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RELAXED ITERATIVE METHODS FOR AN INFINITE FAMILY OF D-ACCRETIVE MAPPINGS IN A BANACH SPACE AND THEIR APPLICATIONS

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Abstract. In this paper, d-accretive mappings, which belong to accretive-type mappings but are different from m-accretive mappings, are studied. Some relaxed projection iterative algorithms for an infinite family of d-accretive mappings are constructed in a real uniformly convex and uniformly smooth Banach space. The iterative sequences are proved to be strongly convergent to a common zero point of the family of d-accretive mappings. Compared to the related work, the construction of the iterative algorithms are simpler and easily realized. Moreover, a kind of generalized (p,q)-Laplacian parabolic systems is exemplified. The example also emphasizes the importance of the study on d-accretive mappings and sets up a relationship between iterative algorithms and nonlinear systems.

Keywords. d-accretive mapping; Iterative algorithm; Metric projection; Normalized duality mapping; Zero point.

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1. Introduction and Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual space of X. Suppose that C is a nonempty closed and convex subset of X. Let $\langle \cdot, \cdot \rangle$ be the duality pairing of X and X^*. In this article, we use "$\to$" and "$\rightharpoonup$" to denote strong and weak convergence, respectively. A Banach space X is strictly convex [1] if $\|x\| = \|y\| = 1, x \neq y$ implies that $\|\frac{x+y}{2}\| < 1$. Also, X is said to be uniformly convex [1] if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that $\|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon$ implies that $\|\frac{x+y}{2}\| \leq 1-\delta$. A Banach space X is said to be smooth [1] if

$$\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t} \tag{1.1}$$

exists for each $x, y \in S(X) := \{u \in X : \|u\| = 1\}$. The norm of X is said to be Fréchet differentiable if, for each $x \in S(X)$, the limit (1.1) is attained uniformly for $y \in S(X)$. The norm of X is said to be uniformly Fréchet differentiable if the limit (1.1) is attained uniformly for $(x,y) \in S(X) \times S(X)$. The space X is

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uniformly smooth if and only if its norm is uniformly Fréchet differentiable. We say that X has Property (H) if for every sequence $\{x_n\} \subset X$ which weakly converges to some $x \in X$ and satisfies $\|x_n\| \to \|x\|$ as $n \to \infty$ necessarily converges to x in the norm.

The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}, \quad \forall x \in X.$$

Lemma 1.1. [1] *The normalized duality mapping $J : X \to 2^{X^*}$ has the following properties:*

(1) if X is a real reflexive and smooth Banach space, then J is single-valued;

(2) if X is reflexive, then J is surjective;

(3) if X is uniformly smooth and uniformly convex, then J^{-1} is also the duality mapping from X^* into X. Moreover, both J and J^{-1} are uniformly continuous on each bounded subset of X or X^*, respectively;

(4) for $x \in X$ and $c \in (0,+\infty)$, $J(cx) = cJ(x)$.

Definition 1.2. [2] Let $A : D(A) \sqsubseteq X \to X$ be a mapping. Then

(1) A is said to be d-accretive if for all $x, y \in D(A)$, $\langle Ax - Ay, j(x) - j(y)\rangle \geq 0$, where $j(x) \in J(x)$, $j(y) \in J(y)$;

(2) A is said to be m-d-accretive if A is d-accretive and $R(I + \lambda A) = X$ for $\forall \lambda > 0$;

(3) A is said to be accretive if for all $x, y \in D(A)$, $\langle Ax - Ay, j(x - y)\rangle \geq 0$, where $j(x - y) \in J(x - y)$;

(4) A is said to be m-accretive if A is accretive and $R(I + \lambda A) = X$ for $\forall \lambda > 0$.

For a mapping $A : D(A) \sqsubseteq X \to X$, we use $A^{-1}0$ to denote the set of zero points of A, that is, $A^{-1}0 = \{x \in D(A) : Ax = 0\}$. We use $F(A)$ to denote the set of fixed points of A, that is, $F(A) = \{x \in D(A) : Ax = x\}$. It is easy to see that d-accretive and accretive mappings are two different types of mappings in non-Hilbertian Banach space. Both the two mappings have been extensively studied via iterative methods in different framework of spaces; see [2]-[9] and the references therein.

Definition 1.3. [10] A mapping $T \subset X \times X^*$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$, for $\forall y_i \in Tx_i, i = 1,2$. The monotone mapping T is said to be maximal monotone if $R(J + \lambda T) = X^*$, $\forall \lambda > 0$.

Lemma 1.4. [10] *Let $T \subset X \times X^*$ be maximal monotone. Then*

(1) *$T^{-1}0$ is closed and convex subset of X;*

(2) *if $x_n \to x$ and $y_n \in Tx_n$ with $y_n \rightharpoonup y$, or $x_n \rightharpoonup x$ and $y_n \in Tx_n$ with $y_n \to y$, then $x \in D(T)$ and $y \in Tx$.*

Definition 1.5. [11] The Lyapunov functional $\varphi : X \times X \to R^+$ is defined as follows:

$$\varphi(x,y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2, \; \forall x, y \in X, j(y) \in J(y).$$

Definition 1.6. Let $B : X \to X$ be a mapping. Then

(1) B is said to be non-expansive if $\|Bx - By\| \leq \|x - y\|$ for $\forall x, y \in X$;

(2) B is said to be generalized non-expansive [12] if $F(B) \neq \emptyset$ and $\varphi(Bx, p) \leq \varphi(x, p)$, for $\forall x \in X$ and $p \in F(B)$.

It is easy to see that non-expansive and generalized non-expansive mappings are two different types of mappings.

Definition 1.7. [1, 13] (1) If X is a reflexive and strictly convex Banach space, then for each $x \in X$ there exists a unique element $v \in C$ such that $\|x - v\| = \inf\{\|x - y\| : y \in C\}$. Such an element v is denoted by $P_C x$ and P_C is called the metric projection of X onto C.

(2) Let X be a real reflexive, strictly convex and smooth Banach space, then for $\forall x \in X$, there exists a unique element $x_0 \in C$ satisfying $\varphi(x_0,x) = \inf\{\varphi(z,x) : z \in C\}$. In this case, $\forall x \in X$, define $\Pi_C : X \to C$ by $\Pi_C x = x_0$, and then Π_C is called the generalized projection from X onto C.

Definition 1.8. [14] Let X be a real smooth Banach space.

(1) Define $G : C \times X^* \to (0,+\infty]$ by:

$$G(x,y) = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2 + 2\rho f(x), \forall x \in C, y \in X^*,$$

where $\rho > 0$ and $f : C \to (-\infty,+\infty]$ is a proper convex and lower-semi-continuous function.

(2) $\Pi_C^f : X \to 2^C$ is called the generalized $f-$projection if

$$\Pi_C^f(y) = \{z \in C : G(z,Jy) \leq G(x,Jy), \forall x \in C\}, \quad \forall y \in X.$$

Definition 1.9. [12] Let Q be a mapping of X onto C. Then Q is said to be sunny if $Q(Q(x)+t(x-Q(x))) = Q(x)$, for all $x \in X$ and $t \geq 0$. A mapping $Q : X \to C$ is said to be a retraction if $Q(z) = z$ for every $z \in C$. If X is a smooth and strictly convex Banach space, then the sunny generalized non-expansive retraction of X onto C is uniquely decided, which is denoted by R_C.

Definition 1.10. [15] Let $\{C_n\}$ be a sequence of nonempty closed and convex subsets of X. Then

(1) $s-\liminf C_n$, which is called a strong lower limit, is defined as the set of all $x \in X$ such that there exists $x_n \in C_n$ for almost all n and it tends to x as $n \to \infty$ in the norm.

