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ON NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYBRID FRACTIONAL SUM-DIFFERENCE EQUATIONS INVOLVING DIFFERENT ORDERS

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Abstract. In this article, we study existence results for Riemann-Liouville fractional sum-difference equations with a nonlocal fractional difference and fractional sum boundary value conditions based on the Sadovskii's fixed point theorem. The investigated equations contain different orders of fractional differences and fractional sums. Finally, we present some examples to show the importance of these results.

Keywords. Boundary value problem; Existence; Fixed point theorem; Fractional sum-difference equation.

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1. Introduction

In this paper, we consider Riemann-Liouville fractional sum-difference equations with a nonlocal fractional difference boundary value conditions of the form

$$\Delta^{\alpha}u(t)=\Delta^{-\theta}f\left(t+\alpha+\theta-1,u(t+\alpha+\theta-1),\Delta^{\beta}u(t+\alpha+\theta-\beta)\right),$$

$$\begin{aligned}&\lambda_1 u(\alpha-2)+u(T+\alpha)=\phi(u),\\&\lambda_2\Delta^{\nu}u(\alpha-\nu-1)+\Delta^{\nu}u(T+\alpha-\nu+1)=\varphi(u),\end{aligned}\tag{1.1}$$

and a nonlocal fractional sum boundary value conditions of the form

$$\begin{aligned}&\mu_1 u(\alpha-2)+u(T+\alpha)=\phi(u),\\&\mu_2\Delta^{-\gamma}u(\alpha+\gamma-2)+\Delta^{-\gamma}u(T+\alpha+\gamma)=\varphi(u),\end{aligned}\tag{1.2}$$

where $t\in\mathbb{N}_{0,T}:=\{0,1,...,T\}$, $1<\alpha\le 2$, $0<\theta,\beta,\gamma,\nu\le 1$ and $\lambda_1,\lambda_2,\mu_1,\mu_2>0$ are given constants, $f\in C(\mathbb{N}_{\alpha-2,T+\alpha}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a given function and $\phi,\varphi:C(\mathbb{N}_{\alpha-2,T+\alpha},\mathbb{R})\to\mathbb{R}$ are given functionals.

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Discrete fractional calculus and fractional difference equations have been interested from many researchers since they can used for describing many problems in the real-world phenomena such as physics, chemistry, mechanics, control systems, flow in porous media, and electrical networks. Mathematicians have employed this fractional calculus in recent years to model and solve various applied problems. Basic definitions and properties of fractional difference calculus can be found in the book [1]. In particular, fractional calculus is a powerful tool for the processes which appears in nature, e.g., biology, ecology and other areas, can be found in [2]-[6] and the references therein. A good account of papers dealing with discrete fractional boundary value problems, which has helped to build up some of the basic theory of this area; see [7]-[23] and references cited therein.

Presently, there is a development of boundary value problems for fractional difference equations which shows an operation of the investigative function. The study may also have another function which is related to our interested one. These creations are incorporating with nonlocal conditions which are both extensive and more complex, for example, Agarwal *et al.* [7] investigated the existence of solutions for two fractional boundary value problems

$$\begin{cases} \Delta^{\mu}_{\mu-2}x(t) = g(t+\mu-1, x(t+\mu-1), \Delta x(t+\mu-1)), \quad t \in \mathbb{N}_{0,b+2}, \\ x(\mu-2) = 0,\ x(\mu+b+1) = \sum_{k=\mu-1}^{\alpha} x(k), \end{cases} \tag{1.3}$$

where $1 < \mu \leq 2$ and $g \in C(\mathbb{N}_{\mu-1,\mu+b+1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function, and

$$\begin{cases} \Delta^{\mu}_{\mu-3}x(t) = g(t+\mu-2, x(t+\mu-2)), \quad t \in \mathbb{N}_{0,b+3}, \\ x(\mu-3) = x(\mu+b+1) = 0,\ x(\alpha) = \sum_{k=\gamma}^{\beta} x(k), \end{cases} \tag{1.4}$$

where $2 < \mu \leq 3,\ \alpha, \beta, \gamma \in \mathbb{N}_{\mu-2,\mu+b}$, and $g \in C(\mathbb{N}_{\mu-2,\mu+b+1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function.

Kang, Li and Chen [9] obtained sufficient conditions for the existence of positive solutions for a nonlocal boundary value problem

$$\begin{cases} -\Delta^{\mu}y(t) = \lambda h(t+\mu-1) f(y(t+\mu-1)), \quad t \in \mathbb{N}_{0,b}, \\ y(\mu-2) = \Psi(y),\ y(\mu+b) = \Phi(y), \end{cases} \tag{1.5}$$

where $1 < \mu \leq 2,\ \ f \in C([0,\infty),[0,\infty)),\ h \in C(\mathbb{N}_{\mu-1,\mu+b-1},[0,\infty))$ are given functions and $\Psi, \Phi : C(\mathbb{N}_{\mu-2,\mu+b}, \mathbb{R}) \to \mathbb{R}$ are given functionals.

Sitthiwirattham [18, 19] examined three-point fractional sum boundary value problems for fractional difference equations of the form

$$\begin{cases} \Delta^{\alpha}_{C}[\phi_p(\Delta^{\beta}_{C}x)](t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \\ \Delta^{\beta}_{C}x(\alpha-1) = 0, \quad x(\alpha+\beta+T) = \rho\Delta^{-\gamma}x(\eta+\gamma), \end{cases} \tag{1.6}$$

$$\begin{cases} \Delta^{\alpha}_{\alpha}(\Delta^{\beta}_{\alpha+\beta-1} + \lambda E_{\beta})x(t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \\ x(\alpha+\beta-2) = 0, \quad x(\alpha+\beta+T) = \rho\Delta^{-\gamma}_{\alpha+\beta-1}x(\eta+\gamma), \end{cases} \tag{1.7}$$

where $t \in \mathbb{N}_{0,T},\ 0 < \alpha, \beta \leq 1,\ 1 < \alpha+\beta \leq 2,\ 0 < \gamma \leq 1,\ \eta \in \mathbb{N}_{\alpha+\beta-1,\alpha+\beta+T-1},\ \rho$ is a constant, $f : \mathbb{N}_{\alpha+\beta-2,\alpha+\beta+T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $E_{\beta}x(t) = x(t+\beta-1)$ and ϕ_p is the p-Laplacian operator.

The organization of this paper is as follows. In Section 2, we recall some definitions and basic lemmas. Also we derive a representation for the solution to (1.1)-(1.2) by converting the problem to an equivalent summation equation. In Sections 3 and 4, using this representation, we prove the existence of the solutions of boundary value problem (1.1)-(1.2) with the help of the Sadovskii's fixed point theorem. Some illustrative examples are presented in Section 5.

2. Preliminaries

In this section, let us first collect some basic definitions and lemmas which play an important role in this article.

Definition 2.1. The generalized falling function defined by

$$t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)},$$

for any t and α for which the right-hand side is defined. If $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.

Lemma 2.2. [17] *Assume the following factorial functions are well defined.*

(i) $(t-\mu)t^{\underline{\mu}} = t^{\underline{\mu+1}}$, *where* $\mu \in \mathbb{R}$.

(ii) *If* $t \leq r$, *then* $t^{\underline{\alpha}} \leq r^{\underline{\alpha}}$ *for any* $\alpha > 0$.

Definition 2.3. For $\alpha > 0$ and f is defined on $\mathbb{N}_a := \{a, a+1, \ldots\}$, the α-order fractional sum of f is defined by

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

for $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 2.4. For $\alpha > 0$ and f is defined on $\mathbb{N}_a := \{a, a+1, \ldots\}$, the α-order Riemann-Liouville fractional difference of f is defined by

$$\Delta^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen such that $0 \leq N-1 < \alpha \leq N$.

Lemma 2.5. [17] *Let* $0 \leq N-1 < \alpha \leq N$. *Then*

$$\Delta^{-\alpha}\Delta^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \ldots + C_N t^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, *with* $1 \leq i \leq N$.

The following lemma deals with the linear variant of the boundary value problem (1.1) and gives a representation of the solution.

