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GENERALIZED YOSIDA APPROXIMATION OPERATOR WITH AN APPLICATION TO A SYSTEM OF YOSIDA INCLUSIONS

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Abstract. In this article, we introduce and study a generalized Yosida approximation operator associated to $H(\cdot,\cdot)$-co-accretive mappings. We show the convergence of generalized Yosida approximation operator based on the concept of graph convergence and resolvent operator convergence. We establish a relationship between the graph convergence for $H(\cdot,\cdot)$-co-accretive operators and generalized Yosida approximation operators. Finally, the existence and uniqueness of solution of a system of generalized Yosida inclusions under some mild conditions is established in q-uniformly smooth Banach spaces.

Keywords. Graph convergence; Resolvent operator; Iterative algorithm; Yosida approximation operator; System of Yosida inclusions.

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1. INTRODUCTION

The theory of classical variational inequalities is one of the most important mathematical subjects. It has many different generalizations with different approaches. One of the remarkable generalizations is the variational inclusion. Since a large number of problems arising in science, engineering, social sciences, management, finance, operations research and optimizations, etc., can be formulated as a variational inclusion; see, for example, [1]-[7] and the references therein, the variational inclusions have recently extensively investigated using novel and innovative techniques in various directions. The notion of monotone operators was first independently introduced and studied by Zarantonello [8] and Minty [9]. A significant interest has been shown by number of researchers as they have firm relations with the

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following evolution equation

$$\begin{cases} \dfrac{dx}{dt} + A(x) = 0; \\ \quad x(0) = x_0, \end{cases}$$

which is the model of many physical problems of practical applications. It is very difficult to solve these types of models, if the involving function A is not continuous. To overcome this problem, a natural step is to find a sequence of Lipschitz functions that approximate A in some sense. This idea was introduced by Yosida. On the other hand, it is well known that two quite useful single-valued lipschiz continuous operators can be associated with a monotone operators, namely its resolvent operator and its Yosida approximation operator. The monotone operators on Hilbert spaces can be regularized into single-valued Lipschitzian monotone operators via a process known as the Yosida approximation [10]-[13]. Further, this process was extended to study the problems in Banach spaces; see, for example, [14]-[17]. The Yosida approximation operators are useful to approximate the solutions of variational inclusion problems using resolvent operators. Recently, many authors implemented Yosida approximation operators to study some of variational inclusion problems using different approaches; see, for example, [18]-[22] and the references therein.

The paper is organized as follows. In Section 2, we recall basic definitions and properties. In Section 3, we introduce a generalized Yosida approximation operator associated to $H(\cdot,\cdot)$-co-accretive mapping and discuss its Lipschitz continuity and strong monotonicity. In Section 4, we investigate the graph convergence of $H(\cdot,\cdot)$-co-accretive mappings and its relationship with the convergence of generalized Yosida approximation operators. An example is constructed to demonstrate the graph convergence of $H(\cdot,\cdot)$-co-accretive mappings and the convergence of generalized Yosida approximation operators. In the last section, we consider a system of generalized Yosida inclusions and establish the existence results. We also suggest an iterative algorithm and discuss its convergence analysis.

2. Preliminaries

In this section, we collect some basic notions and auxiliary results needed in the subsequent sections. Let X be a real Banach space with norm $\|\cdot\|$. Let X^* be the topological dual of X and let d be the metric induced by norm $\|\cdot\|$. Let $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^* and let $CB(X)$ (respectively 2^X) be the family of all nonempty closed and bounded subsets (respectively, all non empty subsets) of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \right\}, \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that $J_q(x) = \|x\|^{q-1} J_2(x), \forall x (\neq 0) \in X$. In the sequel, we assume that X is a real Banach space such that J_q is single-valued. If $X \equiv H$, a real Hilbert space then J_2 becomes the identity mapping on X. The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

A Banach space X is said to be uniformly smooth if $\lim\limits_{t\to 0} \frac{\rho_X(t)}{t} = 0$; X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_X(t) \leq ct^q$, $q > 1$. Note that J_q is single-valued, if X is uniformly smooth. In the study of characterstic inequalities in q-uniformly smooth Banach spaces, Xu [23] proved the following lemma.