(2) $w-\limsup C_n$, which is called a weak upper limit, is defined as the set of all $x \in X$ such that there exists a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and $x_{n_k} \in C_{n_k}$ for every n_k and it tends to x as $n_k \to \infty$ in the weak topology;

(3) if $s-\liminf C_n = w-\limsup C_n$, then the common value is denoted by $\lim C_n$.

Lemma 1.11. [16] *Suppose that X is a real reflexive and strictly convex Banach space. If $\lim C_n$ exists and is not empty, then $\{P_{C_n}x\}$ converges weakly to $P_{\lim C_n}x$ for every $x \in X$. Moreover, if X has Property (H), the convergence is in norm.*

Lemma 1.12. [15] *Let $\{C_n\}$ be a decreasing sequence of closed and convex subsets of X, i.e., $C_n \subset C_m$ if $n \geq m$. Then $\{C_n\}$ converges in X and $\lim C_n = \bigcap_{n=1}^{\infty} C_n$.*

The class of d-accretive mappings has a close relationship with nonlinear evolution equations. A lot of work has been done on accretive mappings, however, fewer research works have been achieved compared to those for accretive mappings. One of the influential research works on d-accretive mappings is presented by Alber and Reich [17] in a real uniformly smooth and uniformly convex Banach space:

$$x_{n+1} = x_n - \alpha_n A x_n \tag{1.2}$$

and

$$x_{n+1} = x_n - \alpha_n \frac{Ax_n}{\|Ax_n\|}. \tag{1.3}$$

However, d-accretive mapping A in both (1.2) and (1.3) are required to be uniformly bounded and demi-continuous. Indeed, only weak convergence is obtained. Recently, Guan [18] removed the condition that "A is uniformly bounded", however, he assumed that "J is weakly sequentially continuous and A satisfies the following condition

$$\varphi(p,(I+r_nA)^{-1}x) \leq \varphi(p,x), \tag{1.4}$$

for $x \in X$ and $p \in A^{-1}0$." To be more precise, Guan studied the following iterative algorithm [18]:

$$\begin{cases} x_1 \in D(A), \\ y_n = (I + r_n A)^{-1} x_n, \\ C_n = \{v \in D(A) : \varphi(v, y_n) \leq \varphi(v, x_n)\}, \\ Q_n = \{v \in D(A) : \langle x_n - v, Jx_1 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \bigcap Q_n} x_1, \quad n \in N. \end{cases} \tag{1.5}$$

It was shown that $\{x_n\}$ converges strongly to an element in $A^{-1}0$. We note here that the restrictions are extremely strong since it is hard for us to find such an m-d-accretive mapping that is both demi-continuous and satisfies (1.4).

In 2014, Wei, Liu and Agarwal [2] made the following two contributions. One is that they removed the condition that the m-d-accretive mapping should be demi-continuous and uniformly bounded or should satisfy condition (1.4). The other one is that they investigated the study on finding zero points of m-d-accretive mappings to common zero points of a finitely many m-d-accretive mappings $\{A_i\}_{i=1}^m \subset X \times X$. One of the iterative algorithms in [2] is the following block combination method:

$$\begin{cases} x_1 \in X, \\ y_n = \sum_{i=1}^m \omega_{n,i} [\alpha_{n,i} x_n + (1 - \alpha_{n,i})(I + r_{n,i} A_i)^{-1} x_n], \\ x_{n+1} = \sum_{i=1}^m \eta_{n,i} [\beta_{n,i} x_n + (1 - \beta_{n,i})(I + s_{n,i} A_i)^{-1} y_n], \quad n \in N. \end{cases} \tag{1.6}$$

They proved that $\{x_n\}$ generated by (1.6) weakly converges to an element in $\bigcap_{i=1}^m A_i^{-1}0$.

In [2], they also studied the following block projection method:

$$\begin{cases} x_1 \in X, \\ u_n = \sum_{i=1}^m \omega_{n,i} [\alpha_{n,i} x_n + (1 - \alpha_{n,i})(I + r_{n,i} A_i)^{-1} x_n], \\ v_{n+1} = \sum_{i=1}^m \eta_{n,i} [\beta_{n,i} x_n + (1 - \beta_{n,i})(I + s_{n,i} A_i)^{-1} y_n], \\ H_1 = X, \\ H_{n+1} = \{z \in H_n : \varphi(v_n, z) \leq \varphi(x_n, z)\}, \\ x_{n+1} = R_{H_{n+1}} x_1, \quad n \in N. \end{cases} \tag{1.7}$$

They proved that $\{x_n\}$ generated by (1.7) strongly converges to an element in $\bigcap_{i=1}^m A_i^{-1}0$.

In 2016, by employing G-function and Π_C^f, Wei and Liu [14] presented two new iterative algorithms for finitely many m-d-accretive mappings $\{A_i\}_{i=1}^m \subset X^* \times X^*$. And, the computational errors are also considered. One of the two iterative algorithms is as follows

$$\begin{cases} x_1 \in X, r_{1,i} > 0, i = 1, 2, \cdots, m, \\ y_{n,i} = (J + r_{n,i} A_i J)^{-1} J x_n, i = 1, 2, \cdots, m, \\ Ju_n = \sum_{i=1}^m \beta_{n,i} J y_{n,i} + \beta_{n,m+1} J e_n, \quad n \in N, \\ x_{n+1} = J^{-1}[(1 - \alpha_n) J x_n + \alpha_n J u_n], \quad n \in N, \end{cases} \tag{1.8}$$

They proved that $\{x_n\}$ generated by (1.8) weakly converges to an element in $\bigcap_{i=1}^{m} A_i^{-1}0$. The other is

$$\begin{cases} x_1 \in X, r_{1,i} > 0, i = 1,2,\cdots,m, \\ y_{n,i} = (J + r_{n,i}A_iJ)^{-1}Jx_n, i = 1,2,\cdots,m, \\ Ju_{n,i} = \beta_{n,i}Jy_{n,i} + (1-\beta_{n,i})Je_n, i = 1,2,\cdots,m, \\ Jz_{n,i} = \alpha_{n,i}Jx_n + (1-\alpha_{n,i})Ju_{n,i}, i = 1,2,\cdots,m, \\ C_{1,i} = X, i = 1,2,\cdots,m, \\ C_1 = \bigcap_{i=1}^{m} C_{1,i}, \\ C_{n+1,i} = \{v \in C_n : G(v, Jz_{n,i}) \le (\alpha_{n,i} + \beta_{n,i} - \alpha_{n,i}\beta_{n,i})G(v, Jx_n) \\ \qquad\qquad +(1-\alpha_{n,i})(1-\beta_{n,i})G(v, Je_n)\}, i = 1,2,\cdots,m, \\ C_{n+1} = \bigcap_{i=1}^{m} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_1, \quad n \in N. \end{cases} \tag{1.9}$$

They also proved that $\{x_n\}$ generated by (1.9) strongly converges to an element in $\bigcap_{i=1}^{m} A_i^{-1}0$.

In Guan [18], Wei, Liu and Agarwal [2] and Wei and Liu [14], one may notice that it is not an easy thing to compute $G(v, Jz_{n,i})$ or $\varphi(v, z_{n,i})$, $\Pi_{C_{n+1}}^{f} x_1$ or $R_{H_{n+1}} x_1$. Can one reduce the computation complexity? In Section 2, we will give an answer to this question. We shall construct some new iterative algorithms and prove the iterative sequences converge strongly to the common zero point of an infinitely family of d-accretive mappings. New proof techniques are used and the restrictions on the parameters are mild compared to the existing works published recently. Moreover, the study on this topic is extended from single or a finite family of d-accretive mappings to that of infinite cases. In Section 3, we shall present a generalized (p,q)-Laplacian parabolic system from which we define m-d-accretive mappings and emphasize the meaningfulness of this topic.