Lemma 2.6. *Let* $\Omega \neq 0$, $1 < \alpha \leq 2$, $0 < \theta, \nu \leq 1$, $h \in C(\mathbb{N}_{\alpha-1,T+\alpha-1}, \mathbb{R})$ *and* $\phi, \varphi : C(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R}) \to \mathbb{R}$ *be given. Then the problem*

$$\begin{aligned} &\Delta^{\alpha} u(t) = \Delta^{-\theta}\left[h(t+\alpha+\theta-1)\right], \quad t \in \mathbb{N}_{0,T}, \\ &\lambda_1 u(\alpha-2) = -u(T+\alpha) + \phi(u), \\ &\lambda_2 \Delta^{\nu} u(\alpha-\nu-1) = -\Delta^{\nu} u(T+\alpha-\nu+1) + \varphi(u), \end{aligned} \tag{2.1}$$

has the unique solution

$$
\begin{aligned}
&u(t)\\
&=\frac{t^{\underline{\alpha-1}}}{\Omega}\Bigg\{\left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\Bigg[\varphi(u)+\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\times\\
&\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}h(\xi)\Bigg]\\
&+\Phi\Bigg[\phi(u)-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}h(\xi)\Bigg]\Bigg\}\\
&+\frac{t^{\underline{\alpha-2}}}{\Omega}\Bigg\{\Psi\Bigg[\phi(u)+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}h(\xi)\Bigg]\\
&+(T+\alpha)^{\underline{\alpha-1}}\Bigg[\varphi(u)-\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}\times\\
&(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}h(\xi)\Bigg]\Bigg\}\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{t-\alpha}\sum_{\xi=\alpha-1}^{s+\alpha-1}(t-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}h(\xi),
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
\Phi &= \frac{1}{\Gamma(-\nu)}\sum_{s=\alpha-2}^{T+\alpha+1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}s^{\underline{\alpha-2}}+\lambda_2\Gamma(\alpha-1)\left[\alpha-\nu-1\right],\\
\Psi &= \frac{1}{\Gamma(-\nu)}\sum_{s=\alpha-1}^{T+\alpha+1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}s^{\underline{\alpha-1}}+\lambda_2\Gamma(\alpha),\\
\Omega &= \Phi(T+\alpha)^{\underline{\alpha-1}}-\Psi\left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right].
\end{aligned}
\tag{2.3}
$$

Proof. Using Lemma 2.5, a general solution for (2.1) can be written in the form

$$
\begin{aligned}
u(t) &= C_1t^{\underline{\alpha-1}}+C_2t^{\underline{\alpha-2}}\\
&\quad+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{t-\alpha}\sum_{\xi=-\theta}^{s-\theta}(t-\sigma(s))^{\underline{\alpha-1}}(s-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1),
\end{aligned}
\tag{2.4}
$$

for $t\in\mathbb{N}_{\alpha-2,\alpha+T}$. Applying the first boundary condition of (2.1) implies

$$
\begin{aligned}
&C_1(T+\alpha)^{\underline{\alpha-1}}+C_2\left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\\
&= \phi(u)-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=-\theta}^{s-\theta}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1).
\end{aligned}
\tag{2.5}
$$

Using the fractional difference of order $0<\nu\leq 1$ for (2.4), we obtain

$$\begin{aligned}\Delta^{\nu}u(t) &= \frac{C_1}{\Gamma(-\nu)}\sum_{s=\alpha-1}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}s^{\underline{\alpha-1}}+\frac{C_2}{\Gamma(-\nu)}\sum_{s=\alpha-2}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}s^{\underline{\alpha-2}}\\ &\quad+\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{t+\nu}\sum_{r=0}^{s-\alpha}\sum_{\xi=-\theta}^{r-\theta}(t-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}}\times\\ &\quad(r-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1),\end{aligned}$$

for $t\in\mathbb{N}_{\alpha-\nu-1,T+\alpha-\nu+1}$. The second condition of (2.1) implies

$$\begin{aligned}&C_1\left[\frac{1}{\Gamma(-\nu)}\sum_{s=\alpha-1}^{T+\alpha+1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}s^{\underline{\alpha-1}}+\lambda_2\Gamma(\alpha)\right]\\ &+C_2\left[\frac{1}{\Gamma(-\nu)}\sum_{s=\alpha-2}^{T+\alpha+1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}s^{\underline{\alpha-2}}+\lambda_2\Gamma(\alpha-1)\left[\alpha-\nu-1\right]\right]\\ =\;&\varphi(u)-\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=-\theta}^{r-\theta}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}}\times\\ &(r-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1).\end{aligned}\tag{2.6}$$

Constants C_1 and C_2 can be obtained by solving equations (2.5) and (2.6), respectively

$$\begin{aligned}C_1 &= \frac{1}{\Omega}\Bigg\{\left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\Bigg[\varphi(u)+\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\times\\ &\quad\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=-\theta}^{r-\theta}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1)\Bigg]\\ &\quad+\Phi\left[\phi(u)-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=-\theta}^{s-\theta}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1)\right]\Bigg\},\end{aligned}$$

$$\begin{aligned}C_2 &= \frac{1}{\Omega}\Bigg\{\Psi\left[\phi(u)+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=-\theta}^{s-\theta}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1)\right]\\ &\quad+(T+\alpha)^{\underline{\alpha-1}}\Bigg[\varphi(u)-\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=-\theta}^{r-\theta}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}\times\\ &\quad(s-\sigma(r))^{\underline{\alpha-1}}(r-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1)\Bigg]\Bigg\},\end{aligned}$$

where Φ,Ψ and Ω are defined by (2.3). Substituting constants C_1,C_2 into (2.4), we obtain (2.2). □

Next, the following lemma deals with linear variant of the boundary value problem (1.2) and gives a representation of the solution.

Lemma 2.7. *Let $\Lambda \neq 0$, $1 < \alpha \leq 2$, $0 < \theta, \gamma \leq 1$, $g : C(\mathbb{N}_{\alpha-1,T+\alpha-1}, \mathbb{R})$ and $\phi, \varphi : C(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R}) \to \mathbb{R}$ be given. Then the problem*

$$\begin{aligned} &\Delta^{\alpha} u(t) = \Delta^{-\theta}[g(t+\alpha+\theta-1)], \quad t \in \mathbb{N}_{0,T}, \\ &\mu_1 u(\alpha-2) = -u(T+\alpha) + \phi(u), \\ &\mu_2 \Delta^{-\gamma} u(\alpha+\gamma-2) = -\Delta^{-\gamma} u(T+\alpha+\gamma) + \varphi(u), \end{aligned} \tag{2.7}$$

has the unique solution

$$\begin{aligned} u(t) = &\frac{t^{\underline{\alpha-1}}}{\Lambda}\Bigg\{\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-2}}\times \\ &\left[\phi(u) - \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}g(\xi)\right] \\ &+\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\Bigg[\varphi(u) - \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\times \\ &\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}g(\xi)\Bigg]\Bigg\} \\ &+\frac{t^{\underline{\alpha-2}}}{\Lambda}\Bigg\{\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-1}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-1}}\times \\ &\left[\phi(u) - \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}g(\xi)\right] \\ &-(T+\alpha)^{\underline{\alpha-1}}\Bigg[\varphi(u) - \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}\times \\ &(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}g(\xi)\Bigg]\Bigg\} \\ &+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{t-\alpha}\sum_{\xi=\alpha-1}^{s+\alpha-1}(t-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}g(\xi), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \Lambda \;=\; &\frac{\left[\mu_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]}{\Gamma(\gamma)}\sum_{s=\alpha-1}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-1}} \\ &-(T+\alpha)^{\underline{\alpha-1}}\left[\mu_2\Gamma(\alpha-1)+\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-2}}\right]. \end{aligned} \tag{2.9}$$

Proof. Using Lemma 2.5, a general solution for (2.7) can be written in the form

$$\begin{aligned} u(t) \;=\; &C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} \\ &+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{t-\alpha}\sum_{\xi=-\theta}^{s-\theta}(t-\sigma(s))^{\underline{\alpha-1}}(s-\sigma(\xi))^{\underline{\theta-1}}h(\xi+\alpha+\theta-1), \end{aligned} \tag{2.10}$$

for $t \in \mathbb{N}_{\alpha-2,\alpha+T}$. Applying the first boundary condition of (2.7) implies

$$\begin{aligned} & C_1(T+\alpha)^{\underline{\alpha-1}} + C_2\left[\mu_1\Gamma(\alpha-1) + (T+\alpha)^{\underline{\alpha-2}}\right] \\ = \; & \phi(u) - \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=-\theta}^{s-\theta}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s-\sigma(\xi))^{\underline{\theta-1}}g(\xi+\alpha+\theta-1). \end{aligned} \tag{2.11}$$

