo fractional differential equations in Banach spaces. Motivated

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by the research going on in this direction, we discuss the existence of solutions for the following coupled system of Hilfer fractional differential equations

$$\begin{cases} (D_0^{\alpha_1,\beta_1}u)(t) = f_1(t,u(t),v(t)) \\ (D_0^{\alpha_2,\beta_2}v)(t) = f_2(t,u(t),v(t)) \end{cases} ;\ t \in I := [0,T], \tag{1.1}$$

with the following initial conditions

$$\begin{cases} (I_0^{1-\gamma_1}u)(0) = \phi_1, \\ (I_0^{1-\gamma_2}v)(0) = \phi_2, \end{cases} \tag{1.2}$$

where $T > 0,\ \alpha_i \in (0,1),\ \beta_i \in [0,1],\ \gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i,\ \phi_i \in E,\ f_i : I \times E \times E \to E;\ i = 1,2,$ are given functions, E is a real (or complex) Banach space with a norm $\|\cdot\|$, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the generalized Riemann-Liouville derivative (Hilfer) operator of order α_i and type $\beta_i :\ i = 1,2$.

Next, we consider the following coupled system of Hilfer-Hadamard fractional differential equations

$$\begin{cases} ({}^HD_1^{\alpha_1,\beta_1}u)(t) = g_1(t,u(t),v(t)) \\ ({}^HD_1^{\alpha_2,\beta_2}v)(t) = g_2(t,u(t),v(t)), \end{cases} ;\ t \in [1,T], \tag{1.3}$$

with the following initial conditions

$$\begin{cases} ({}^HI_1^{1-\gamma_1}u)(1) = \psi_1 \\ ({}^HI_1^{1-\gamma_2}v)(1) = \psi_2, \end{cases} \tag{1.4}$$

where $T > 1,\ \alpha_i \in (0,1),\ \beta_i \in [0,1],\ \gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i,\ \psi_i \in E,\ g_i : [1,T] \times E \times E \to E;\ i = 1,2$ are given functions, ${}^HI_1^{1-\gamma_i}$ is the left-sided mixed Hadamard integral of order $1-\gamma_i$, and ${}^HD_1^{\alpha_i,\beta_i}$ is the Hilfer-Hadamard fractional derivative of order α_i and type β_i; $i = 1,2$.

2. Preliminaries

Let C be the Banach space of all continuous functions v from I into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} \|v(t)\|.$$

As usual, by $AC(I)$ we denote the space of absolutely continuous functions from I into E. We denote by $AC^1(I)$ the space defined by

$$AC^1(I) := \{w : I \to E : \frac{d}{dt}w(t) \in AC(I)\}.$$

By $L^1(I)$, we denote the space of measurable functions $v : I \to E$ which are Bochner integrable, with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w : (0,T] \to E : t^{1-\gamma}w(t) \in C\},$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t\in I} |t^{1-\gamma}w(t)|,$$

and

$$C^1_\gamma(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},$$

with the norm

$$\|w\|_{C^1_\gamma} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

Also, by $\mathscr{C} := C_{\gamma_1} \times C_{\gamma_2}$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathscr{C}} = \|u\|_{C_{\gamma_1}} + \|v\|_{C_{\gamma_2}}.$$

Let $\mathscr{M}_X$ denote the class of all bounded subsets of a metric space X.

Definition 2.1. Let X be a complete metric space. A map $\mu : \mathscr{M}_X \to [0,\infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B, B_1, B_2 \in \mathscr{M}_X$.

(a) $\mu(B) = 0$ if and only if B is precompact (Regularity),
(b) $\mu(B) = \mu(\overline{B})$ (Invariance under closure),
(c) $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$ (Semi-additivity).

Example 2.2. In every metric space X, the map $\phi : \mathscr{M}_X \to [0,\infty)$ with $\phi(B) = 0$ if B is relatively compact and $\phi(B) = 1$ otherwise is a measure of noncompactness, the so-called discrete measure of noncompactness; see Example 1 in [19].

Definition 2.3. [20] Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0,\infty)$ defined by

$$\mu(M) = \inf\{\varepsilon > 0 : M \subset \cup_{j=1}^m M_j, \operatorname{diam}(M_j) \le \varepsilon\}$$

where $M \in \Omega_E$.