The following lemma is needed in our paper.

Lemma 1.13. [19] *Let X be a real uniformly convex Banach space and $r \in (0,+\infty)$. Then there exists a continuous, strictly increasing and convex function $h : [0,2r] \to [0,+\infty)$ with $h(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)h(\|x-y\|),$$

for $\alpha \in [0,1]$, $x, y \in X$ with $\|x\| \le r$ and $\|y\| \le r$.

2. Strong Convergence Theorems

2.1. **Results for m-d-accretive mappings $\{A_i\}_{i=1}^{\infty} \subset X \times X$.** In this subsection, we always assume that the following conditions are satisfied:

(C_1) X is a real uniformly convex and uniformly smooth Banach space and $J : X \to X^*$ is the normalized duality mapping;

(C_2) $A_i : X \to X$ is an m-d-accretive mapping, for $i \in N$;

(C_3) $v \in X^*$ is a fixed element;

(C_4) $\{s_{n,i}\}$ is a real number sequence in $(0,+\infty)$ and $\{\tau_n\}$ is a real number sequence in $(0,1)$, for $i, n \in N$.

Algorithm 2.1.

Step 1. Choose $u_1 = v \in X^*$ and let $s_{1,i}$ and τ_1 be any positive constants, for $i \in N$. Set $n = 1$, and go to Step 2.

Step 2. Compute $w_{n,i} = (I + s_{n,i}JA_iJ^{-1})^{-1}u_n$, for $i \in N$. If $u_n = w_{n,i}$, for all $i \in N$, then stop; otherwise, construct the sets U_n and V_n as follows:

$$\begin{cases} U_1 = X^*, \\ U_{n+1,i} = \{z \in X^* : \langle w_{n,i} - z, J^{-1}(u_n - w_{n,i})\rangle \geq 0\}, \\ U_{n+1} = (\bigcap_{i=1}^{\infty} U_{n+1,i}) \bigcap U_n, \\ V_{n+1} = \{z \in U_{n+1} : \|v - z\|^2 \leq P_{U_{n+1}}^2(v) + \tau_{n+1}\}, \ n \in N, \end{cases}$$

and go to Step 3.

Step 3. Choose any element $u_{n+1} \in V_{n+1}$ and compute $\overline{u_n} = J^{-1}u_n$, for $n \in N$.

Step 4. Set $n = n + 1$, and return to Step 2.

Lemma 2.1. *If, in Algorithm 2.1, $u_n = w_{n,i}$, $\forall i \in N$, then $\overline{u_n} \in \bigcap_{i=1}^{\infty} A_i^{-1}0$.*

Proof. It is easy to check from Algorithm 2.1 that $u_n = w_{n,i}$, $\forall i \in N$ is equivalent to $u_n = (I + s_{n,i}JA_iJ^{-1})^{-1}u_n$. Then $s_{n,i}JA_iJ^{-1}u_n = 0$, which implies that from Lemma 1.1, $A_iJ^{-1}u_n = 0$, that is, $\overline{u_n} \in \bigcap_{i=1}^{\infty} A_i^{-1}0$. This completes the proof. □

Theorem 2.2. *If $\bigcap_{i=1}^{\infty} A_i^{-1}0 \neq \emptyset$, then under the following assumptions that $\liminf_n s_{n,i} > 0$ for $i \in N$ and $\limsup_{n\to\infty} \tau_n = 0$, the iterative sequence $\overline{u_n} \to u_0 \in \bigcap_{i=1}^{\infty} A_i^{-1}0$, where $u_0 = J^{-1}v_0$ and $v_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}v$, as $n \to \infty$.*

Proof. We split the proof into eight steps.

Step 1. $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \neq \emptyset$.

Since $\bigcap_{i=1}^{\infty} A_i^{-1}0 \neq \emptyset$, then there exists $q \in X$ such that $A_iq = 0$, for $i \in N$. In view of Lemma 1.1, there exists $p \in X^*$ such that $J^{-1}p = q$. Thus $(A_iJ^{-1})p = A_iq = 0$, for $i \in N$, which implies that $p \in \bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0$.

Step 2. $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset U_n$.

To this end, we shall use the mathematical induction. For $n = 1$, it is obvious that $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset U_1 = X^*$. Now, $\forall p \in \bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0$, since $w_{1,i} = (I + s_{1,i}JA_iJ^{-1})^{-1}u_1$, we find that $w_{1,i} + s_{1,i}JA_iJ^{-1}w_{1,i} = u_1$, which implies that from Lemma 1.1 that $s_{1,i}A_iJ^{-1}w_{1,i} = J^{-1}(u_1 - w_{1,i})$, for $i \in N$. From the definition of m-d-accretive mappings, we have

$$\begin{aligned} \langle w_{1,i} - p, J^{-1}(u_1 - w_{1,i})\rangle &= \langle w_{1,i} - p, s_{1,i}A_iJ^{-1}w_{1,i}\rangle \\ &= s_{1,i}\langle J(J^{-1}w_{1,i}) - J(J^{-1}p), A_iJ^{-1}w_{1,i} - A_iJ^{-1}p\rangle \\ &\geq 0. \end{aligned}$$

Then $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset U_{2,i}$, which ensures that $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset U_2$.

Suppose the result is true for $n = k + 1$. If $n = k + 2$, we have $\forall p \in \bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0$. Since

$$w_{k+1,i} = (I + s_{k+1,i}JA_iJ^{-1})^{-1}u_{k+1},$$

we have

$$w_{k+1,i} + s_{k+1,i}JA_iJ^{-1}w_{k+1,i} = u_{k+1},$$

which implies from Lemma 1.1 that $s_{k+1,i}A_iJ^{-1}w_{k+1,i} = J^{-1}(u_{k+1} - w_{k+1,i})$, for $i \in N$. It follows that

$$\begin{aligned}\langle w_{k+1,i} - p, J^{-1}(u_{k+1} - w_{k+1,i})\rangle &= \langle w_{k+1,i} - p, s_{k+1,i}A_iJ^{-1}w_{k+1,i}\rangle \\ &= s_{k+1,i}\langle J(J^{-1}w_{k+1,i}) - J(J^{-1}p), A_iJ^{-1}w_{k+1,i} - A_iJ^{-1}p\rangle \\ &\geq 0.\end{aligned}$$

Then $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset U_{k+2,i}$, which ensures that $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset U_{k+2}$. Therefore, $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset U_n$.

Step 3. $\{u_n\}$ is well-defined.

It is easy to check that

$$\{z \in X^* : \langle w_{n,i} - z, J^{-1}(u_n - w_{n,i})\rangle \geq 0\}$$

is closed and convex, which implies that U_n is closed and convex. Thus $P_{U_{n+1}}(v)$ is well-defined.

Since $P_{U_{n+1}}(v) = \inf_{z\in U_{n+1}} \|z - v\|$, we find from the definition of infimum that $V_n \neq \emptyset$. Then $\{u_n\}$ is well-defined.

Step 4. Let $v_n = P_{U_n}v$, for $n \in N$. Then $v_n \to v_0 = P_{\bigcap_{n=1}^{\infty} U_n}v$, as $n \to \infty$.

In fact, from Step 1 and Step 2, we know that $\emptyset \neq \bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset \bigcap_{n=1}^{\infty} U_n$. Lemma 1.12 ensures that $\lim U_n = \bigcap_{n=1}^{\infty} U_n \neq \emptyset$. Since X is uniformly convex and uniformly smooth, we see that X has Property (H) (see [1]). Using Lemma 1.11, $v_n \to v_0 = P_{\bigcap_{n=1}^{\infty} U_n}v$, as $n \to \infty$.