Lemma 2.1. *Let $q > 1$ be a real number and let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q, \quad \forall x, y \in X.$$

Definition 2.2. A mapping $A : X \to X$ is said to be

(i) accretive if

$$\langle Ax - Ay, J_q(x-y)\rangle \geq 0,\ \forall x, y \in X;$$

(ii) strictly accretive if

$$\langle Ax - Ay, J_q(x-y)\rangle > 0,\ \forall x, y \in X,$$

and the equality holds if and only if $x = y$;

(iii) δ-strongly accretive if there exists a constant $\delta > 0$ such that

$$\langle Ax - Ay, J_q(x-y)\rangle \geq \delta\|x-y\|^q,\ \forall x, y \in X;$$

(iv) β-relaxed accretive if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, J_q(x-y)\rangle \geq (-\beta)\|x-y\|^q,\ \forall x, y \in X;$$

(v) μ-cocoercive if there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, J_q(x-y)\rangle \geq \mu\|Ax-Ay\|^q,\ \forall x, y \in X;$$

(vi) γ-relaxed cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, J_q(x-y)\rangle \geq (-\gamma)\|Ax-Ay\|^q,\ \forall x, y \in X;$$

(vii) σ-Lipschitz continuous if there exists a constant $\sigma > 0$ such that

$$\|Ax - Ay\| \leq \sigma\|x-y\|,\ \forall x, y \in X;$$

(viii) η-expansive, if there exists a constant $\eta > 0$ such that

$$\|Ax - Ay\| \geq \eta\|x-y\|,\ \forall x, y \in X;$$

if $\eta = 1$, then it is expansive.

Definition 2.3. Let $H : X \times X \to X$ and $A, B : X \to X$ be three single-valued mappings. Then

(i) $H(A,\cdot)$ is said to be μ_1-cocoercive with respect to A if there exists a constant $\mu_1 > 0$ such that $\langle H(Ax,u) - H(Ay,u), J_q(x-y)\rangle \geq \mu_1\|Ax-Ay\|^q,\ \forall x, y, u \in X;$

(ii) $H(\cdot,B)$ is said to be γ_1-relaxed cocoercive with respect to B if there exists a constant $\gamma_1 > 0$ such that $\langle H(u,Bx) - H(u,By), J_q(x-y)\rangle \geq (-\gamma_1)\|Bx-By\|^q,\ \forall x, y, u \in X;$

(iii) $H(A,B)$ is said to be symmetric cocoercive with respect to A and B if $H(A,\cdot)$ is cocoercive with respect to A and $H(\cdot,B)$ is relaxed cocoercive with respect to B;

(iv) $H(A,\cdot)$ is said to be α_1-strongly accretive with respect to A if there exists a constant $\alpha_1 > 0$ such that $\langle H(Ax,u) - H(Ay,u), J_q(x-y)\rangle \geq \alpha_1\|x-y\|^q,\ \forall x, y, u \in X;$

(v) $H(\cdot,B)$ is said to be β_1-relaxed accretive with respect to B if there exists a constant $\beta_1 > 0$ such that
$\langle H(u,Bx)-H(u,By),J_q(x-y)\rangle \geq (-\beta_1)\|x-y\|^q,\ \forall\, x,y,u \in X;$

(vi) $H(A,B)$ is said to be symmetric accretive with respect to A and B if $H(A,\cdot)$ is strongly accretive with respect to A and $H(\cdot,B)$ is relaxed accretive with respect to B;

(vii) $H(A,\cdot)$ is said to be ξ_1-Lipschitz continuous with respect to A if there exists a constant $\xi_1 > 0$ such that
$\|H(Ax,u)-H(Ay,u)\| \leq \xi_1\|x-y\|,\ \forall x,y,u \in X;$

(viii) $H(\cdot,B)$ is said to be ξ_2-Lipschitz continuous with respect to B if there exists a constant $\xi_2 > 0$ such that
$\|H(u,Bx)-H(u,By)\| \leq \xi_2\|x-y\|,\ \forall x,y,u \in X.$

Definition 2.4. Let $f,g : X \to X$ be two single-valued mappings and let $M : X\times X \to 2^X$ be a multi-valued mapping. Then

(i) $M(f,\cdot)$ is said to be α-strongly accretive with respect to f, if there exists a constant $\alpha > 0$ such that
$\langle u-v,J_q(x-y)\rangle \geq \alpha\|x-y\|^q,\ \forall x,y,w \in X$ and $\forall u \in M(f(x),w),\ v \in M(f(y),w);$

(ii) $M(\cdot,g)$ is said to be β-relaxed accretive with respect to g, if there exists a constant $\beta > 0$ such that
$\langle u-v,J_q(x-y)\rangle \geq (-\beta)\|x-y\|^q,\ \forall x,y,w \in X$ and $\forall u \in M(w,g(x)), v \in M(w,g(y));$

(iii) $M(f,g)$ is said to be symmetric accretive with respect to f and g, if $M(f,\cdot)$ is strongly accretive with respect to f and $M(\cdot,g)$ is relaxed accretive with respect to g.