Using the fractional sum of order $0 < \gamma \leq 1$ for (2.10), we obtain

$$\begin{aligned} & \Delta^{-\gamma}u(t) \\ = \; & \frac{C_1}{\Gamma(\gamma)}\sum_{s=\alpha-1}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-1}} + \frac{C_2}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-2}} \\ & + \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{t-\gamma}\sum_{r=0}^{s-\alpha}\sum_{\xi=0}^{r-\theta}(t-\sigma(s))^{\underline{\gamma-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r-\sigma(\xi))^{\underline{\theta-1}}g(\xi+\alpha+\theta-1), \end{aligned}$$

for $t \in \mathbb{N}_{\alpha+\gamma-2,T+\alpha+\gamma}$. The second condition of (2.7) implies

$$\begin{aligned} & \frac{C_1}{\Gamma(\gamma)}\sum_{s=\alpha-1}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-1}} \\ & + C_2\left[\mu_2\Gamma(\alpha-1) + \frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-2}}\right] \\ = \; & \varphi(u) - \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=-\theta}^{r-\theta}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}(s-\sigma(r))^{\underline{\alpha-1}}\times \\ & (r-\sigma(\xi))^{\underline{\theta-1}}g(\xi+\alpha+\theta-1). \end{aligned} \tag{2.12}$$

Constants C_1 and C_2 can be obtained by solving the equations (2.11) and (2.12), respectively

$$\begin{aligned} C_1 \; = \; & \frac{1}{\Lambda}\Bigg\{\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-2}}\times \\ & \left[\phi(u) - \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=-\theta}^{s-\theta}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s-\sigma(\xi))^{\underline{\theta-1}}g(\xi+\alpha+\theta-1)\right] \\ & + \mu_1\left[\Gamma(\alpha-1) + (T+\alpha)^{\underline{\alpha-2}}\right]\times \\ & \Bigg[\varphi(u) - \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=-\theta}^{r-\theta}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}(s-\sigma(r))^{\underline{\alpha-1}}\times \\ & (r-\sigma(\xi))^{\underline{\theta-1}}g(\xi+\alpha+\theta-1)\Bigg]\Bigg\}, \end{aligned}$$

$$
\begin{aligned}
C_2 \;=\; & \frac{1}{\Lambda}\Bigg\{\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-1}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-1}}\times \\
& \left[\phi(u)-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=-\theta}^{s-\theta}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s-\sigma(\xi))^{\underline{\theta-1}}g(\xi+\alpha+\theta-1)\right] \\
& -(T+\alpha)^{\underline{\alpha-1}}\Bigg[\varphi(u)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=-\theta}^{r-\theta}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}\times \\
& (s-\sigma(r))^{\underline{\alpha-1}}(r-\sigma(\xi))^{\underline{\theta-1}}g(\xi+\alpha+\theta-1)\Bigg]\Bigg\},
\end{aligned}
$$

where Λ are defined by (2.9). Substituting constants C_1, C_2 into (2.10), we obtain (2.8) immediately. □

In the following we will give two existence results for problems (1.1)-(1.2) with help of Sadovskii's fixed point theorem.

Definition 2.8. Let M be a bounded set in metric space (X,d). Then Kuratowskii measure of noncompactness, $\alpha(M)$ is defined as

$$\inf\{\varepsilon : \text{M covered by a finitely many sets such that the diameter of each set} \leq \varepsilon\}.$$

Definition 2.9. Let $\Phi : D(\Phi) \subseteq X \to X$ be a bounded and continuous operator on a Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset D(\Phi)$, where α denotes the Kuratowski measure of noncompactness.

Lemma 2.10. [24] *The map $K+C$ is a k-set contraction with $0 \leq k < 1$, and thus also condensing if*
(i) $K, C : D \subseteq X \to X$ are operators on the Banach space X,
(ii) K is k-contractive, i.e., $\|Kx - Ky\| \leq k\|x-y\|$, for all $x, y \in D$ and fixed $k \in [0,1)$,
(iii) C is compact.

Theorem 2.11. [25] *(Sadovskii's fixed point theorem) Let B be a convex, bounded and closed subset of a Banach space X and let $\Phi : B \to B$ be a condensing map. Then Φ has a fixed point.*

3. Existence of solutions for fractional difference BVP 1.1

Now, we wish to establish the existence result for problem (1.1). To accomplish this, we denote that $\mathscr{C} = C(\mathbb{N}_{\alpha-2,\alpha+T}, \mathbb{R})$, the Banach space of all function u with the norm defined by

$$\|u\|_{\mathscr{C}} = \max\{\|u\|, \|\Delta^{\beta}u\|\},$$

where $\|u\| = \max_{t\in\mathbb{N}_{\alpha-2,\alpha+T}} |u(t)|$ and $\|\Delta^{\beta}u\| = \max_{t\in\mathbb{N}_{\alpha-2,\alpha+T}} |\Delta^{\beta}u(t-\beta+1)|$. We also define an operator $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ by

$$(\mathscr{F}u)(t) = (\mathscr{F}_1 u)(t) + (\mathscr{F}_2 u)(t), \tag{3.1}$$

where

$$
\begin{aligned}
&(\mathscr{F}_1 u)(t)\\
&= \frac{t^{\underline{\alpha-1}}}{\Omega}\Bigg\{ \left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\Bigg[\varphi(u)+\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\times\\
&\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\times\\
&f(\xi,u(\xi),\Delta^{\beta}u(\xi-\beta+1))\Bigg]+\Phi\Bigg[\phi(u)-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\times\\
&(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}f(\xi,u(\xi),\Delta^{\beta}u(\xi-\beta+1))\Bigg]\Bigg\}\\
&+\frac{t^{\underline{\alpha-2}}}{\Omega}\Bigg\{\Psi\Bigg[\phi(u)+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\times\\
&f(\xi,u(\xi),\Delta^{\beta}u(\xi-\beta+1))\Bigg]\\
&+(T+\alpha)^{\underline{\alpha-1}}\Bigg[\varphi(u)-\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}\times\\
&(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}f(\xi,u(\xi),\Delta^{\beta}u(\xi-\beta+1))\Bigg]\Bigg\},
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&(\mathscr{F}_2 u)(t)\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{t-\alpha}\sum_{\xi=\alpha-1}^{s+\alpha-1}(t-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}f(\xi,u(\xi),\Delta^{\beta}u(\xi-\beta+1)),
\end{aligned}
\tag{3.3}
$$

with Φ,Ψ ad Ω are defined on (2.3). Hence, problem (1.1) has solutions if and only if the operator $\mathscr{F}$ has fixed points.

Theorem 3.1. *Assume that $f:\mathbb{N}_{\alpha-2,T+\alpha}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and $\phi,\varphi:C(\mathbb{N}_{\alpha-2,T+\alpha},\mathbb{R})\to\mathbb{R}$ are given functionals, satisfying the conditions:*

(H_1) *There exist constants $L_1,L_2>0$ such that for each $t\in\mathbb{N}_{\alpha-2,\alpha+T}$ and $u_1,u_2,v_1,v_2\in\mathbb{R}$*

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\le L_1|u_1-v_1|+L_2|u_2-v_2|.$$

(H_2) *There exists constants $\psi,\omega>0$ such that for each $u,v\in\mathscr{C}$*

$$|\phi(u)-\phi(v)|\le\psi\|u-v\|_{\mathscr{C}} \text{ and } |\varphi(u)-\varphi(v)|\le\omega\|u-v\|_{\mathscr{C}}.$$

(H_3) $\chi:=\tau\Theta+\psi\Xi+\omega\Upsilon<1,$

where

$$
\begin{aligned}
\tau &= \max\{L_1+L_2\},\\
\Theta &= \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|}\cdot\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\times\\
&\quad\left|\left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\cdot\frac{\Gamma(T-\nu+2)}{\Gamma(1-\nu)\Gamma(T+2)}-\Phi\right| \qquad (3.4)\\
&\quad+\frac{(T+\alpha)^{\underline{\alpha-2}}}{|\Omega|}\cdot\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\cdot\left|\Psi-(T+\alpha)^{\underline{\alpha-1}}\cdot\frac{\Gamma(T-\nu+2)}{\Gamma(1-\nu)\Gamma(T+2)}\right|\\
\Xi &= \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|}\left[\Phi+\frac{\Psi}{T+2}\right],\\
\Upsilon &= \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|}\left[\lambda_1\Gamma(\alpha-1)+(\lambda_1+1)(T+\alpha)^{\underline{\alpha-2}}\right].
\end{aligned}
$$

Then problem (1.1) *has at least one solution on* $\mathbb{N}_{\alpha-2,\alpha+T}$.

Proof. Let $B_R=\{u\in\mathscr{C}:\|u\|_{\mathscr{C}}\le R\}$ be a closed bounded, and convex subset of $\mathscr{C}:=C(\mathbb{N}_{\alpha-2,\alpha+T},\mathbb{R})$, where R will be fixed later. We define a map $\mathscr{F}:B_R\to C$ as $(\mathscr{F}u)(t)=(\mathscr{F}_1u)(t)+(\mathscr{F}_2u)(t)$, where $\mathscr{F}_1$ and $\mathscr{F}_2$ are defined by (3.2) and (3.3) respectively. Notice that the problem (1.1) is equivalent to a fixed point problem $\mathscr{F}(u)=u$.