Step 5. $\|u_n - v_n\| \leq h(\tau_n)$ for $n \in N$, where v_n is the same as that in Step 4.

Since $u_n \in V_n \subset U_n$ and $d(v, U_n) = \|v - v_n\|$, we find from Lemma 1.13 and the fact that U_n is convex that, for $\forall k \in (0,1)$,

$$\begin{aligned}\|v_n - v\|^2 &\leq \|kv_n + (1-k)u_n - v\|^2 \\ &\leq k\|v_n - v\|^2 + (1-k)\|u_n - v\|^2 - k(1-k)h(\|v_n - u_n\|).\end{aligned}$$

Therefore,

$$kh(\|v_n - u_n\|) \leq \|u_n - v\|^2 - \|v_n - v\|^2 \leq \tau_n.$$

Letting $k \to 1$, one has $\|v_n - u_n\| \leq h^{-1}(\tau_n)$.

Step 6. $u_n \to v_0$ and $w_{n,i} \to v_0$ as $n \to \infty$, for $i \in N$, where v_0 is the same as that in Step 4.

Since $v_{n+1} \in U_{n+1}$, we find that $\langle w_{n,i} - v_{n+1}, J^{-1}(u_n - w_{n,i})\rangle \geq 0$. Therefore,

$$\begin{aligned}&\langle u_n - v_{n+1}, J^{-1}(u_n - w_{n,i})\rangle \\ &= \langle u_n - w_{n,i}, J^{-1}(u_n - w_{n,i})\rangle + \langle w_{n,i} - v_{n+1}, J^{-1}(u_n - w_{n,i})\rangle \\ &\geq \|u_n - w_{n,i}\|^2.\end{aligned}$$

Then

$$\|u_n - w_{n,i}\| \leq \|u_n - v_{n+1}\| \leq \|u_n - v_n\| + \|v_n - v_{n+1}\| \leq h(\tau_n) + \|v_n - v_{n+1}\|.$$

Since $v_n \to v_0$ and $\limsup_{n\to\infty} \tau_n = 0$, we find that $\limsup_{n\to\infty} \|u_n - w_{n,i}\| \leq 0$, for $i \in N$. Thus $u_n - w_{n,i} \to 0$, as $n \to \infty$, for $i \in N$. Moreover, from Step 5, $u_n - v_n \to 0$, as $n \to \infty$. Therefore, $u_n \to v_0$ and $w_{n,i} \to v_0$, as $n \to \infty$, for $i \in N$.

Step 7. $v_0 = P_{\bigcap_{n=1}^{\infty} U_n}v = P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}v$. This means that the best approximation of v in both $\bigcap_{n=1}^{\infty} U_n$ and $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0$ coincide.

First, we show $v_0 = P_{\bigcap_{n=1}^{\infty} U_n} v \in \bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0$. Since $w_{n,i} = (I + s_{n,i}JA_iJ^{-1})^{-1}u_n$, one has $w_{n,i} + s_{n,i}JA_iJ^{-1}w_{n,i} = u_n$, which implies that $s_{n,i}A_iJ^{-1}w_{n,i} = J^{-1}(u_n - w_{n,i})$, for $i,n \in N$. From Step 6, both $\{u_n\}$ and $\{w_{n,i}\}$ are bounded, for $i,n \in N$. Then Lemma 1.1 implies that $J^{-1}(u_n - w_{n,i}) \to 0$, as $n \to \infty$. Since $\liminf_n s_{n,i} > 0$, one has $A_iJ^{-1}w_{n,i} \to 0$, as $n \to \infty$. Since A_iJ^{-1} is maximal monotone from [2], we find that Lemma 1.4 implies that $v_0 \in \bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0$.

Secondly, we show $v_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0} v$. Since $\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0 \subset \bigcap_{n=1}^{\infty} U_n$, one has $\|P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}v - v\| \geq \|P_{\bigcap_{n=1}^{\infty} U_n}v - v\| = \|v_0 - v\|$. On the other hand, since $v_0 \in \bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0$, one has $\|P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}v - v\| \leq \|v_0 - v\|$. Therefore, $\|P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}v - v\| = \|P_{\bigcap_{n=1}^{\infty} U_n}v - v\|$. Since $P_{\bigcap_{n=1}^{\infty} U_n}v$ is unique, one has $P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}v = P_{\bigcap_{n=1}^{\infty} U_n}v$.

Step 8. $\overline{u_n} \to u_0 = J^{-1}v_0$, as $n \to \infty$, where $v_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}v$ as that in Step 7.

Following from Steps 7 and 6, we have $u_n \to v_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}v$, as $n \to \infty$. Then Lemma 1.1 implies that $\overline{u_n} = J^{-1}u_n \to J^{-1}v_0 = u_0$, as $n \to \infty$. This completes the proof. □

Theorem 2.3. *Let $\{\overline{u_n}\}$ be generated by Algorithm 2.1. Set $\lambda_n = \frac{\sum_{i=1}^{n+1} a_i\overline{u_i}}{\sum_{i=1}^{n+1} a_i}$ for $i \in N$ and suppose $\sum_{i=1}^{n} a_i \to \infty$, as $n \to \infty$. Under the assumptions of Theorem 2.2, we obtain the result of ergodic convergence in the sense that $\lambda_n \to u_0 \in \bigcap_{i=1}^{\infty} A_i^{-1}0$, where $u_0 = J^{-1}v_0$ and $v_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ^{-1})^{-1}0}(v)$, as $n \to \infty$.*

Proof. The proof is similar to that of Step 5 of Theorem 2.2 in [20]. □

2.2. **Results for m-d-accretive mappings $\{A_i\}_{i=1}^{\infty} \subset X^* \times X^*$.** In this subsection, we always assume that the following conditions are satisfied:

(A_1) X is a real uniformly convex and uniformly smooth Banach space and $J : X \to X^*$ is the normalized duality mapping;

(A_2) $A_i \subset X^* \times X^*$ is m-d-accretive, for $i \in N$;

(A_3) $u \in X$ is a fixed element;

(A_4) $\{r_{n,i}\}$ is a real number sequence in $(0, +\infty)$ and $\{\delta_n\}$ is a real number sequence in $(0,1)$, for $i,n \in N$.

Algorithm 2.2.

Step 1. Set $x_1 = u \in X$ and let $r_{1,i}$ and δ_1 be any positive constants for $i \in N$. Set $n = 1$, and go to Step 2.

Step 2. Compute $y_{n,i} = (I + r_{n,i}J^{-1}A_iJ)^{-1}x_n$, for each $i \in N$. If $x_n = y_{n,i}$, for $i \in N$, then stop. Otherwise, construct the sets X_n and Y_n as follows:

$$\begin{cases} X_1 = X, \\ X_{n+1,i} = \{z \in X : \langle y_{n,i} - z, J(x_n - y_{n,i})\rangle \geq 0\}, \\ X_{n+1} = (\bigcap_{i=1}^{\infty} X_{n+1,i})\bigcap X_n, \\ Y_{n+1} = \{z \in X_{n+1} : \|u - z\|^2 \leq P^2_{X_{n+1}}(u) + \delta_{n+1}\},\ n \in N, \end{cases}$$

and go to Step 3.

Step 3. Choose any element $x_{n+1} \in Y_{n+1}$ and compute $\overline{x_n} = Jx_n$, for $n \in N$.

Step 4. Set $n = n + 1$ and return to Step 2.