Definition 2.5. Let $A,B,f,g : X \to X$ and $H : X\times X \to X$ be the single-valued mappings. Let $M : X\times X \to 2^X$ be a multi-valued mapping. The mapping M is said to be $H(\cdot,\cdot)$-co-accretive with respect to A,B,f and g, if $H(A,B)$ is symmetric cocoercive with respect to A and B, $M(f,g)$ is symmetric accretive with respect to f and g and $(H(A,B)+\lambda M(f,g))(X) = X$, for every $\lambda > 0$.

Lemma 2.6. [24] *Let $A,B,f,g : X \to X$ and $H : X\times X \to X$ be the single-valued mappings. Let $M : X\times X \to 2^X$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A,B,f and g. Let A be η-expansive and B be σ-Lipschitz continuous such that $\alpha > \beta, \mu > \gamma$ and $\eta > \sigma$. Then the mapping $[H(A,B)+\lambda M(f,g)]^{-1}$ is single-valued, for all $\lambda > 0$.*

Definition 2.7. [24] Let $A,B,f,g : X \to X$ and $H : X\times X \to X$ be the single-valued mappings. Let $M : X\times X \to 2^X$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A,B,f and g. The resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : X \to X$ is defined by

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) = [H(A,B)+\lambda M(f,g)]^{-1}(u),\ \forall u \in X,\ \lambda > 0. \tag{2.1}$$

Note that the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$, defined in (2.1) is θ-Lipschitz continuous, where

$$\theta = \frac{1}{\lambda(\alpha-\beta)+(\mu\eta^q-\gamma\sigma^q)}. \tag{2.2}$$

3. GENERALIZED YOSIDA APPROXIMATION OPERATORS

In this section, we define the generalized Yosida approximation operator associated to $H(\cdot,\cdot)$-co-accretive mappings and discuss some of its properties.

Definition 3.1. The generalized Yosida approximation operator $J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : X \to X$ is defined as

$$J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) = \frac{1}{\lambda}\left[I - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}\right](u),\ \forall u \in X,\ \lambda > 0, \tag{3.1}$$

where, I is the identity mapping on X and $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ is the resolvent operator defined by (2.1).

Lemma 3.2. *Let $A,B,f,g : X \to X$ and $H : X \times X \to X$ be the single-valued mappings. Suppose that $M : X \times X \to 2^X$ is an $H(\cdot,\cdot)$-co-accretive mapping with respect to A,B,f and g. Let A be η-expansive and let B be σ-Lipschitz continuous such that $\alpha > \beta, \mu > \gamma$ and $\eta > \sigma$. Then the generalized Yosida approximation operator $J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ defined in (3.1) is*

(i) κ-Lipschitz continuous;

(ii) ς-strongly monotone;

where $\kappa = \frac{1}{\lambda}(1+\theta),\ \varsigma = \frac{1}{\lambda}(1-\theta),\ \theta = \frac{1}{\lambda(\alpha-\beta)+(\mu\eta^q-\gamma\sigma^q)}$ and $\lambda > 0$.

Proof. (i). Let u,v be any given points in X. It follows from the definition of the generalized Yosida approximation operator and the Lipschitz continuity of resolvent operators that

$$\begin{aligned}
\left\|J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\right\| &= \frac{1}{\lambda}\left\|\left[I(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)\right] - \left[I(v) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\right]\right\| \\
&\leq \frac{1}{\lambda}\left[\|u-v\| + \left\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\right\|\right] \\
&\leq \frac{1}{\lambda}\left[\|u-v\| + \theta\|u-v\|\right] \\
&= \frac{1}{\lambda}(1+\theta)\|u-v\|,
\end{aligned}$$

i.e.,

$$\|J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\| \leq \kappa\|u-v\|.$$

Thus, generalized Yosida approximation operator $J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ is κ-Lipschitz continuous.