Here, we list some important properties.

(1) $\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).
(2) $\mu(M) = \mu(\overline{M})$.
(3) $M_1 \subset M_2 \Rightarrow \mu(M_1) \le \mu(M_2)$.
(4) $\mu(M_1 + M_2) \le \mu(M_1) + \mu(M_2)$.
(5) $\mu(cM) = |c|\mu(M)$, $c \in \mathbb{R}$.
(6) $\mu(\operatorname{conv} M) = \mu(M)$.

Now, we give some results and properties of fractional calculus.

Definition 2.4. [3, 5, 6] The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$(I_0^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s) ds; \ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt; \ \xi > 0.$$

For all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t); \ for\ a.e.\ t \in I.$$

Definition 2.5. [3, 5, 6] The Riemann-Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}(D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s) ds; \ for\ a.e.\ t \in I.\end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows

$$(D_0^r I_0^r w)(t) = w(t); \ for\ all\ t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C_{1-\gamma}^1(I)$, then the following composition is proved in [5]

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1}; \ for\ all\ t \in (0,T].$$

Definition 2.6. [3, 6, 5] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}({}^c D_0^r w)(t) &= \left(I_0^{1-r} \frac{d}{dt} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} \frac{d}{ds} w(s) ds; \ for\ a.e.\ t \in I.\end{aligned}$$

In [1], Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as specific cases (see also [14, 15].

Definition 2.7. (Hilfer derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, and

$$I_0^{(1-\alpha)(1-\beta)} w \in AC^1(I).$$

The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} w\right)(t); \ for\ a.e.\ t \in I. \tag{2.1}$$

Here, we also list some properties.

Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} w\right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w\right)(t); \ for\ a.e.\ t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta,\ 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (2.1) for $\beta = 0$ coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative

$$D_0^{\alpha,0} = D_0^{\alpha},\ and\ D_0^{\alpha,1} = {}^c D_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)} w$ exists and is in $L^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t); \ for\ a.e.\ t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^\alpha w)(t) = w(t);\ for\ a.e.\ t \in I.$$

4. If $D_0^\gamma w$ exists and is in $L^1(I)$, then

$$(I_0^\alpha D_0^{\alpha,\beta} w)(t) = (I_0^\gamma D_0^\gamma w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1};\ for\ a.e.\ t \in I.$$

Corollary 2.8. *Let $h \in C_\gamma(I)$. Then the Cauchy problem*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t);\ t \in I, \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{cases}$$

has the following unique solution

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha h)(t).$$

Lemma 2.9. [22] *If Y is a bounded subset of a Banach space X, then for each $\varepsilon > 0$, there is a sequence $\{y_k\}_{k=1}^\infty \subset Y$ such that*

$$\mu(Y) \le 2\mu(\{y_k\}_{k=1}^\infty) + \varepsilon.$$

Lemma 2.10. [23] *If $\{u_k\}_{k=1}^\infty \subset L^1(I)$ is uniformly integrable, then $\mu(\{u_k\}_{k=1}^\infty)$ is measurable and for each $t \in I$,*

$$\mu\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^\infty\right) \le 2\int_0^t \mu(\{u_k(s)\}_{k=1}^\infty)ds.$$

In the sequel, we will make use of the following fixed point theorem.

Theorem 2.11. [24] *Let F be a closed and convex subset of a real Banach space. Let $G : F \to F$ be a continuous operator and Let $G(F)$ be bounded. If there exists a constant $k \in [0,1)$ such that for each bounded subset $B \subset F$,*

$$\mu(G(B)) \le k\mu(B),$$

then G has a fixed point in F.

3. A Coupled System of Hilfer Fractional Differential Equations

In this section, we are concerned with the existence of solutions of system (1.1)-(1.2).

Definition 3.1. By a solution of problem (1.1)-(1.2), we mean a coupled measurable functions $(u,v) \in C_{\gamma_1} \times C_{\gamma_2}$ those satisfy equations (1.1) on I, and conditions $(I_0^{1-\gamma_1} u)(0^+) = \phi_1$, and $(I_0^{1-\gamma_2} v)(0^+) = \phi_2$.

The following hypotheses will be used in the sequel.