Lemma 2.4. *If, in Algorithm 2.2, $x_n = y_{n,i}$, $\forall i \in N$, then $\overline{x_n} \in \bigcap_{i=1}^{\infty} A_i^{-1}0$.*

Proof. It is easy to check from Algorithm 2.2 that $x_n = y_{n,i} = (I + r_{n,i}J^{-1}A_iJ)^{-1}x_n$. Then $r_{n,i}J^{-1}A_iJx_n = 0$, which implies from Lemma 1.1 that $\overline{x_n} = Jx_n \in \bigcap_{i=1}^{\infty} A_i^{-1}0$. This completes the proof. □

Theorem 2.5. *If $\bigcap_{i=1}^{\infty} A_i^{-1}0 \neq \emptyset$, then, under the following assumptions, $\liminf_n r_{n,i} > 0$ for $i \in N$ and $\limsup_{n\to\infty} \delta_n = 0$, the iterative sequence $\overline{x_n} \to x_0 \in \bigcap_{i=1}^{\infty} A_i^{-1}0$, where $x_0 = Jz_0$ and $z_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0}u$, as $n \to \infty$.*

Proof. Similar to Theorem 2.2, we also split the proof into eight steps.

Step 1. $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \neq \emptyset$.

Since $\bigcap_{i=1}^{\infty} A_i^{-1}0 \neq \emptyset$, there exists $v^* \in X^*$ such that $A_iv^* = 0$, for $i \in N$. In view of Lemma 1.1, there exists $y \in X$ such that $Jy = v^*$. Thus $(A_iJ)y = A_iv^* = 0$, for $i \in N$, which implies that $y \in \bigcap_{i=1}^{\infty}(A_iJ)^{-1}0$.

Step 2. $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \subset X_n$.

To this end, we shall use the mathematical induction. For $n = 1$, it is obvious that $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \subset X_1 = X$. Now, $\forall p \in \bigcap_{i=1}^{\infty}(A_iJ)^{-1}0$, since $y_{1,i} = (I + r_{1,i}J^{-1}A_iJ)^{-1}x_1$, we have $y_{1,i} + r_{1,i}J^{-1}A_iJy_{1,i} = x_1$, which implies that from Lemma 1.1, $r_{1,i}J(J^{-1}A_iJ)y_{1,i} = J(x_1 - y_{1,i})$, for $i \in N$. So, from the definition of m-d-accretive mappings, we know that

$$\begin{aligned}\langle y_{1,i} - p, J(x_1 - y_{1,i})\rangle &= \langle y_{1,i} - p, r_{1,i}(A_iJy_{1,i} - A_iJp)\rangle \\ &= r_1\langle J^{-1}(Jy_{1,i}) - J^{-1}(Jp), A_iJy_{1,i} - A_iJp\rangle \\ &\geq 0.\end{aligned}$$

Then $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \subset X_{2,i}$, which implies that $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \subset X_2$. Suppose that the result is true for $n = k+1$. If $n = k+2$, we have $\forall p \in \bigcap_{i=1}^{\infty}(A_iJ)^{-1}0$. Since $y_{k+1,i} = (I + r_{k+1,i}J^{-1}A_iJ)^{-1}x_{k+1}$, we have $y_{k+1,i} + r_{k+1,i}J^{-1}A_iJy_{k+1,i} = x_{k+1}$, which implies that from Lemma 1.1, $r_{k+1,i}J(J^{-1}A_iJ)y_{k+1,i} = J(x_{k+1} - y_{k+1,i})$, for $i \in N$. It follows that

$$\begin{aligned}\langle y_{k+1,i} - p, J(x_{k+1} - y_{k+1,i})\rangle &= \langle y_{k+1,i} - p, r_{k+1,i}(A_iJy_{k+1,i} - A_iJp)\rangle \\ &= r_{k+1,i}\langle J^{-1}(Jy_{k+1,i}) - J^{-1}(Jp), A_iJy_{k+1,i} - A_iJp\rangle \\ &\geq 0.\end{aligned}$$

Then $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \subset X_{k+2,i}$, which implies that $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \subset X_{k+2}$.

Step 3. $\{x_n\}$ is well-defined.

It is easy to check that

$$\{z \in X : \langle y_{n,i} - z, J(x_n - y_{n,i})\rangle \geq 0\}$$

is closed and convex, which implies that X_n is closed and convex. Thus $P_{X_{n+1}}(u)$ is well-defined. Since $P_{X_{n+1}}(u) = inf_{z\in X_{n+1}}\|z - u\|$, it follows from the definition of infimum that $Y_n \neq \emptyset$. Then $\{x_n\}$ is well-defined.

Step 4. Let $z_n = P_{X_n}u$, for $\forall n \in N$. Then $z_n \to z_0 = P_{\bigcap_{n=1}^{\infty} X_n}u$, as $n \to \infty$.

In fact, from Step 1 and Step 2, we know that $\emptyset \neq \bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \subset \bigcap_{n=1}^{\infty} X_n$. Lemma 1.12 ensures that $\lim X_n \neq \emptyset$. Fromrom Lemma 1.11, $z_n \to z_0 = P_{\bigcap_{n=1}^{\infty} X_n}u$, as $n \to \infty$.

Step 5. $\|x_n - z_n\| \leq h^{-1}(\delta_n)$ for $n \in N$, where z_n is the same as that in Step 4.

Since $x_n \in Y_n \subset X_n$ and $d(u,X_n) = \|u - z_n\|$, we find from Lemma 1.13 and the fact that X_n is convex that, $\forall \alpha \in (0,1)$,

$$\begin{aligned}\|z_n - u\|^2 &\leq \|\alpha z_n + (1-\alpha)x_n - u\|^2 \\ &\leq \alpha\|z_n - u\|^2 + (1-\alpha)\|x_n - u\|^2 - \alpha(1-\alpha)h(\|z_n - x_n\|).\end{aligned}$$

Therefore, $\alpha h(\|z_n - x_n\|) \leq \delta_n$. Letting $\alpha \to 1$, one has $\|z_n - x_n\| \leq h^{-1}(\delta_n)$.

Step 6. $x_n \to z_0$ and $y_{n,i} \to z_0$ as $n \to \infty$, for $i \in N$, where z_0 is the same as that in Step 4.

Since $z_{n+1} \in X_{n+1}$, one has $\langle y_{n,i} - z_{n+1}, J(x_n - y_{n,i})\rangle \geq 0$. Therefore,

$$\langle x_n - z_{n+1}, J(x_n - y_{n,i})\rangle = \langle x_n - y_{n,i}, J(x_n - y_{n,i})\rangle + \langle y_{n,i} - z_{n+1}, J(x_n - y_{n,i})\rangle \geq \|x_n - y_{n,i}\|^2.$$

Then

$$\begin{aligned}\|x_n - y_{n,i}\| &\leq \|x_n - z_{n+1}\| \\ &\leq \|x_n - z_n\| + \|z_n - z_{n+1}\| \\ &\leq h^{-1}(\delta_n) + \|z_n - z_{n+1}\|.\end{aligned}$$

Since $z_n \to z_0$ and $\limsup_{n\to\infty} \delta_n = 0$, one has $\limsup_{n\to\infty} \|x_n - y_{n,i}\| = 0$, for $i \in N$ and $x_n - y_{n,i} \to 0$ as $n \to \infty$, for $i \in N$. Following Steps 4 and 5, $x_n \to z_0$ and $y_{n,i} \to z_0$, as $n \to \infty$, for $i \in N$.

Step 7. $z_0 = P_{\bigcap_{n=1}^{\infty} X_n} u = P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0} u$. This means that the best approximation of v in both $\bigcap_{n=1}^{\infty} X_n$ and $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0$ are coincided.