Step I. $(\mathscr{F}u)(B_R)\subset B_R$.

Set $\max\limits_{t\in\mathbb{N}_{\alpha-2,T+\alpha}}|f(t,0,0)|=M$, $\max\limits_{u\in B_R}|\phi(u)|=A$, $\max\limits_{u\in B_R}|\varphi(u)|=B$ and choose a constant R such that

$$
R\ge\frac{A\Xi+B\Upsilon+M\left(\Theta+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right)}{1-\psi\Xi-\omega\Upsilon+\tau\left(\Theta+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right)}.
$$

Denote $\mathscr{N}(u,0)=|f(t,u(t),\Delta^{\beta}u(t-\beta+1))-f(t,0,0)|+|f(t,0,0)|$. For any $u\in B_R$, we have

$$
\begin{aligned}
&|(\mathscr{F}u)(t)|\\
\le\ &\frac{t^{\underline{\alpha-1}}}{|\Omega|}\Bigg|\left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\Bigg[(|\varphi(u)-\varphi(0)|+|\varphi(0)|)+\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\times\\
&\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\times\\
&\mathscr{N}(u,0)\Bigg]+\Phi\Bigg[(|\phi(u)-\phi(0)|+|\phi(0)|)-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\times\\
&(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\Bigg]\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t^{\underline{\alpha-2}}}{|\Omega|}\Bigg|\Psi\Bigg[(|\phi(u)-\phi(0)|+|\phi(0)|)+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\times\\
&(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\Bigg]+(T+\alpha)^{\underline{\alpha-1}}\Bigg[(|\varphi(u)-\varphi(0)|+|\varphi(0)|)\\
&-\frac{1}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}\times\\
&(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\Bigg]\Bigg|\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\Bigg|\sum_{s=0}^{t-\alpha}\sum_{\xi=\alpha-1}^{s+\alpha-1}(t-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\Bigg|\\
\leq\quad&(\psi\|u\|_{\mathscr{C}}+A)\left\{\frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|}\left[\Phi+\frac{\Psi}{T+2}\right]\right\}\\
&+(\omega\|u\|_{\mathscr{C}}+B)\left\{\frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|}\left[\lambda_1\Gamma(\alpha-1)+(\lambda_1+1)(T+\alpha)^{\underline{\alpha-2}}\right]\right\}\\
&+(\tau\|u\|_{\mathscr{C}}+M)\Bigg\{\frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|}\cdot\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\times\\
&\left|\left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\cdot\frac{\Gamma(T-\nu+2)}{\Gamma(1-\nu)\Gamma(T+2)}-\Phi\right|\\
&+\frac{(T+\alpha)^{\underline{\alpha-2}}}{|\Omega|}\cdot\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\cdot\left|\Psi-(T+\alpha)^{\underline{\alpha-1}}\cdot\frac{\Gamma(T-\nu+2)}{\Gamma(1-\nu)\Gamma(T+2)}\right|\\
&+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\Bigg\}\\
\leq\quad&(\psi R+A)\Xi+(\omega R+B)\Upsilon+(\tau R+M)\left(\Theta+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&|(\mathscr{F}\Delta^{\beta}u)(t-\beta+1)|\\
\leq\quad&\frac{1}{|\Omega|\Gamma(-\beta)}\sum_{s=\alpha-1}^{t+1}(t-\beta+1-\sigma(s))^{\underline{-\beta-1}}s^{\underline{\alpha-1}}\left|\left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\times\right.\\
&\Big[(\omega\|u\|_{\mathscr{C}}+B)+\frac{\big(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M\big)}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\times\\
&\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\Phi\Big[(\psi\|u\|_{\mathscr{C}}+A)-\frac{\big(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M\big)}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\times\\
&(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\Big]\Bigg|\\
&+\frac{1}{\Gamma(-\beta)}\sum_{s=\alpha-1}^{t+1}(t-\beta+1-\sigma(s))^{\underline{-\beta-1}}s^{\underline{\alpha-2}}\Bigg|\Psi\Big[(\psi\|u\|_{\mathscr{C}}+A)\\
&+\frac{\big(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M\big)}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\Big]\\
&+(T+\alpha)^{\underline{\alpha-1}}\Big[(\omega\|u\|_{\mathscr{C}}+B)-\frac{\big(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M\big)}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)}\times\\
&\sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\Big]\Bigg|\\
&+\Big(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M\Big)\sum_{s=\alpha}^{t+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{s+\alpha-1}(t-\beta+1-\sigma(s))^{\underline{-\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}}\times\\
&(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\\
<\quad&(\psi\|u\|_{\mathscr{C}}+A)\left\{\Xi\left(\frac{\Gamma(T-\beta+2)}{\Gamma(1-\beta)\Gamma(T+2)}\right)\right\}+(\omega\|u\|_{\mathscr{C}}+B)\left\{\Upsilon\left(\frac{\Gamma(T-\beta+2)}{\Gamma(1-\beta)\Gamma(T+2)}\right)\right\}\\
&+(\tau\|u\|_{\mathscr{C}}+M)\left\{\Theta\left(\frac{\Gamma(T-\beta+2)}{\Gamma(1-\beta)\Gamma(T+2)}\right)+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\times\right.\\
&\left.\left(\frac{\Gamma(T-\beta+2)}{\Gamma(1-\beta)\Gamma(T+2)}\right)\right\}\\
<\quad&(\psi R+A)\Xi+(\omega R+B)\Upsilon+(\tau R+M)\left(\Theta+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right).
\end{aligned}
$$

Therefore, we have that $\|\mathscr{F}u\|_{\mathscr{C}}\le R$. Hence $(\mathscr{F}u)(B_R)\subset B_R$.

Step II. $\mathscr{F}_1$ is continuous and χ-contractive.

Let $\varepsilon>0$ be given. Since f is continuous function, there exists a positive constant $\delta=\min\{\delta_1,\delta_2,\delta_3\}$ such that for each $t\in\mathbb{N}_{\alpha-2,\alpha+T}$ and for any $u,v,\Delta^{\beta}u,\Delta^{\beta}v\in B_{\rho}$ $|f(t,u(t),\Delta^{\beta}u(t-\beta+1))-f(t,v(t),\Delta^{\beta}v(t-\beta+1))|<\frac{\varepsilon}{3\Theta}$ whenever $\|u-v\|_{\mathscr{C}}<\delta_1$, $|\phi(u)-\phi(v)|<\frac{\varepsilon}{3\Xi}$ whenever $\|u-v\|_{\mathscr{C}}<\delta_2$, and $|\varphi(u)-\varphi(v)|<\frac{\varepsilon}{3\Upsilon}$ whenever $\|u-v\|_{\mathscr{C}}<\delta_3$. Thus, we obtain

$$
\begin{aligned}
|(\mathscr{F}_1u)(t)-(\mathscr{F}_1v)(t)|\quad\le\quad&|f(t,u(t),\Delta^{\beta}u(t-\beta+1))-f(t,v(t),\Delta^{\beta}v(t-\beta+1))|\Theta\\
&+|\phi(u)-\phi(v)|\Xi+|\varphi(u)-\varphi(v)|\Upsilon\\
<\quad&\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\;=\;\varepsilon.
\end{aligned}
$$

Similar to the proof above and Step I, we obtain $\|(\Delta^{\beta}\mathscr{F}_1)u-(\Delta^{\beta}\mathscr{F}_1)v\|<\|\mathscr{F}_1u-\mathscr{F}_1v\|<\varepsilon$. Hence, $\|\mathscr{F}_1u-\mathscr{F}_1v\|_{\mathscr{C}}<\varepsilon$. This means that $\mathscr{F}_1$ is continuous on B_R.