(ii) Let u,v be any given points in X. Using the definition of the generalized Yosida approximation operator, we get

$$\begin{aligned}
&\left\langle J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v), J_q(u-v)\right\rangle \\
&= \frac{1}{\lambda}\left\langle \left(I(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)\right) - \left(I(v) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\right), J_q(u-v)\right\rangle \\
&= \frac{1}{\lambda}\left[\left\langle u-v, J_q(u-v)\right\rangle - \left\langle R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v), J_q(u-v)\right\rangle\right] \\
&\geq \frac{1}{\lambda}\left[\|u-v\|^q - \left\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\right\|\|u-v\|^{q-1}\right].
\end{aligned}$$

Now using the Lipschitz continuity of resolvent operators, we have

$$\begin{aligned}\left\langle J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v), J_q(u-v)\right\rangle &\geq \frac{1}{\lambda}\left[\|u-v\|^q - \theta\|u-v\|^q\right]\\ &= \frac{1}{\lambda}(1-\theta)\|u-v\|^q,\end{aligned}$$

i.e.,

$$\left\langle J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v), J_q(u-v)\right\rangle \geq \varsigma\|u-v\|^q.$$

Thus, generalized Yosida approximation operator $J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ is ς-strongly monotone. This completes the proof. □

4. Convergence of the generalized Yosida approximation operator

In this section, we discuss the convergence of the generalized Yosida approximation operator associated to $H(\cdot,\cdot)$-co-accretive mapping. We also illustrate an example to support the convergence result.

Let $M : X\times X \to 2^X$ be a multi-valued mapping. The graph of the mapping M is defined by

$$graph(M) = \{((x,y),z) : z \in M(x,y)\}.$$

Definition 4.1. [24] Let $A,B,f,g : X \to X$ and $H : X\times X \to X$ be the single-valued mappings. Let $M_n, M : X\times X \to 2^X$ be $H(\cdot,\cdot)$-co-accretive mappings, for $n = 0,1,2,\cdots$. The sequence $\{M_n\}$ is said to be graph convergence to M, denoted by $M_n \underset{\to}{G} M$, if for every $\Big((f(x),g(x)),z\Big) \in graph(M)$, there exists a sequence $\{(f(x_n),g(x_n)),z_n\} \in graph(M_n)$ such that

$$f(x_n) \to f(x), g(x_n) \to g(x) \text{ and } z_n \to z \text{ as } n \to \infty.$$

Theorem 4.2. [24] *Let $A,B,f,g : X \to X$ be the single-valued mappings and let $M_n,M : X\times X \to 2^X$ be $H(\cdot,\cdot)$-co-accretive mappings with respect to A,B,f and g. Let $H : X\times X \to X$ be a single-valued mapping such that*

(i) $H(A,B)$ is ξ_1-Lipschitz continuous with respect to A and ξ_2-Lipschitz continuous with respect to B;

(ii) f is τ-expansive.

Then $M_n \underset{\to}{G} M$, if and only if

$$R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(u) \to R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u), \forall u \in X, \lambda > 0,$$

where

$$R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(u) = [H(A,B)+\lambda M_n(f,g)]^{-1}(u), \text{ and } R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) = [H(A,B)+\lambda M(f,g)]^{-1}(u).$$

In the following theorem, we establish the relationship between the convergence of the generalized Yosida approximation operator and the graph convergence for the $H(\cdot,\cdot)$-co-accretive mapping.

Theorem 4.3. *Let $A,B,f,g : X \to X$ be the single-valued mappings and let $M_n,M : X\times X \to 2^X$ be $H(\cdot,\cdot)$-co-accretive mappings with respect to A,B,f and g. Let $H : X\times X \to X$ be a single-valued mapping such that the conditions (i) and (ii) of the Theorem 4.2 hold. Then $M_n \underset{\to}{G} M$, if and only if*

$$J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) \to J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x), \forall x \in X, \lambda > 0,$$

where

$$J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) = \Big[I - R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}\Big](x), J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x) = \Big[I - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}\Big](x),$$

and $R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}, R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ *are same as defined in Theorem 4.2.*

Proof. Suppose that $M_n \underset{\to}{G} M$. For any given $x \in X$, let

$$z_n = J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) \text{ and } z = J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x).$$