First, we show that $z_0 = P_{\bigcap_{n=1}^{\infty} X_n} u \in \bigcap_{i=1}^{\infty}(A_iJ)^{-1}0$. Since $y_{n,i} = (I + r_{n,i}J^{-1}A_iJ)^{-1}x_n$, one has $y_{n,i} + r_{n,i}J^{-1}A_iJy_{n,i} = x_n$, which implies that $r_{n,i}A_iJy_{n,i} = J(x_n - y_{n,i})$, for $i,n \in N$. From Step 6, both $\{x_n\}$ and $\{y_{n,i}\}$ are bounded, for $i,n \in N$. Then Lemma 1.1 implies that $J(x_n - y_{n,i}) \to 0$, as $n \to \infty$. Since $\liminf_{n\to\infty} r_{n,i} > 0$, one has $A_iJy_{n,i} \to 0$, as $n \to \infty$. Since A_iJ is maximal monotone from [14], we see that Lemma 1.4 implies that $z_0 \in \bigcap_{i=1}^{\infty}(A_iJ)^{-1}0$. Since $\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0 \subset \bigcap_{n=1}^{\infty} X_n$, one has $\|P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0}u - u\| \geq \|P_{\bigcap_{n=1}^{\infty} X_n}u - u\| = \|z_0 - u\|$. On the other hand, since $z_0 \in \bigcap_{i=1}^{\infty}(A_iJ)^{-1}0$, one has $\|P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0}u - u\| \leq \|z_0 - u\|$. Therefore

$$\|P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0}u - u\| = \|P_{\bigcap_{n=1}^{\infty} X_n}u - u\|.$$

Since $P_{\bigcap_{n=1}^{\infty} X_n}u$ is unique, one has $P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0}u = P_{\bigcap_{n=1}^{\infty} X_n}u$.

Step 8. $\overline{x_n} \to x_0 = Jz_0$, as $n \to \infty$, where $z_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0}u$.

Follows from Steps 7 and 6, we have $x_n \to z_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0}u$, as $n \to \infty$. Using Lemma 1.1, we find that $\overline{x_n} = Jx_n \to Jz_0 = x_0$, as $n \to \infty$. This completes the proof. □

Theorem 2.6. *Let $\{\overline{x_n}\}$ be generated by Algorithm 2.2. Set $\eta_n = \frac{\sum_{i=1}^{n+1} b_i\overline{x_i}}{\sum_{i=1}^{n+1} b_i}$, for $i \in N$ and suppose $\sum_{i=1}^{n} b_i \to \infty$, as $n \to \infty$. Then under the assumptions of Theorem 2.5, we obtain the result of ergodic convergence in the sense that $\eta_n \to x_0 \in \bigcap_{i=1}^{\infty} A_i^{-1}0$, where $x_0 = Jz_0$ and $z_0 = P_{\bigcap_{i=1}^{\infty}(A_iJ)^{-1}0}(u)$, as $n \to \infty$.*

Proof. The proof is similar to that of Step5 of Theorem 2.2 in [20]. □

Remark 2.7. We notice that C_n in (1.5) and H_n in (1.7) involve the calculation of the value of Lyapunov functional φ. And, $C_{n+1,i}$ in (1.9) involves the calculation of the value of bifunction G defined in Definition 1.8. However, our algorithms 2.1 and 2.2 avoid such expensive labor.

Remark 2.8. $\{x_n\}$ in (1.5) involves the evaluation of the generalized projection, $\{x_n\}$ in (1.7) involves the evaluation of the sunny generalized retraction and $\{x_n\}$ in (1.9) involves the evaluation of the f-projection. However, in our algorithms 2.1 and 2.2, only metric projection is involved. From the viewpoint of theory, Algorithms 2.1 and 2.2 are easy for computation and realization.

Remark 2.9. The normalized duality mapping J is no longer needed to be weakly sequentially continuous as that in [18], or [2] or [14]. The m-d-accretive mapping is no longer needed to be uniformly bounded and demi-continuous as that in [17], and it need not to satisfy condition (1.4) as that in [18].

Remark 2.10. Iterative constructions of zero points of m-d-accretive mappings or a finite family of m-d-accretive mappings in [17], [18], [14] and [2] are extended to an infinite family of m-d-accretive mappings in this paper.

Remark 2.11. From Theorems 2.2, 2.3, 2.5 and 2.6, we may find the restrictions on parameters are rather weaker than corresponding studies, see, e.g., [17], [18], [14] and [2].

3. Applications

In this section, we shall present some applications of the results presented in Section 2.

3.1. **Applications in special Banach spaces.** Let $X = L^p(\Omega)$, l^p, or $W^{1,p}(\Omega)$, $1 < p < +\infty$, and let X^* be the dual space of X. It is known that X is a uniformly convex and uniformly smooth Banach space. Therefore, Theorems 2.2, 2.3, 2.5 and 2.6 are applicable in these special Banach spaces.

3.2. **Applications in Hilbert space.** In a Hilbert space, Theorems 2.2 and 2.3 are identical and Theorems 2.5 and 2.6 are identical, too. Then we have the following theorems.

Theorem 3.1. *Let H be a Hilbert space and let $A_i \subset H \times H$ be m-d-accretive mappings, for $i \in N$. Let $u \in H$ be a fixed element. Assume that $\{r_{n,i}\}$ is a real number sequence in $(0,+\infty)$ and $\{\delta_n\}$ is a real number sequence in $(0,1)$, for $i,n \in N$. Let $\{x_n\}$ be a sequence generated in the following process*

$$\begin{cases} x_1 = u \in H, \\ X_1 = H, \\ y_{n,i} = (I + r_{n,i}A_i)^{-1}x_n, i \in N, \\ X_{n+1,i} = \{z \in H : \langle y_{n,i} - z, x_n - y_{n,i}\rangle \geq 0\}, i \in N, \\ X_{n+1} = (\bigcap_{i=1}^{\infty} X_{n+1,i}) \bigcap X_n, \\ Y_{n+1} = \{z \in X_{n+1} : \|u - z\|^2 \leq P^2_{X_{n+1}}(u) + \delta_{n+1}\}, \\ x_{n+1} \in Y_{n+1},\ n \in N. \end{cases} \tag{3.1}$$

If $\bigcap_{i=1}^{\infty} A_i^{-1}0 \neq \emptyset$, $\liminf_n r_{n,i} > 0$, for $i \in N$ and $\limsup_{n\to\infty} \delta_n = 0$, then $x_n \to z_0$, where $z_0 = P_{\bigcap_{i=1}^{\infty} A_i^{-1}0} u$, as $n \to \infty$.

Theorem 3.2. *Let $\{x_n\}$ be generated by (3.1). Set $\eta_n = \frac{\sum_{i=1}^{n+1} b_i x_i}{\sum_{i=1}^{n+1} b_i}$, for $i \in N$ and suppose that $\sum_{i=1}^{n} b_i \to \infty$, as $n \to \infty$. Under the assumptions of Theorem 3.1, we obtain the result of ergodic convergence in the sense that $\eta_n \to z_0$ and $z_0 = P_{\bigcap_{i=1}^{\infty} A_i^{-1}0}(u)$, as $n \to \infty$.*

3.3. Parabolic systems involving the p-Laplacian.

Remark 3.3. In [2], we demonstrate an example of m-d-accretive mappings which has connection with the generalized p-Laplacian elliptic boundary value problem. Now, we present another example involving (p,q)-Laplacian parabolic systems which is a special case in [21]:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - div[(C_1(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] + \varepsilon_1|u|^{r-2}u + g_1(x,u,\nabla u) = f_1(x,t),\ (x,t)\in\Omega\times(0,T) \\ \frac{\partial v(x,t)}{\partial t} - div[(C_2(x,t)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v] + \varepsilon_2|v|^{s-2}v + g_2(x,v,\nabla v) = f_2(x,t),\ (x,t)\in\Omega\times(0,T) \\ -<\vartheta,(C_1(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u> \in \beta_x(u),\ (x,t)\in\Gamma\times(0,T) \\ -<\vartheta,(C_2(x,t)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v> \in \beta_x(v),\ (x,t)\in\Gamma\times(0,T) \\ u(x,0)=u(x,T), v(x,0)=v(x,T),\ x\in\Omega. \end{cases} \tag{3.2}$$