Next, we show that $\mathscr{F}_1$ is χ-contractive. For any $u,v,\Delta^\beta u,\Delta^\beta v \in B_R$ and for each $t \in \mathbb{N}_{\alpha-2,\alpha+T}$, we have

$$
\begin{aligned}
& \|\mathscr{F}_1 u - \mathscr{F}_1 v\| \\
= \; & \max_{t\in\mathbb{N}_{\alpha-2,\alpha+T}} \Bigg\{ \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|} \Bigg| \left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right] \Bigg[\omega\|u-v\|_{\mathscr{C}} \\
& + \frac{\tau\|u-v\|_{\mathscr{C}}}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)} \sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1} (T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}}(s-\sigma(r))^{\underline{\alpha-1}} \times \\
& (r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}} \Bigg] + \Phi \Bigg[\psi\|u-v\|_{\mathscr{C}} - \frac{\tau\|u-v\|_{\mathscr{C}}}{\Gamma(\alpha)\Gamma(\theta)} \times \\
& \sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1} (T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}} \Bigg] \Bigg| \\
& + \frac{(T+\alpha)^{\underline{\alpha-2}}}{|\Omega|} \Bigg| \Psi \Bigg[\psi\|u-v\|_{\mathscr{C}} + \frac{\tau\|u-v\|_{\mathscr{C}}}{\Gamma(\alpha)\Gamma(\theta)} \times \\
& \sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1} (T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}} \Bigg] + (T+\alpha)^{\underline{\alpha-1}} \times \\
& \Bigg[\omega\|u-v\|_{\mathscr{C}} - \frac{\tau\|u-v\|_{\mathscr{C}}}{\Gamma(-\nu)\Gamma(\alpha)\Gamma(\theta)} \sum_{s=\alpha}^{T+\alpha+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1} (T+\alpha-\nu+1-\sigma(s))^{\underline{-\nu-1}} \times \\
& (s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}} \Bigg] \Bigg| \Bigg\} \\
\leq \; & \psi\|u-v\|_{\mathscr{C}} \Bigg\{ \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|} \left[\Phi + \frac{\Psi}{T+2}\right] \Bigg\} \\
& + \omega\|u-v\|_{\mathscr{C}} \Bigg\{ \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|} \left[\lambda_1\Gamma(\alpha-1)+(\lambda_1+1)(T+\alpha)^{\underline{\alpha-2}}\right] \Bigg\} \\
& + \tau\|u-v\|_{\mathscr{C}} \Bigg\{ \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Omega|} \cdot \frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)} \times \\
& \Bigg| \left[\lambda_1\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right] \cdot \frac{\Gamma(T-\nu+2)}{\Gamma(1-\nu)\Gamma(T+2)} - \Phi \Bigg| + \frac{(T+\alpha)^{\underline{\alpha-2}}}{\Omega} \times \\
& \frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)} \cdot \Bigg| \Psi - (T+\alpha)^{\underline{\alpha-1}} \cdot \frac{\Gamma(T-\nu+2)}{\Gamma(1-\nu)\Gamma(T+2)} \Bigg| \Bigg\} \\
= \; & \left[\psi\Xi + \omega\Upsilon + \tau\Theta\right] \|u-v\|_{\mathscr{C}} = \chi\|u-v\|_{\mathscr{C}}.
\end{aligned}
$$

Similar to the proof above and Step I, we obtain $\|\Delta^\beta\mathscr{F}_1 u - \Delta^\beta\mathscr{F}_1 v\| < \|\mathscr{F}_1 u - \mathscr{F}_1 v\| \leq \chi\|u-v\|_{\mathscr{C}}$. Hence, $\|\mathscr{F}_1 u - \mathscr{F}_1 v\|_{\mathscr{C}} \leq \chi\|u-v\|_{\mathscr{C}}$. In view of assumption $\chi < 1$, we find that $\mathscr{F}_1$ is χ-contractive.

Step III. $\mathscr{F}_2$ is compact.

In Step I, it has been shown that $\mathscr{F}_2$ is uniformly bounded. Now we show that $\mathscr{F}_2$ maps bounded sets into equicontinuous sets of $\mathscr{C}$. For any $\varepsilon > 0$, there exists $\delta \in \mathbb{N}$ such that for $t_1, t_2 \in \mathbb{N}_{\alpha-2,T+\alpha}$

$$\left|\frac{\Gamma(t_2+1)}{\Gamma(t_2-\alpha+1)}-\frac{\Gamma(t_1+1)}{\Gamma(t_1-\alpha+1)}\right| \quad < \quad \frac{\varepsilon\Gamma(\alpha+1)}{\|f\|_{\mathscr{C}}} \text{ whenever } |t_2-t_1|<\delta.$$

For any $u \in B_R$, we have

$$\begin{aligned}
|\mathscr{F}_2u(t_2)-\mathscr{F}_2u(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\Bigg|\sum_{s=\alpha-1}^{t_2-1}(t_2+\alpha-1-\sigma(s))^{\underline{\alpha-1}}|f(s,u(s),\Delta^\beta u(s-\beta+1))| \\
&\quad -\sum_{s=\alpha-1}^{t_1-1}(t_1+\alpha-1-\sigma(s))^{\underline{\alpha-1}}|f(s,u(s),\Delta^\beta u(s-\beta+1))|\Bigg| \\
&\leq \frac{\|f\|_{\mathscr{C}}}{\Gamma(\alpha+1)}\left|\frac{\Gamma(t_2+1)}{\Gamma(t_2-\alpha+1)}-\frac{\Gamma(t_1+1)}{\Gamma(t_1-\alpha+1)}\right| \\
&< \varepsilon.
\end{aligned}$$

Similar to the proof above and Step I, we obtain $\|\Delta^\beta\mathscr{F}_2u(t_2)-\Delta^\beta\mathscr{F}_2u(t_1)\| < \|\mathscr{F}_2u(t_2)-\mathscr{F}_2u(t_1)\| < \varepsilon$. Thus $(\mathscr{F}_2)(B_R)$ is an equicontinuous set. Therefore it follows by the Arzelá-Ascoli theorem that $\mathscr{F}_2$ is completely continuous. Hence $\mathscr{F}_2$ is compact on B_R.

Step IV. $\mathscr{F}$ is is condensing.

Since $\mathscr{F}_1$ is continuous, χ-contraction and $\mathscr{F}_2$ is compact, we find from Lemma 2.10, $\mathscr{F}: B_R \to B_R$ with $\mathscr{F}=\mathscr{F}_1+\mathscr{F}_2$ is a condensing map on B_R. Consequently, by Theorem 2.11, the map $\mathscr{F}$ has a fixed point which implies that problem (1.1) has a solution. □

4. Existence of solutions for fractional difference BVP 1.2

Next, we wish to establish the existence results for problem (1.2). To accomplish this, we define an operator $\mathscr{P}:\mathscr{C}\to\mathscr{C}$ by

$$(\mathscr{P}u)(t)=(\mathscr{P}_1u)(t)+(\mathscr{P}_2u)(t), \tag{4.1}$$

where

$$\begin{aligned}
(\mathscr{P}_1u)(t) =& \frac{t^{\underline{\alpha-1}}}{\Lambda}\Bigg\{\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-2}}\times \\
&\Bigg[\phi(u)-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\times \\
&f(\xi,u(\xi),\Delta^\beta u(\xi-\beta+1))\Bigg]+\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\times \\
&\Bigg[\varphi(u)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}(s-\sigma(r))^{\underline{\alpha-1}}\times \\
&(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}f(\xi,u(\xi),\Delta^\beta u(\xi-\beta+1))\Bigg]\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^{\underline{\alpha-2}}}{\Lambda}\Bigg\{\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-1}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-1}}\Bigg[\phi(u)-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\times\\
&\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}f(\xi,u(\xi),\Delta^{\beta}u(\xi-\beta+1))\Bigg]\\
&-(T+\alpha)^{\underline{\alpha-1}}\Bigg[\varphi(u)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}\times\\
&(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}f(\xi,u(\xi),\Delta^{\beta}u(\xi-\beta+1))\Bigg]\Bigg\},
\end{aligned}
\tag{4.2}
$$

and

$$
\begin{aligned}
(\mathscr{P}_2u)(t)=&\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{t-\alpha}\sum_{\xi=\alpha-1}^{s+\alpha-1}(t-\sigma(s))^{\underline{\alpha-1}}\times\\
&(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}f(\xi,u(\xi),\Delta^{\beta}u(\xi-\beta+1)),
\end{aligned}
\tag{4.3}
$$

where Λ are defined on (2.9). Hence, problem (1.2) has solutions if and only if the operator F has fixed points.

Theorem 4.1. *Assume that $f:\mathbb{N}_{\alpha-2,T+\alpha}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and $\phi,\varphi:C(\mathbb{N}_{\alpha-2,T+\alpha},\mathbb{R})\to\mathbb{R}$ are given functionals. In addition, suppose that $(H_1)-(H_2)$ hold, and (H_4) $\wp:=\tau\widetilde{\Theta}+\psi\widetilde{\Xi}+\omega\widetilde{\Upsilon}<1$ where*

$$
\begin{aligned}
\widetilde{\Theta} &= \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Lambda|}\cdot\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\cdot\frac{\Gamma(T+\gamma+1)}{\Gamma(\gamma+1)\Gamma(T+1)}\times\\
&\quad\left[\frac{(T+\gamma+2)(T+\gamma+1)\Gamma(T+\alpha+1)}{(T+2)(T+1)\Gamma(T+3)}+\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\right]\\
&\quad+\frac{(T+\alpha)^{\underline{\alpha-2}}}{|\Lambda|}\cdot\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\cdot\frac{\Gamma(T+\gamma+1)}{\Gamma(\gamma+1)\Gamma(T+1)}\times\\
&\quad\left|(T+\alpha)^{\underline{\alpha-1}}-\frac{(T+\gamma+1)\Gamma(T+\alpha+1)}{(T+1)\Gamma(T+2)}\right|,\\
\widetilde{\Xi} &= \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Lambda|}\cdot\frac{\Gamma(T+\gamma+2)\Gamma(T+\alpha+1)}{\Gamma(\gamma+1)\Gamma(T+2)\Gamma(T+3)}\left[\frac{T+\gamma+2}{T+2}+1\right],\\
\widetilde{\Upsilon} &= \left|\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]-(T+\alpha)^{\underline{\alpha-1}}\right|.
\end{aligned}
\tag{4.4}
$$

Then problem (1.2) has at least one solution on $\mathbb{N}_{\alpha-2,\alpha+T}$.