- (H_1) The functions $t \mapsto f_i(t,u,v);\ i = 1,2$ are measurable on I for each $(u,v) \in C_{\gamma_1} \times C_{\gamma_2}$, and the functions $(u,v) \mapsto f_i(t,u,v)$ are continuous on $C_{\gamma_1} \times C_{\gamma_2}$ for a.e. $t \in I$,
- (H_2) There exist continuous functions $p_i : I \to (0,\infty);\ i = 1,2$ such that

$$\|f_i(t,u,v) - f_i(t,w,z)\| \le \frac{p_i(t)\|u-w\|}{1+\|u-w\|+\|v-z\|};\ for\ a.e.\ t \in I,\ and\ each\ u,v,w,z \in E,$$

- (H_3) For each bounded and measurable set $B \subset C_{\gamma_i};\ i = 1,2$ and for each $t \in I$, we have

$$\mu(f_i(t,B,B)) \le p_i(t)\mu(B),$$

Set

$$p_i^* = \sup_{t\in I} p_i(t), \text{ and } f_i^* = \sup_{t\in I} \|f_i(t,0,0)\|; \ i = 1,2.$$

Now, we shall prove the following theorem concerning the existence of solutions of system (1.1)-(1.2).

Theorem 3.2. *Assume that hypotheses $(H_1)-(H_3)$ hold. If*

$$\ell := \frac{4p_1^* T^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{4p_2^* T^{\alpha_2}}{\Gamma(1+\alpha_2)} < 1, \tag{3.1}$$

then coupled system (1.1)-(1.2) has at least one solution defined on I.

Proof. Define the operators $N_1 : C_{\gamma_1} \to C_{\gamma_1}$ and $N_2 : C_{\gamma_2} \to C_{\gamma_2}$ by

$$(N_1 u)(t) = \frac{\phi_1}{\Gamma(\gamma_1)} t^{\gamma_1 - 1} + \int_0^t (t-s)^{\alpha_1 - 1} \frac{f(s,u(s),v(s))}{\Gamma(\alpha_1)} ds, \tag{3.2}$$

and

$$(N_2 v)(t) = \frac{\phi_2}{\Gamma(\gamma_2)} t^{\gamma_2 - 1} + \int_0^t (t-s)^{\alpha_2 - 1} \frac{f(s,u(s),v(s))}{\Gamma(\alpha_2)} ds. \tag{3.3}$$

Consider the continuous operator $N : \mathscr{C} \to \mathscr{C}$ defined by

$$(N(u,v))(t) = ((N_1 u)(t), (N_2 v)(t)). \tag{3.4}$$

Clearly, the fixed points of the operator N are solutions of system (1.1)-(1.2). For any $u \in C_{\gamma_1}$, and each $t \in I$, we have

$$\begin{aligned}
\|t^{1-\gamma_1}(N_1 u)(t)\| &\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} \|f_1(s,u(s),v(s))\| ds \\
&\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} \|f_1(s,0,0)\| ds \\
&\quad + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} \|f_1(s,u(s),v(s)) - f_1(s,0,0)\| ds \\
&\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} (\|f_1(s,0,0)\| + p_1(s)) ds \\
&\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{(f_1^* + p_1^*) T^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} ds \\
&\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{(f_1^* + p_1^*) T^{1-\gamma_1+\alpha_1}}{\Gamma(1+\alpha_1)}.
\end{aligned}$$

Also, for any $v \in C_{\gamma_2}$, and each $t \in I$, we get

$$\|t^{1-\gamma_1}(N_1 u)(t)\| \leq \frac{\|\phi_2\|}{\Gamma(\gamma_2)} + \frac{(f_2^* + p_2^*) T^{1-\gamma_2+\alpha_2}}{\Gamma(1+\alpha_2)}.$$

Thus

$$\|N(u,v)\|_{\mathscr{C}} \leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{\|\phi_2\|}{\Gamma(\gamma_2)} + \frac{(f_1^* + p_1^*) T^{1-\gamma_1+\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{(f_2^* + p_2^*) T^{1-\gamma_2+\alpha_2}}{\Gamma(1+\alpha_2)} := R. \tag{3.5}$$

This proves that N transforms the ball

$$B_R := B(0,R) = \{(u,v) \in \mathscr{C} : \|(u,v)\|_{\mathscr{C}} \leq R\}$$

into itself. We next show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.11. The proof will be given in several steps.