In (3.2), Ω is a bounded conical domain of a Euclidean space R^N $(N\geq 1)$ with its boundary $\Gamma\in C^1$, (see [21]). ϑ is the exterior normal derivative of Γ and T is a positive constant. $0\leq C_1(x,t)\in V_1 = L^p(0,T;W^{1,p}(\Omega))$, $0\leq C_2(x,t)\in V_2 = L^q(0,T;W^{1,q}(\Omega))$, $f_1(x,t)\in W_1 = L^{max\{p,p'\}}(0,T;L^{max\{p,p'\}}(\Omega))$ and $f_2(x,t)\in W_2 = L^{max\{q,q'\}}(0,T;L^{max\{q,q'\}}(\Omega))$ are given functions. ε_1 and ε_2 are nonnegative constants. Moreover, β_x is the subdifferential of φ_x, where $\varphi_x=\varphi(x,\cdot):R\to R$ for $x\in\Gamma$ and $\varphi:\Gamma\times R\to R$ is a given function. $\frac{2N}{N+1}<r\leq\min\{p,p'\}$ and $\frac{2N}{N+1}<s\leq\min\{q,q'\}$.

To discuss (3.2), the following assumptions are considered in [21]:

Assumption 1. Green's formula is available.

Assumption 2. For each $x\in\Gamma$, $\varphi_x=\varphi(x,\cdot):R\to R$ is proper, convex and lower-semi-continuous function and $\varphi_x(0)=0$.

Assumption 3. $0\in\beta_x(0)$ and for each $t\in R$, the function $x\in\Gamma\to(I+\lambda\beta_x)^{-1}(t)\in R$ is measurable for $\lambda>0$.

Assumption 4. Suppose that $g_i:\Omega\times R^{N+1}\to R$ $(i=1,2)$ satisfies the following conditions:

(a) Carathéodory's conditions.

$$x\to g_i(x,r)\ is\ measurable\ on\ \Omega,\ for\ all\ r\in R^{N+1};$$

$$r\to g_i(x,r)\ is\ continuous\ on\ R^{N+1},\ for\ almost\ all\ x\in\Omega.$$

(b) Growth condition.

$$|g_1(x,r_1,\cdots,r_{N+1})|^{\max\{p,p'\}}\leq|h_1(x,t)|^p+b_1|r_1|^p,$$

where $(r_1,r_2,\cdots,r_{N+1})\in R^{N+1}$, $h_1(x,t)\in W_1$ and b_1 *is a positive constant*;

$$|g_2(x,r_1,\cdots,r_{N+1})|^{\max\{q,q'\}}\leq|h_2(x,t)|^q+b_2|r_1|^q,$$

where $(r_1,r_2,\cdots,r_{N+1})\in R^{N+1}$, $h_2(x,t)\in W_2$ and b_2 *is a positive constant*;

(c) Monotone Condition. g_i is monotone in the following sense:

$$(g_i(x,r_1,\cdots,r_{N+1})-g_i(x,t_1,\cdots,t_{N+1}))\geq(r_1-t_1),$$

for all $x\in\Omega$ and $(r_1,\cdots,r_{N+1}),(t_1,\cdots,t_{N+1})\in R^{N+1}$.

Lemma 3.4. [10] *Let X be a real Banach space. If $B : X \to 2^{X^*}$ is an everywhere defined, monotone and hemi-continuous mapping, then B is maximal monotone.*

Definition 3.5. Let $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Define the mapping $\widehat{B_{p,r}} : W^{1,p'}(\Omega) \to (W^{1,p'}(\Omega))^*$ by

$$\langle w,\widehat{B_{p,r}}u\rangle$$
$$=\int_\Omega <(C_1(x,t)+|\nabla(|u|^{p'-1}sgnu\|u\|_{p'}^{2-p'})|^2)^{\frac{p-2}{2}}\nabla(|u|^{p'-1}sgnu\|u\|_{p'}^{2-p'}),\nabla(|w|^{p'-1}sgnw\|w\|_{p'}^{2-p'})>dx,$$

for any $u,w\in W^{1,p'}(\Omega)$, where $<\cdot,\cdot>$ denotes inner-product in R^N. Similarly, define the mapping $\widehat{B_{q,s}} : W^{1,q'}(\Omega)\to (W^{1,q'}(\Omega))^*$ by

$$\langle w,\widehat{B_{q,s}}u\rangle$$
$$=\int_\Omega <(C_2(x,t)+|\nabla(|u|^{q'-1}sgnu\|u\|_{q'}^{2-q'})|^2)^{\frac{q-2}{2}}\nabla(|u|^{q'-1}sgnu\|u\|_{q'}^{2-q'}),\nabla(|w|^{q'-1}sgnw\|w\|_{q'}^{2-q'})>dx,$$

for any $v,w\in W^{1,q'}(\Omega)$.

Proposition 3.6. *The mapping $\widehat{B_{p,r}} : W^{1,p'}(\Omega)\to(W^{1,p'}(\Omega))^*$ is maximal monotone. And, the mapping $\widehat{B_{q,s}} : W^{1,q'}(\Omega)\to(W^{1,q'}(\Omega))^*$ is maximal monotone.*

Proof. Step 1. $\widehat{B_{p,r}}$ is everywhere defined.

In fact, for $u,v\in W^{1,p'}(\Omega)$, we have

$$|\langle v,\widehat{B_{p,r}}u\rangle|$$
$$\le\int_\Omega|\nabla(|u|^{p'-1}sgnu\|u\|_{p'}^{2-p'})|^{p-1}|\nabla(|v|^{p'-1}sgnv\|v\|_{p'}^{2-p'})|dx$$
$$\le Const.\|u\|_{p'}^{(p-1)(2-p')}\|v\|_{p'}^{2-p'}\int_\Omega|\nabla u|^{p-1}|u|^{2-p}|\nabla v||v|^{p'-2}dx$$
$$\le Const.\|u\|_{p'}^{(p-1)(2-p')}\|v\|_{p'}^{2-p'}(\int_\Omega|\nabla u|^{p}|u|^{p'-p}dx)^{\frac{1}{p'}}(\int_\Omega|\nabla v|^{p}|v|^{p'-p}dx)^{\frac{1}{p}}$$
$$\le Const.\|u\|_{p'}^{(p-1)(2-p')}\|v\|_{p'}^{2-p'}(\int_\Omega|\nabla u|^{p'}dx)^{\frac{p}{(p')^2}}(\int_\Omega|u|^{p'})^{\frac{p'-p}{p'^2}}(\int_\Omega|\nabla v|^{p'})^{\frac{1}{p'}}(\int_\Omega|v|^{p'}dx)^{\frac{p'-p}{p'p}}$$
$$\le Const.\|u\|_{1,p'}^{p-1}\|v\|_{1,p'}.$$

Thus $\widehat{B_{p,r}}$ is everywhere defined.

Step 2. $\widehat{B_{p,r}}$ is monotone.

The monotonicity of $\widehat{B_{p,r}}$ follows from its definition.

Step 3.$\widehat{B_{p,r}}$ is hemi-continuous.

To show that $\widehat{B_{p,r}}$ is hemi-continuous, it suffices to prove that for $u,v,w\in W^{1,p'}(\Omega)$ and $t\in[0,1]$, $\langle w,\widehat{B_{p,r}}(u+tv)-\widehat{B_{p,r}}u\rangle\to 0$, as $t\to 0$.