Proof. Let $B_\rho=\{u\in\mathscr{C}:\|u\|\le\rho\}$ be a closed bounded and convex subset of $\mathscr{C}:=C(\mathbb{N}_{\alpha-2,\alpha+T},\mathbb{R})$, where ρ will be fixed later. We define a map $\mathscr{P}:B_\rho\to C$ as $(\mathscr{P}u)(t)=(\mathscr{P}_1u)(t)+(\mathscr{P}_2u)(t)$, where $\mathscr{P}_1$ and $\mathscr{P}_2$ are defined by (4.2) and (4.3) respectively. Notice that the problem (1.2) is equivalent to a fixed point problem $\mathscr{P}(u)=u$.

Step I. $(\mathscr{P}u)(B_\rho)\subset B_\rho$.

Set $\max\limits_{t\in\mathbb{N}_{\alpha-2,T+\alpha}}|f(t,0,0)|=M$, $\max\limits_{u\in B_R}|\phi(u)|=A$, $\max\limits_{u\in B_R}|\varphi(u)|=B$ and choose a constant ρ such that

$$\rho\geq\frac{A\widetilde{\Xi}+B\widetilde{\Upsilon}+M\left(\widetilde{\Theta}+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right)}{1-\psi\widetilde{\Xi}-\omega\widetilde{\Upsilon}+\tau\left(\widetilde{\Theta}+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right)}.$$

Denote $\mathscr{N}(u,0)=|f(t,u(t),\Delta^{\beta}u(t-\beta+1))-f(t,0,0)|+|f(t,0,0)|$, for any $u\in B_{\rho}$,

$$\begin{aligned}
|(\mathscr{P}u)(t)|\leq{}&\frac{t^{\underline{\alpha-1}}}{|\Lambda|}\Bigg\{\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-2}}\Bigg[(|\phi(u)-\phi(0)|+|\phi(0)|)\\
&-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\Bigg]\\
&+\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\Bigg[(|\varphi(u)-\varphi(0)|+|\varphi(0)|)\\
&-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}(s-\sigma(r))^{\underline{\alpha-1}}\times\\
&(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\Bigg]\Bigg\}\\
&+\frac{t^{\underline{\alpha-2}}}{|\Lambda|}\Bigg\{\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-1}^{T+\alpha}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}s^{\underline{\alpha-1}}\Bigg[(|\phi(u)-\phi(0)|+|\phi(0)|)\\
&-\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{T}\sum_{\xi=\alpha-1}^{s+\alpha-1}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\Bigg]\\
&-(T+\alpha)^{\underline{\alpha-1}}\Bigg[(|\varphi(u)-\varphi(0)|+|\varphi(0)|)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\theta)}\times\\
&\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{r+\alpha-1}(T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}}(s-\sigma(r))^{\underline{\alpha-1}}(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\Bigg]\Bigg\}\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\theta)}\sum_{s=0}^{t-\alpha}\sum_{\xi=\alpha-1}^{s+\alpha-1}(t-\sigma(s))^{\underline{\alpha-1}}(s+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\mathscr{N}(u,0)\\
<{}&\frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Lambda|}\Bigg|\frac{\Gamma(T+\gamma+3)\Gamma(T+\alpha+1)}{\Gamma(\gamma+1)\Gamma(T+3)\Gamma(T+3)}\Bigg[(\psi\|u\|_{\mathscr{C}}+A)-(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\times\\
&\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\Bigg]+\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\Bigg[(\omega\|u\|_{\mathscr{C}}+B)\\
&-(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\frac{\Gamma(T+\gamma+1)\Gamma(T+\alpha+\theta+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\theta+1)\Gamma(T+1)\Gamma(T+1)}\Bigg]\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(T+\alpha)^{\underline{\alpha-2}}}{|\Lambda|}\Bigg|\frac{\Gamma(T+\gamma+2)\Gamma(T+\alpha+1)}{\Gamma(\gamma+1)\Gamma(T+2)\Gamma(T+2)}\Bigg[(\psi\|u\|_{\mathscr{C}}+A)-(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\times\\
&\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\Bigg]-(T+\alpha)^{\underline{\alpha-1}}\Bigg[(\omega\|u\|_{\mathscr{C}}+B)-(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\times\\
&\frac{\Gamma(T+\gamma+1)\Gamma(T+\alpha+\theta+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\theta+1)\Gamma(T+1)\Gamma(T+1)}\Bigg]\Bigg|\\
&+(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\\
\leq\ &(\psi\|u\|_{\mathscr{C}}+A)\left\{\frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Lambda|}\cdot\frac{\Gamma(T+\gamma+2)\Gamma(T+\alpha+1)}{\Gamma(\gamma+1)\Gamma(T+2)\Gamma(T+3)}\left[\frac{T+\gamma+2}{T+2}+1\right]\right\}\\
&+(\omega\|u\|_{\mathscr{C}}+B)\left\{\left|\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]-(T+\alpha)^{\underline{\alpha-1}}\right|\right\}\\
&+(\tau\|u\|_{\mathscr{C}}+M)\Bigg\{\frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Lambda|}\cdot\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\cdot\frac{\Gamma(T+\gamma+1)}{\Gamma(\gamma+1)\Gamma(T+1)}\times\\
&\left[\frac{(T+\gamma+2)(T+\gamma+1)\Gamma(T+\alpha+1)}{(T+2)(T+1)\Gamma(T+3)}+\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\right]\\
&+\frac{(T+\alpha)^{\underline{\alpha-2}}}{|\Lambda|}\cdot\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\cdot\frac{\Gamma(T+\gamma+1)}{\Gamma(\gamma+1)\Gamma(T+1)}\times\\
&\left|(T+\alpha)^{\underline{\alpha-1}}-\frac{(T+\gamma+1)\Gamma(T+\alpha+1)}{(T+1)\Gamma(T+2)}\right|+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\Bigg\}\\
\leq\ &(\psi R+A)\widetilde{\Xi}+(\omega R+B)\widetilde{\Upsilon}+(\tau R+M)\left(\widetilde{\Theta}+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&|(\mathscr{P}\Delta^{\beta}u)(t-\beta+1)|\\
\leq\ &\frac{1}{|\Lambda|\Gamma(-\beta)}\sum_{s=\alpha-1}^{t+1}(t-\beta+1-\sigma(s))^{\underline{-\beta-1}}s^{\underline{\alpha-1}}\Bigg|\frac{\Gamma(T+\gamma+3)\Gamma(T+\alpha+1)}{\Gamma(\gamma+1)\Gamma(T+3)\Gamma(T+3)}\times\\
&\left[(\psi\|u\|_{\mathscr{C}}+A)-(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right]\\
&+\mu_1\left[\Gamma(\alpha-1)+(T+\alpha)^{\underline{\alpha-2}}\right]\Bigg[(\omega\|u\|_{\mathscr{C}}+B)-(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\times\\
&\frac{\Gamma(T+\gamma+1)\Gamma(T+\alpha+\theta+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\theta+1)\Gamma(T+1)\Gamma(T+1)}\Bigg]\Bigg|\\
&+\frac{1}{|\Lambda|\Gamma(-\beta)}\sum_{s=\alpha-1}^{t+1}(t-\beta+1-\sigma(s))^{\underline{-\beta-1}}s^{\underline{\alpha-2}}\Bigg|\frac{\Gamma(T+\gamma+2)\Gamma(T+\alpha+1)}{\Gamma(\gamma+1)\Gamma(T+2)\Gamma(T+2)}\times\\
&\left[(\psi\|u\|_{\mathscr{C}}+A)-(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-(T+\alpha)^{\underline{\alpha-1}}\Big[(\omega\|u\|_{\mathscr{C}}+B)-(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)\times \\
&\frac{\Gamma(T+\gamma+1)\Gamma(T+\alpha+\theta+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\theta+1)\Gamma(T+1)\Gamma(T+1)}\Big]\Bigg| \\
&+\frac{(L_1\|u\|+L_2\|\Delta^{\beta}u\|+M)}{\Gamma(-\beta)\Gamma(\alpha)\Gamma(\theta)}\Bigg|\sum_{s=\alpha}^{t+1}\sum_{r=0}^{s-\alpha}\sum_{\xi=\alpha-1}^{s+\alpha-1}(t-\beta+1-\sigma(s))^{\underline{-\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}} \\
&(r+\alpha+\theta-1-\sigma(\xi))^{\underline{\theta-1}}\Bigg| \\
\leq\ & (\psi\|u\|_{\mathscr{C}}+A)\left\{\widetilde{\Xi}\left(\frac{\Gamma(T-\beta+2)}{\Gamma(1-\beta)\Gamma(T+2)}\right)\right\}+(\omega\|u\|_{\mathscr{C}}+B)\left\{\widetilde{\Upsilon}\left(\frac{\Gamma(T-\beta+2)}{\Gamma(1-\beta)\Gamma(T+2)}\right)\right\} \\
&+(\tau\|u\|_{\mathscr{C}}+M)\left\{\widetilde{\Theta}\left(\frac{\Gamma(T-\beta+2)}{\Gamma(1-\beta)\Gamma(T+2)}\right)\right]+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\times \\
&\left(\frac{\Gamma(T-\beta+2)}{\Gamma(1-\beta)\Gamma(T+2)}\right)\Bigg\} \\
<\ & (\psi R+A)\widetilde{\Xi}+(\omega R+B)\widetilde{\Upsilon}+(\tau R+M)\left(\widetilde{\Theta}+\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)}\right).
\end{aligned}
$$