Step 1. $N : B_R \to B_R$ is continuous.

Let $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ be a sequence such that $(u_n, v_n) \to (u, v)$ in B_R. Then, for each $t \in I$, we have

$$\begin{aligned}
&\|t^{1-\gamma_1}(N_1u_n)(t) - t^{1-\gamma_1}(N_1u)(t)\| \\
&\leq \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u_n(s),v_n(s)) - f_1(s,u(s),v(s))\|ds \\
&\leq \frac{p_1^*T^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|u_n(s)-u(s)\|ds.
\end{aligned}$$

Also, $\|t^{1-\gamma_2}(N_2v_n)(t) - t^{1-\gamma_2}(N_2v)(t)\| \leq \frac{p_2^*T^{1-\gamma_2}}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\|u_n(s)-u(s)\|ds$. Hence

$$\begin{aligned}
&\|t^{1-\gamma_1}(N_1u_n)(t) - t^{1-\gamma_1}(N_1u)(t)\| + \|t^{1-\gamma_2}(N_2v_n)(t) - t^{1-\gamma_2}(N_2v)(t)\| \\
&\leq \frac{p_1^*T^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|u_n(s)-u(s)\|ds + \frac{p_2^*T^{1-\gamma_2}}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\|u_n(s)-u(s)\|ds.
\end{aligned} \tag{3.6}$$

Since $u_n \to u$ as $n \to \infty$, we see that equation (3.6) implies $\|N(u_n,v_n) - N(u,v)\|_{\mathscr{C}} \to 0$ as $n \to \infty$.

Step 2. $N(B_R)$ is bounded.

Since $N(B_R) \subset B_R$ and B_R is bounded, we find that $N(B_R)$ is bounded.

Step 3. For each bounded subset D of B_R, $\mu(N(D)) \leq \ell\mu(D)$.

From Lemmas 2.9 and 2.10, for any $D \subset B_R$ and any $\varepsilon > 0$, there exists a sequence $\{(u_n,v_n)\}_{n=0}^{\infty} \subset D$, such that, for all $t \in I$,

$$\begin{aligned}
\mu((ND)(t)) &= \mu\left(\left\{\frac{\phi_1}{\Gamma(\gamma_1)}t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1}\frac{f_1(s,u(s),v(s))}{\Gamma(\alpha_1)}ds\right.\right. \\
&\quad \left.\left. + \frac{\phi_2}{\Gamma(\gamma_2)}t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1}\frac{f_2(s,u(s),v(s))}{\Gamma(\alpha_2)}ds;\ (u,v)\in D\right\}\right) \\
&\leq 2\mu\left(\left\{\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}f_1(s,u_n(s),v_n(s))ds\right.\right. \\
&\quad \left.\left. + \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}f_2(s,u_n(s),v_n(s))ds\right\}_{n=1}^{\infty}\right) + \varepsilon \\
&\leq 4\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu\left(\{f_1(s,u_n(s),v_n(s))\}_{n=1}^{\infty}\right)ds \\
&\quad + 4\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu\left(\{f_2(s,u_n(s),v_n(s))\}_{n=1}^{\infty}\right)ds + \varepsilon \\
&\leq 4\left(\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}p_1(s) + \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}p_2(s)\right)\mu\left(\{(u_n(s),v_n(s))\}_{n=1}^{\infty}\right)ds + \varepsilon \\
&\leq 4\left(\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}p_1(s)ds + \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}p_2(s)ds\right)\mu\left(\{(u_n,v_n)\}_{n=1}^{\infty}\right) + \varepsilon \\
&\leq 4\left(\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}p_1(s)ds + \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}p_2(s)ds\right)\mu(D) + \varepsilon \\
&\leq 4\left(\frac{p_1^*T^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{p_2^*T^{\alpha_2}}{\Gamma(1+\alpha_2)}\right)\mu(D) + \varepsilon \\
&\leq \ell\mu(D) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we ahve

$$\mu(N(B)) \leq \ell\mu(B).$$

As a consequence of Steps 1 to 3 together with Theorem 2.11, we can conclude that N has at least one fixed point in B_R which is a solution of system (1.1)- (1.2). □

4. A Coupled System of Hilfer-Hadamard Fractional Differential Equations

Now, we are concerned with the existence of solutions of coupled system (1.3)-(1.4).