Then $z = J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x) = \frac{1}{\lambda}\Big[I - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}\Big](x)$. It follows that

$$x - \lambda z = R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x) = [H(A,B) + \lambda M(f,g)]^{-1}(x),$$

which implies that

$$H(A,B)(x-\lambda z) + \lambda M(f,g)(x-\lambda z) = x$$

$$\text{i.e., } \frac{1}{\lambda}\Big[x - H(A,B)(x-\lambda z)\Big] \in M\Big(f(x-\lambda z), g(x-\lambda z)\Big)$$

or

$$\Big((f(x-\lambda z), g(x-\lambda z)), \frac{1}{\lambda}(x - H(A(x-\lambda z), B(x-\lambda z)\Big) \in graph(M).$$

Then by the definition of graph convergence, there exists a sequence $\Big\{\Big(f(z'_n), g(z'_n)\Big), y'_n\Big\} \in graph(M_n)$ such that

$$f(z'_n) \to f(x-\lambda z), g(z'_n) \to g(x-\lambda z) \text{ and } y'_n \to \frac{1}{\lambda}\Big(x - H(A(x-\lambda z), B(x-\lambda z)\Big) \text{ as } n \to \infty. \quad (4.1)$$

Since $y'_n \in M_n\Big(f(z'_n), g(z'_n)\Big)$, we have

$$H\Big(A(z'_n), B(z'_n)\Big) + \lambda y'_n \in \Big[H(A,B) + \lambda M_n(f,g)\Big](z'_n).$$

It follows that

$$\begin{aligned} z'_n &= R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}\Big[H\Big(A(z'_n), B(z'_n)\Big) + \lambda y'_n\Big] \\ &= \Big(I - \lambda J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}\Big)\Big[H\Big(A(z'_n), B(z'_n)\Big) + \lambda y'_n\Big], \end{aligned}$$

which implies that

$$\frac{1}{\lambda} z'_n = \frac{1}{\lambda} H\Big(A(z'_n), B(z'_n)\Big) + y'_n - J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}\Big[H\Big(A(z'_n), B(z'_n)\Big) + \lambda y'_n\Big].$$

Note that

$$\begin{aligned} \|z_n - z\| &= \Big\|J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) - z\Big\| \\ &= \Big\|J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) + \frac{1}{\lambda} z'_n - \frac{1}{\lambda} z'_n - z\Big\| \\ &= \Big\|J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) + \frac{1}{\lambda}\Big(H(A(z'_n), B(z'_n))\Big) + y'_n \\ &\quad - J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}\Big[H\Big(A(z'_n), B(z'_n)\Big) + \lambda y'_n\Big] - \frac{1}{\lambda} z'_n - z\Big\| \\ &\le \Big\|J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) - J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}\Big[H\Big(A(z'_n), B(z'_n)\Big) + \lambda y'_n\Big]\Big\| \\ &\quad + \Big\|\frac{1}{\lambda} H\Big(A(z'_n), B(z'_n)\Big) + y'_n - \frac{1}{\lambda} z'_n - z\Big\|. \end{aligned}$$

Using the Lipschitz continuity of the generalized Yosida approximation operator, we get

$$\begin{aligned}\|z_n - z\| &\leq \kappa\|x - H(A(z'_n),B(z'_n)) - \lambda y'_n\| + \left\|\tfrac{1}{\lambda}H\Big(A(z'_n),B(z'_n)\Big) + y'_n - \tfrac{1}{\lambda}x\right\| \\ &+ \left\|\tfrac{1}{\lambda}z'_n - \tfrac{1}{\lambda}x + z\right\| \\ &= \left(\kappa + \tfrac{1}{\lambda}\right)\left\|x - H(A(z'_n),B(z'_n)) - \lambda y'_n\right\| + \tfrac{1}{\lambda}\|z'_n - x + \lambda z\| \\ &= \left(\kappa + \tfrac{1}{\lambda}\right)\left\|x - H(A(z'_n),B(z'_n)) + H(A,B)(x-\lambda z) - H(A,B)(x-\lambda z) - \lambda y'_n\right\| \\ &+ \tfrac{1}{\lambda}\|z'_n - x + \lambda z\| \\ &\leq \left(\kappa + \tfrac{1}{\lambda}\right)\left\|x - H(A,B)(x-\lambda z) - \lambda y'_n\right\| + \left(\kappa + \tfrac{1}{\lambda}\right)\Big\|H(A,B)(x-\lambda z) \\ &- H\Big(A(z'_n),B(z'_n)\Big)\Big\| + \tfrac{1}{\lambda}\|z'_n - x + \lambda z\|.\end{aligned} \tag{4.2}$$