Therefore, $\|\mathscr{P}u\|_{\mathscr{C}}\leq\rho$. It follows that $(\mathscr{P}u)(B_\rho)\subset B_\rho$.

Step II. $\mathscr{P}_1$ is continuous and $\wp$-contractive.

Let $\varepsilon>0$ be given. Since f is a continuous function, there exists a positive constant $\delta=\min\{\delta_1,\delta_2,\delta_3\}$ such that for each $t\in\mathbb{N}_{\alpha-2,\alpha+T}$ and for any $u,v,\Delta^{\beta}u,\Delta^{\beta}v\in B_\rho$

$$|f(t,u(t),\Delta^{\beta}u(t-\beta+1))-f(t,v(t),\Delta^{\beta}v(t-\beta+1))|<\frac{\varepsilon}{3\widetilde{\Theta}}\text{ whenever}$$

$\|u-v\|_{\mathscr{C}}<\delta_1$,

$$|\phi(u)-\phi(v)|<\frac{\varepsilon}{3\widetilde{\Xi}}\text{ whenever }\|u-v\|_{\mathscr{C}}<\delta_2,$$

$$\text{and }|\varphi(u)-\varphi(v)|<\frac{\varepsilon}{3\widetilde{\Upsilon}}\text{ whenever }\|u-v\|_{\mathscr{C}}<\delta_3.$$

Thus, we obtain

$$
\begin{aligned}
|(\mathscr{P}_1u)(t)-(\mathscr{P}_1v)(t)| &\leq |f(t,u(t),\Delta^{\beta}u(t-\beta+1))-f(t,v(t),\Delta^{\beta}v(t-\beta+1))|\,\widetilde{\Theta} \\
&\quad+|\phi(u)-\phi(v)|\,\widetilde{\Xi}+|\varphi(u)-\varphi(v)|\,\widetilde{\Upsilon} \\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\ =\ \varepsilon.
\end{aligned}
$$

Similar to the proof above and Step I, we obtain

$$\|(\Delta^{\beta}\mathscr{P}_1)u-(\Delta^{\beta}\mathscr{F}_1)v\|<\|\mathscr{P}_1u-\mathscr{F}_1v\|<\varepsilon.$$

Hence, $\|\mathscr{P}_1u-\mathscr{P}_1v\|_{\mathscr{C}}<\varepsilon$. This means that $\mathscr{P}_1$ is continuous on B_ρ.

Next, we show that $\mathscr{P}_1$ is $\wp$-contractive. Denote $\mathscr{S}(u,v) = f(t,u(t),\Delta^\beta u(t-\beta+1)) - f(t,v(t),\Delta^\beta v(t-\beta+1))$, for any $u,v,\Delta^\beta u,\Delta^\beta v \in B_\rho$ and for each $t \in \mathbb{N}_{\alpha-2,\alpha+T}$, we have

$$
\begin{aligned}
&\|(\mathscr{P}_1 u)(t) - (\mathscr{P}_1 v)(t)\| \\
= &\max_{t\in\mathbb{N}_{\alpha-2,\alpha+T}} \Bigg\{ \frac{(T+\alpha)^{\underline{\alpha-1}}}{|\Lambda|} \Bigg| \frac{\Gamma(T+\gamma+3)\Gamma(T+\alpha+1)}{\Gamma(\gamma+1)\Gamma(T+3)\Gamma(T+3)} \Big[|\phi(u)-\phi(v)| - |\mathscr{S}(u,v)| \times \\
&\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)} \Big] + \mu_1 \left[\Gamma(\alpha-1) + (T+\alpha)^{\underline{\alpha-2}}\right] \Big[|\varphi(u)-\varphi(v)| - |\mathscr{S}(u,v)| \times \\
&\frac{\Gamma(T+\gamma+1)\Gamma(T+\alpha+\theta+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\theta+1)\Gamma(T+1)\Gamma(T+1)} \Big] \Bigg| \\
&+ \frac{(T+\alpha)^{\underline{\alpha-2}}}{|\Lambda|} \Bigg| \frac{\Gamma(T+\gamma+2)\Gamma(T+\alpha+1)}{\Gamma(\gamma+1)\Gamma(T+2)\Gamma(T+2)} \Big[|\phi(u)-\phi(v)| - |\mathscr{S}(u,v)| \times \\
&\frac{\Gamma(T+\alpha+\theta+1)}{\Gamma(\alpha+\theta+1)\Gamma(T+1)} \Big] - (T+\alpha)^{\underline{\alpha-1}} \Big[|\varphi(u)-\varphi(v)| - |\mathscr{S}(u,v)| \times \\
&\frac{\Gamma(T+\gamma+1)\Gamma(T+\alpha+\theta+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\theta+1)\Gamma(T+1)\Gamma(T+1)} \Big] \Bigg| \Bigg\} \\
\leq\ & \psi\|u-v\|_{\mathscr{C}} \widetilde{\Xi} + \omega\|u-v\|_{\mathscr{C}} \widetilde{\Upsilon} + \tau\|u-v\|_{\mathscr{C}} \widetilde{\Theta} \\
=\ & \left(\psi\widetilde{\Xi} + \omega\widetilde{\Upsilon} + \tau\widetilde{\Theta}\right) \|u-v\|_{\mathscr{C}} \\
=\ & \wp\|u-v\|_{\mathscr{C}}.
\end{aligned}
$$

Similar to the proof above and Step I, we obtain

$$\|\Delta^\beta \mathscr{P}_1 u - \Delta^\beta \mathscr{P}_1 v\| < \|\mathscr{P}_1 u - \mathscr{P}_1 v\| \leq \wp\|u-v\|_{\mathscr{C}}.$$

Hence, $\|\mathscr{P}_1 u - \mathscr{P}_1 v\|_{\mathscr{C}} \leq \wp\|u-v\|_{\mathscr{C}}$. In view of assumption that $\wp < 1$, we obtain that $\mathscr{P}_1$ is $\wp$-contractive.

Step III. $\mathscr{P}_2$ is compact.