Set $C := C([1,T])$, and denote the weighted space of continuous functions defined by

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma} w(t) \in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]} |(\ln t)^{1-r} w(t)|.$$

Also, by $\mathscr{C}_{\gamma_1,\gamma_2,\ln}([1,T]) := C_{\gamma_1,\ln}([1,T]) \times C_{\gamma_2,\ln}([1,T])$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathscr{C}_{\gamma_1,\gamma_2,\ln}([1,T])} = \|u\|_{C_{\gamma_1,\ln}} + \|v\|_{C_{\gamma_2,\ln}}.$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [6] for a more detailed analysis.

Definition 4.1. [6] (Hadamard fractional integral). The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,T])$, is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

Example 4.2. Let $0 < q < 1$ and $g(x) = \ln x$, $x \in [0,e]$. Then

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(2+q)} (\ln x)^{1+q};\ for\ a.e.\ x \in [0,e].$$

Set

$$\delta = x\frac{d}{dx},\ q > 0,\ n = [q]+1,$$

and

$$AC_\delta^n := \{u : [1,T] \to E : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Analogous to the Riemann-Liouville fractional derivative, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

Definition 4.3. [6] (Hadamard fractional derivative). The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^H D_1^q w)(x) = \delta^n ({}^H I_1^{n-q} w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^H D_1^q w)(x) = \delta ({}^H I_1^{1-q} w)(x).$$

Example 4.4. Let $0 < q < 1$. Let $w(x) = \ln x$, $x \in [0,e]$. Then

$$({}^{H}D_1^q w)(x) = \frac{1}{\Gamma(2-q)}(\ln x)^{1-q};\ for\ a.e.\ x \in [0,e].$$

It has been proved (see, e.g., Kilbas [[25], Theorem 4.8]) that in the space $L^1(I)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.,

$$({}^{H}D_1^q)({}^{H}I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [6], we have

$$({}^{H}I_1^q)({}^{H}D_1^q w)(x) = w(x) - \frac{({}^{H}I_1^{1-q} w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo-Hadamard fractional derivative is defined in the following way.

Definition 4.5. (Caputo-Hadamard fractional derivative) The Caputo-Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^{Hc}D_1^q w)(x) = ({}^{H}I_1^{n-q}\delta^n w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^{Hc}D_1^q w)(x) = ({}^{H}I_1^{1-q}\delta w)(x).$$

From the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative (introduced for the first time in [26]) is defined in the following way.

Definition 4.6. (Hilfer-Hadamard fractional derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $w \in L^1(I)$, and ${}^{H}I_1^{(1-\alpha)(1-\beta)} w \in AC^1(I)$. The Hilfer-Hadamard fractional derivative of order α and type β applied to the function w is defined as

$$\begin{aligned}({}^{H}D_1^{\alpha,\beta} w)(t) &= \left({}^{H}I_1^{\beta(1-\alpha)}({}^{H}D_1^{\gamma} w)\right)(t)\\ &= \left({}^{H}I_1^{\beta(1-\alpha)}\delta({}^{H}I_1^{1-\gamma} w)\right)(t);\ for\ a.e.\ t \in [1,T].\end{aligned} \tag{4.1}$$

This new fractional derivative (4.1) may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative and when $\beta = 1$, we recover the Caputo-Hadamard fractional derivative.

$${}^{H}D_1^{\alpha,0} = {}^{H}D_1^{\alpha},\ and\ {}^{H}D_1^{\alpha,1} = {}^{Hc}D_1^{\alpha}.$$

From Theorem 21 of [27], we concluded the following lemma.