Since H is ξ_1-Lipschitz continuous with respect to A and ξ_2-Lipschitz continuous with respect to B, we have

$$\begin{aligned}\|H(A,B)(x-\lambda z) - H(A(z'_n),B(z'_n))\| &= \|H(A(x-\lambda z),B(x-\lambda z)) - H(A(z'_n),B(z'_n))\| \\ &\leq \|H(A(x-\lambda z),B(x-\lambda z)) - H(A(x-\lambda z),B(z'_n))\| \\ &+ \|H(A(x-\lambda z),B(z'_n)) - H(A(z'_n),B(z'_n))\| \\ &\leq (\xi_1 + \xi_2)\|x - \lambda z - z'_n\|.\end{aligned} \tag{4.3}$$

Thus, it follows from (4.2) and (4.3) that

$$\begin{aligned}\|z_n - z\| &\leq \left(\kappa + \tfrac{1}{\lambda}\right)\left\|x - H(A(x-\lambda z),B(x-\lambda z)) - \lambda y'_n\right\| \\ &+ \left[\left(\kappa + \tfrac{1}{\lambda}\right)(\xi_1+\xi_2) + \tfrac{1}{\lambda}\right]\left\|x - \lambda z - z'_n\right\|.\end{aligned} \tag{4.4}$$

Since f is τ-expansive, we have

$$\|f(z'_n) - f(x-\lambda z)\| \geq \tau\|z'_n - (x-\lambda z)\| \geq 0. \tag{4.5}$$

Since $f(z'_n) \to f(x-\lambda z)$ as $n \to \infty$. By (4.5), we have $z'_n \to x - \lambda z$ as $n \to \infty$. Also from (4.1), we have $y'_n \to \frac{1}{\lambda}\Big(x - H(A(x-\lambda z),B(x-\lambda z))\Big)$ as $n \to \infty$. Thus, it follows from (4.4) that $\|z_n - z\| \to 0$ as $n \to \infty$, which implies that

$$J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) \to J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x).$$

Conversely, suppose that

$$J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) \to J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x),\ \forall x \in X, \lambda > 0.$$

For any $\Big((f(x),g(x)),y\Big) \in graph(M)$, we have $y \in M\Big(f(x),g(x)\Big)$, which in turn implies

$$H(A(x),B(x)) + \lambda y \in \Big[H(A,B) + \lambda M(f,g)\Big](x).$$

Hence

$$x = R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}\Big[H(A(x),B(x)) + \lambda y\Big] = \Big[I - \lambda J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}\Big][H(A(x),B(x)) + \lambda y]. \tag{4.6}$$

Letting

$$x_n = \Big[I - \lambda J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}\Big][H(A(x),B(x)) + \lambda y], \tag{4.7}$$

we have

$$\frac{1}{\lambda}\Big[H(A(x),B(x)) - H(A(x_n),B(x_n)) + \lambda y\Big] \in M_n\Big(f(x_n),g(x_n)\Big).$$

Let $y_n^{'} = \frac{1}{\lambda}\Big[H(A(x),B(x)) - H(A(x_n),B(x_n)) + \lambda y\Big]$. Note that

$$\begin{aligned}\|y_n^{'} - y\| &= \Big\|\tfrac{1}{\lambda}[H(A(x),B(x)) - H(A(x_n),B(x_n)) + \lambda y] - y\Big\| \\ &= \tfrac{1}{\lambda}\Big\|H(A(x),B(x)) - H(A(x_n),B(x_n))\Big\|.\end{aligned}$$

Then by using the same argument as in (4.3), we get

$$\|y_n^{'} - y\| \leq \frac{(\xi_1 + \xi_2)}{\lambda}\|x_n - x\|. \tag{4.8}$$

From (4.6) and (4.7), we have

$$\begin{aligned}\|x_n - x\| &= \Big\|\Big[I - \lambda J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}\Big][H(A(x),B(x)) + \lambda y] - \Big[I - \lambda J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}\Big][H(A(x),B(x)) + \lambda y]\Big\| \\ &= \lambda\Big\|\Big[J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)} - J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}\Big][H(A(x),B(x)) + \lambda y]\Big\|.\end{aligned}$$

Since $J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)} \to J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$, we have $\|x_n - x\| \to 0$ as $n \to \infty$. From (4.8), we have $\|y_n^{'} - y\| \to 0$ as $n \to \infty$. Hence $M_n \underset{\to}{G} M$. This completes the proof. □

Remark 4.4. One can easily verify that the convergence of the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)} \to R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ and the convergence of generalized Yosida approximation operator $J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)} \to J^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ are equivalent, if and only if $M_n \underset{\to}{G} M$.