In Step I, it has been shown that $\mathscr{P}_2$ is uniformly bounded. Now we show that $\mathscr{P}_2$ maps bounded sets into equicontinuous sets of $\mathscr{C}$. For any $\varepsilon > 0$, there exists $\delta \in \mathbb{N}$ such that for $t_1, t_2 \in \mathbb{N}_{\alpha-2,T+\alpha}$

$$\left| \frac{\Gamma(t_2+1)}{\Gamma(t_2-\alpha+1)} - \frac{\Gamma(t_1+1)}{\Gamma(t_1-\alpha+1)} \right| \ < \ \frac{\varepsilon\Gamma(\alpha+1)}{\|f\|_{\mathscr{C}}},$$

whenever $|t_2 - t_1| < \delta$. For any $u \in B_\rho$, we have

$$
\begin{aligned}
&\|\mathscr{P}_2 u(t_2) - \mathscr{P}_2 u(t_1)\| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \Bigg| \sum_{s=\alpha-1}^{t_2-1} (t_2+\alpha-1-\sigma(s))^{\underline{\alpha-1}} |f(s,u(s),\Delta^\beta u(s-\beta+1))| \\
&- \sum_{s=\alpha-1}^{t_1-1} (t_1+\alpha-1-\sigma(s))^{\underline{\alpha-1}} |f(s,u(s),\Delta^\beta u(s-\beta+1))| \Bigg| \\
\leq\ & \frac{\|f\|_{\mathscr{C}}}{\Gamma(\alpha+1)} \left| \frac{\Gamma(t_2+1)}{\Gamma(t_2-\alpha+1)} - \frac{\Gamma(t_1+1)}{\Gamma(t_1-\alpha+1)} \right| < \varepsilon.
\end{aligned}
$$

Similar to the proof above and Step I, we obtain

$$\|\Delta^{\beta}\mathscr{P}_2u(t_2)-\Delta^{\beta}\mathscr{P}_2u(t_1)\|<\|\mathscr{P}_2u(t_2)-\mathscr{P}_2u(t_1)\|_{\mathscr{C}}<\varepsilon.$$

Thus $(\mathscr{P}_2)(B_R)$ is an equicontinuous set. Therefore it follows by the Arzelá-Ascoli theorem that $\mathscr{P}_2$ is completely continuous. Hence $\mathscr{P}_2$ is compact on B_R.

Step IV. $\mathscr{P}$ is is condensing.

Since $\mathscr{P}_1$ is continuous, $\wp$-contraction and $\mathscr{P}_2$ is compact, therefore, by Lemma 2.10, $\mathscr{P}:B_\rho\to B_\rho$ with $\mathscr{P}=\mathscr{P}_1+\mathscr{P}_2$ is a condensing map on B_ρ. Consequently, by Theorem 2.11, map $\mathscr{P}$ has a fixed point which, implies that problem (1.2) has a solution. □

5. Some examples

In order to illustrate our result, we consider some examples in this section.

Example 5.1. Consider the following fractional difference boundary value problem

$$\begin{aligned}
&\Delta^{\frac{3}{2}}u(t)=\Delta^{-\frac{1}{4}}\left[\frac{e^{-\sin^2\pi(t+\frac{3}{4})}}{\left(t+\frac{203}{4}\right)^2}\cdot\frac{|u|+\arctan\left(\sin^2\pi(t+\frac{3}{4})\right)\Delta^{\frac{3}{4}}u(t+1)}{|u|+\cos^2u}\right],\quad t\in N_{0,4},\\
&2u\left(-\frac{1}{2}\right)=-u\left(\frac{11}{2}\right)+\frac{1}{20\pi^3}\sum_{i=0}^{6}C_iu(t_i),\ t_i=i-\frac{1}{2},\\
&3\Delta^{\frac{1}{3}}\left(\frac{1}{6}\right)=-\Delta^{\frac{1}{3}}\left(\frac{37}{6}\right)+\frac{(\|u\|^2+1)^{\frac{(\|u\|^2+1)-\pi}{}}}{\|u\|\Gamma(u^2+1)},
\end{aligned}\tag{5.1}$$

where C_i are are given positive constants with $\sum_{i=0}^{6}C_i<2$. Here $\alpha=\frac{3}{2},\ T=4,\ \theta=\frac{1}{4},\ \beta=\frac{3}{4},\ \nu=\frac{1}{3},\ \lambda_1=2,\ \lambda_2=3,\ \phi(u)=\frac{1}{20\pi^3}\sum_{i=0}^{6}C_iu(t_i),\ \varphi(u)=\frac{(\|u\|^2+1)^{\frac{(\|u\|^2+1)-\pi}{}}}{\|u\|\Gamma(u^2+1)}$ and $f(t,u(t),\Delta^{\beta}u(t-\beta+1))=\frac{e^{-\sin^2\pi t}}{(t+50)^2}\cdot\frac{|u|+\arctan\left(\sin^2(\pi t)\right)\Delta^{\frac{3}{4}}u(t-\frac{1}{4})}{|u|+\cos^2u}$. Let $t\in\mathbb{N}_{-\frac{1}{2},\frac{11}{2}}$. Clearly

$$\begin{aligned}
|f(t,u_1,v_1)-f(t,u_2,v_2)| &< \frac{|u_1-u_2|}{e(t+50)^2}+\frac{\arctan(1)|v_1-v_2|}{e(t+50)^2}\\
&\le \frac{4}{9801e}|u_1-u_2|+\frac{\pi}{9801e}|v_1-v_2|.
\end{aligned}$$

So, (H_1) holds with $L_1=\frac{4}{9801e}$, $L_2=\frac{\pi}{9801e}$ and $\tau=\frac{\pi+4}{9801e}$.

$$\begin{aligned}
|\phi(u)-\phi(v)| &= \left|\frac{\sum_{i=0}^{6}C_iu(t_i)}{(20\pi^3)}-\frac{\sum_{i=0}^{6}C_iv(t_i)}{(20\pi^3)}\right|\le\frac{\sum_{i=0}^{6}C_i}{20\pi^3}\|u-v\|_{\mathscr{C}}<\frac{1}{10\pi^3}\|u-v\|_{\mathscr{C}},\\
|\varphi(u)-\varphi(v)| &= \left|\frac{\Gamma(\|u\|^2+2)}{\Gamma(1+\pi)\|u\|\Gamma(\|u\|^2+1)}-\frac{\Gamma(\|v\|^2+2)}{\Gamma(1+\pi)\|v\|\Gamma(\|v\|^2+1)}\right|\\
&\le \frac{1}{\Gamma(1+\pi)}\left|\frac{\|u\|^2+1}{\|u\|}-\frac{\|v\|^2+1}{\|v\|}\right|\le\frac{1}{\Gamma(1+\pi)}\|u-v\|_{\mathscr{C}}.
\end{aligned}$$

So (H_2) holds with $\psi=\frac{1}{10\pi^3}$ and $\omega=\frac{1}{\Gamma(1+\pi)}$.

Also, we get $\Phi=1.0458,\ \Psi=3.6723$ and $|\Omega|=11.9774$.

We can show that $\Theta = 2.7579,\ \Xi = 0.3369,\ \Upsilon = 0.9504,$ and

$$\chi = \tau\Theta + \psi\Xi + \omega\Upsilon \approx 0.1340 < 1.$$

Hence, by Theorem 3.1, boundary value problem (5.1) has at least one solution on $\mathbb{N}_{-\frac{1}{2},\frac{11}{2}}$.

Example 5.2. Consider the following fractional sum boundary value problem

$$\begin{aligned}
&\Delta^{\frac{3}{2}}u(t) = \Delta^{-\frac{1}{4}}\left[\frac{e^{-\sin^2\pi(t+\frac{3}{4})}}{\left(t+\frac{203}{4}\right)^2}\cdot\frac{|u|+\arctan\left(\sin^2\pi(t+\frac{3}{4})\right)\Delta^{\frac{3}{4}}u(t-1)}{|u|+\cos^2 u}\right], \quad t\in N_{0,4},\\
&3u\left(-\frac{1}{2}\right) = -u\left(\frac{11}{2}\right) + \frac{1}{20\pi^3}\sum_{i=0}^{6}C_i u(t_i),\ t_i = i-\frac{1}{2},\\
&2\Delta^{-\frac{2}{3}}\left(-\frac{5}{6}\right) = -\Delta^{-\frac{2}{3}}\left(\frac{35}{6}\right) + \frac{(\|u\|^2+1)^{\frac{(\|u\|^2+1)-\pi}{}}}{\|u\|\Gamma(u^2+1)},
\end{aligned} \tag{5.2}$$

where C_i are are given positive constants with $\sum_{i=0}^{6}C_i < 2$. Here $\alpha = \frac{3}{2},\ T = 4,\ \theta = \frac{1}{4},\ \beta = \frac{3}{4},\ \gamma = \frac{2}{3},\ \mu_1 = 2$, and $\mu_2 = 3$. By Example 5.1, for $t \in \mathbb{N}_{-\frac{1}{2},\frac{11}{2}}$, we have that (H_1) and (H_2) hold with $\tau = \frac{\pi+4}{9801e}$, $\psi = \frac{1}{10\pi^3}$ and $\omega = \frac{1}{\Gamma(1+\pi)}$. Also, we get that $|\Lambda| = 30.1506,\ \widetilde{\Theta} = 14.5034,\ \widetilde{\Xi} = 0.2412,\ \widetilde{\Upsilon} = 1.9458,$ and

$$\wp = \tau\widetilde{\Theta} + \psi\widetilde{\Xi} + \omega\widetilde{\Upsilon} \approx 0.2754 < 1.$$

Hence, by Theorem 4.1, boundary value problem 5.2 has at least one solution on $\mathbb{N}_{-\frac{1}{2},\frac{11}{2}}$.

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