Lemma 4.7. *Let $g : [1,T] \times E \to E$ be such that $g(\cdot,u(\cdot)) \in C_{\gamma,\ln}([1,T])$ for any $u \in C_{\gamma,\ln}([1,T])$. Then problem (1.3) is equivalent to the following Volterra integral equation*

$$u(t) = \frac{\phi_0}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^{H}I_1^{\alpha} g(\cdot,u(\cdot)))(t).$$

Definition 4.8. By a solution of coupled system (1.3)-(1.4), we mean a coupled measurable functions $(u,v) \in C_{\gamma_1,\ln} \times C_{\gamma_2,\ln}$ those satisfy the conditions (1.4) and the equations (1.3) on $[1,T]$.

Now we give (without proof) similar existence results for system (1.3)-(1.4). Let us introduce the following hypotheses:

(H_1') The functions $t \mapsto g_i(t,u,v)$; $i=1,2$ are measurable on $[1,T]$ for each $(u,v) \in C_{\gamma_1,\ln} \times C_{\gamma_2,\ln}$, and the function $(u,v) \mapsto g_i(t,u,v)$ is continuous on $C_{\gamma_1,\ln} \times C_{\gamma_2,\ln}$ for a.e. $t \in [1,T]$,

(H_2') There exist continuous functions $q_i : [1,T] \to (0,\infty)$ such that

$$\|g_i(t,u,v) - g_i(t,w,z)\| \leq \frac{q_i(t)\|u-w\|}{1+\|u-w\|+\|v-z\|}; \ for\ a.e.\ t \in [1,T],\ and\ each\ u,v,w,z \in E,$$

(H_3') For each bounded and measurable set $B \subset C_{\gamma_1,\gamma_2,\ln}$ and for each $t \in [1,T]$, we have

$$\mu(g_i(t,B,B)) \leq q_i(t)\mu(B);\ i=1,2.$$

Theorem 4.9. *Assume that hypotheses $(H_1') - (H_3')$ hold. If*

$$\ell^* := \frac{4q_1^*(\ln T)^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{4q_2^*(\ln T)^{\alpha_2}}{\Gamma(1+\alpha_2)} < 1, \tag{4.2}$$

where $q_i^* = \sup_{t\in I} q_i(t)$; $i=1,2$, then system (1.3) -(1.4) has at least one solution defined on $[1,T]$.

5. An Example

Let

$$E = l^1 = \left\{ u = (u_1,u_2,\ldots,u_n,\ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider the following coupled system of Hilfer fractional differential equation of the form

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}} u_n)(t) = f_n(t,u(t),v(t)); \\ (D_0^{\frac{1}{2},\frac{1}{2}} v_n)(t) = g_n(t,u(t),v(t)); \\ (I_0^{\frac{1}{4}} u_n)(t)|_{t=0} = (1,0,\ldots,0,\ldots), \\ (I_0^{\frac{1}{4}} v_n)(t)|_{t=0} = (1,0,\ldots,0,\ldots), \end{cases} ;\ t \in [0,1], \tag{5.1}$$

where

$$\begin{cases} f_n(t,u,v) = \dfrac{t^{\frac{-1}{4}} u_n(t)\sin t}{64(1+\sqrt{t})(1+\|u\|_E+\|v\|_E)};\ t \in (0,1], \\ f_n(0,u,v) = 0, \end{cases}$$

$$g_n(t,u,v) = \frac{u_n(t)\cos t}{64(1+\|u\|_E+\|v\|_E)};\ t \in (0,1],$$

with

$$f = (f_1,f_2,\ldots,f_n,\ldots),\ g = (g_1,g_2,\ldots,g_n,\ldots),\ u = (u_1,u_2,\ldots,u_n,\ldots),$$

$$v = (v_1,v_2,\ldots,v_n,\ldots),\ and\ c := \frac{e^3}{8}\Gamma\left(\frac{1}{2}\right).$$

Setting $\alpha_i = \beta_i = \frac{1}{2}$; $i = 1,2$, one has $\gamma_i = \frac{3}{4}$; $i = 1,2$. The hypothesis (H_2) is satisfied with

$$\begin{cases} p_1(t) = \dfrac{t^{\frac{-1}{4}}|\sin t|}{64(1+\sqrt{t})}; \ t \in (0,1], \\ p_1(0) = 0, \end{cases}$$

and

$$p_2(t) = \frac{|\cos t|}{64}; \ t \in (0,1].$$

Hence, Theorem 3.2 implies that system (5.1) has at least one solution defined on $[0,1]$.

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