In fact, by using Lebesque's dominated convergence theorem, we find that

$$0\le\lim_{t\to 0}|\langle w,\widehat{B_{p,r}}(u+tv)-\widehat{B_{p,r}}u\rangle|$$
$$\le\int_\Omega\lim_{t\to 0}|(C_1(x,t)+|\nabla(|u+tv|^{p'-1}sgn(u+tv)\|u+tv\|_{p'}^{2-p'})|^2)^{\frac{p-2}{2}}\nabla(|u+tv|^{p'-1}sgn(u+tv)\|u+tv\|_{p'}^{2-p'})$$
$$-(C_1(x,t)+|\nabla(|u|^{p'-1}sgnu\|u\|_{p'}^{2-p'})|^2)^{\frac{p-2}{2}}\nabla(|u|^{p'-1}sgnu\|u\|_{p'}^{2-p'})|\times|\nabla(|w|^{p'-1}sgnw\|w\|_{p'}^{2-p'})|dx=0.$$

Step 4. $\widehat{B_{p,r}}$ is maximal monotone.

Lemma 3.4 implies that $\widehat{B_{p,r}}$ is maximal monotone. Similarly, we get the result that $\widehat{B_{q,s}}$ is maximal monotone. This completes the proof. □

Proposition 3.7. *Let $1 < p \leq 2$ and $1 < q \leq 2$. $B_{p,r} : L^{p'}(0,T;W^{1,p'}(\Omega)) \to (L^{p'}(0,T;W^{1,p'}(\Omega)))^*$ defined by*

$$\langle w, B_{p,r}u\rangle = \int_0^T \langle w, \widehat{B_{p,r}}u\rangle dt,$$

for $u,w \in L^{p'}(0,T;W^{1,p'}(\Omega))$, is maximal monotone. And, the mapping $\widehat{B_{q,s}} : L^{q'}(0,T;W^{1,q'}(\Omega)) \to (L^{q'}(0,T;W^{1,q'}(\Omega)))^*$ defined by

$$\langle w, B_{q,s}v\rangle = \int_0^T \langle w, \widehat{B_{q,s}}v\rangle dt,$$

for $v,w \in L^{q'}(0,T;W^{1,q'}(\Omega))$, is maximal monotone.

Proof. The result follows from Proposition 3.6. □

Lemma 3.8. [10] *If B_1 and B_2 are two maximal monotone mappings in X such that $(intD(B_1)) \bigcap D(B_2) \neq \emptyset$, then $B_1 + B_2$ is maximal monotone. Here $(intD(B_1))$ indicates the interior of $D(B_1)$.*

Proposition 3.9. [21] *Define $S_1 : D(S_1) = \{u(x,t) \in L^{p'}(0,T;W^{1,p'}(\Omega)) : \frac{\partial u}{\partial t} \in (L^{p'}(0,T;W^{1,p'}(\Omega)))^*, u(x,0) = u(x,T)\} \to (L^{p'}(0,T;W^{1,p'}(\Omega)))^*$ by*

$$S_1 u = \frac{\partial u}{\partial t},$$

for $u \in D(S_1)$. Define $S_2 : D(S_2) = \{v(x,t) \in L^{q'}(0,T;W^{1,q'}(\Omega)) : \frac{\partial v}{\partial t} \in (L^{q'}(0,T;W^{1,q'}(\Omega)))^*, v(x,0) = v(x,T)\} \to (L^{p'}(0,T;W^{1,p'}(\Omega)))^*$ by

$$S_2 v = \frac{\partial v}{\partial t},$$

for $v \in D(S_2)$. Then both S_1 and S_2 are linear maximal monotone.

By using Lemma 3.8 and Propositions 3.7 and 3.9, we have the following result immediately.

Proposition 3.10. *Let $1 < p \leq 2$ and $1 < q \leq 2$. The mapping $U_1 : D(U_1) \subset L^{p'}(0,T;W^{1,p'}(\Omega)) \to (L^{p'}(0,T;W^{1,p'}(\Omega)))^*$ defined by $U_1 w = B_{p,r}w + S_1 w$, for $w \in D(U_1)$, is maximal monotone. And, the mapping $U_2 : D(U_2) \subset L^{q'}(0,T;W^{1,q'}(\Omega)) \to (L^{q'}(0,T;W^{1,q'}(\Omega)))^*$ defined by*

$$U_2 v = B_{q,s}v + S_2 v,$$

for $v \in D(U_2)$, is maximal monotone.

Remark 3.11. For $1 < p \leq 2$ and $1 < q \leq 2$, there exists a maximal monotone extension of U_1 from $L^{p'}(0,T;W^{1,p'}(\Omega))$ to $L^p(0,T;W^{1,p}(\Omega))$, which is denoted by $\widetilde{U_1}$. And, there exists a maximal monotone extension of U_2 from $L^{q'}(0,T;W^{1,q'}(\Omega))$ to $L^q(0,T;W^{1,q}(\Omega))$, which is denoted by $\widetilde{U_2}$.

The following two theorems can be obtained as those in [2].

Theorem 3.12. *For $1 < p \le 2$, define $A_1 : L^p(0,T;W^{1,p}(\Omega)) \to L^p(0,T;W^{1,p}(\Omega))$ by*

$$A_1 u = \widetilde{U_1} J_1^{-1} u,$$

for $u \in L^p(0,T;W^{1,p}(\Omega))$. Then A_1 is m-d-accretive. For $1 < q \le 2$, define $A_2 : L^q(0,T;W^{1,q}(\Omega)) \to L^q(0,T;W^{1,q}(\Omega))$ by

$$A_2 v = \widetilde{U_2} J_2^{-1} v,$$

for $v \in L^q(0,T;W^{1,q}(\Omega))$. Then A_2 is m-d-accretive. Here $J_1 : L^{p'}(0,T;W^{1,p'}(\Omega)) \to (L^{p'}(0,T;W^{1,p'}(\Omega)))^*$ and $J_2 : L^{q'}(0,T;W^{1,q'}(\Omega)) \to (L^{q'}(0,T;W^{1,q'}(\Omega)))^*$ are the normalized duality mappings.

Theorem 3.13. $A_1^{-1}0 = \{u \in L^p(0,T;W^{1,p}(\Omega)) : u(x,t) \equiv Const.\}$ *and* $A_2^{-1}0 = \{v \in L^q(0,T;W^{1,q}(\Omega)) : v(x,t) \equiv Const.\}$.

Remark 3.14. From Theorems 3.12 and 3.13, we know the restriction that $\bigcap_i A_i^{-1}0 \neq \emptyset$ imposed on the m-d-accretive mapping in Theorems 2.2, 2.3, 2.5 and 2.6 is valid.

Remark 3.15. If (3.2) is reduced to the following one

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - div[(C_1(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] = 0, \ (x,t) \in \Omega \times (0,T) \\ \frac{\partial v(x,t)}{\partial t} - div[(C_2(x,t) + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v] = 0, \ (x,t) \in \Omega \times (0,T) \\ - < \vartheta, (C_1(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u >= 0, \ (x,t) \in \Gamma \times (0,T) \\ - < \vartheta, (C_2(x,t) + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v >= 0, \ (x,t) \in \Gamma \times (0,T) \\ u(x,0) = u(x,T), v(x,0) = v(x,T), \ x \in \Omega, \end{cases} \tag{3.3}$$

then it is not difficult to see that $\{u \in L^p(0,T;W^{1,p}(\Omega)), v \in L^q(0,T;W^{1,q}(\Omega)) : u(x,t) \equiv Const., v(x,t) \equiv Const.\} = A_1^{-1}0 \bigcap A_2^{-1}0$ is exactly the solution of (3.3), from which we can not only see the connections between the zeros of an m-d-accretive mapping and the nonlinear systems but also see that the work on designing the iterative algorithms to approximate zeros of nonlinear m-d-accretive mappings is meaningful.

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