Example 4.5. Letting $X = \mathbb{R}^2$, we define the inner product by

$$\langle (x_1,x_2),(y_1,y_2)\rangle = x_1y_1 + x_2y_2,\ \forall (x_1,x_2),(y_1,y_2) \in \mathbb{R}^2.$$

Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be the mappings defined by

$$A(x_1,x_2) = \Big(\frac{x_1}{3}, \frac{x_2}{4}\Big),\ \forall (x_1,x_2) \in \mathbb{R}^2,$$

$$B(x_1,x_2) = \Big(-x_1, -\frac{3}{2}x_2\Big),\ \forall (x_1,x_2) \in \mathbb{R}^2.$$

Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by $H(A(x),B(x)) = A(x) + B(x)$, $\forall x \in \mathbb{R}^2$. Then for any $u \in \mathbb{R}^2$, we have

$$\begin{aligned}\langle H(A(x),u) - H(A(y),u), x - y\rangle &= \langle A(x) - A(y), x - y\rangle \\ &= \Big\langle \Big(\frac{1}{3}(x_1 - y_1), \frac{1}{4}(x_2 - y_2)\Big), (x_1 - y_1, x_2 - y_2)\Big\rangle \\ &= \frac{1}{3}(x_1 - y_1)^2 + \frac{1}{4}(x_2 - y_2)^2,\end{aligned}$$

and

$$\begin{aligned}\|A(x) - A(y)\|^2 &= \langle A(x) - A(y), A(x) - A(y)\rangle \\ &= \Big\langle \Big(\frac{1}{3}(x_1 - y_1), \frac{1}{4}(x_2 - y_2)\Big), \Big(\frac{1}{3}(x_1 - y_1), \frac{1}{4}(x_2 - y_2)\Big)\Big\rangle \\ &= \frac{1}{9}(x_1 - y_1)^2 + \frac{1}{16}(x_2 - y_2)^2,\end{aligned}$$

which imply that

$$\begin{aligned}\langle H(A(x),u)-H(A(y),u),x-y\rangle &= \frac{1}{3}(x_1-y_1)^2+\frac{1}{4}(x_2-y_2)^2\\ &= \frac{48(x_1-y_1)^2+36(x_2-y_2)^2}{144}\\ &= 3\Big[\frac{16(x_1-y_1)^2+12(x_2-y_2)^2}{144}\Big]\\ &\geq 3\Big[\frac{16(x_1-y_1)^2+9(x_2-y_2)^2}{144}\Big]\\ &= 3\|A(x)-A(y)\|^2,\end{aligned}$$

i.e.,

$$\langle H(A(x),u)-H(A(y),u),x-y\rangle \geq 3\|A(x)-A(y)\|^2.$$

Hence, $H(A,B)$ is 3-cocoercive with respect to A and

$$\begin{aligned}\langle H(u,B(x))-H(u,B(y)),x-y\rangle &= \langle B(x)-B(y),x-y\rangle\\ &= \Big\langle\Big(-(x_1-y_1),-\frac{3}{2}(x_2-y_2)\Big),(x_1-y_1,x_2-y_2)\Big\rangle\\ &= -\Big[(x_1-y_1)^2+\frac{3}{2}(x_2-y_2)^2\Big],\end{aligned}$$

and

$$\begin{aligned}\|B(x)-B(y)\|^2 &= \langle B(x)-B(y),B(x)-B(y)\rangle\\ &= \Big\langle\Big(-(x_1-y_1),-\frac{3}{2}(x_2-y_2)\Big),\Big(-(x_1-y_1),-\frac{3}{2}(x_2-y_2)\Big)\Big\rangle\\ &= (x_1-y_1)^2+\frac{9}{4}(x_2-y_2)^2,\end{aligned}$$

which imply that

$$\begin{aligned}\langle H(u,B(x))-H(u,B(y)),x-y\rangle &= -\Big[(x_1-y_1)^2+\frac{3}{2}(x_2-y_2)^2\Big]\\ &= -\Big[\frac{4(x_1-y_1)^2+6(x_2-y_2)^2}{4}\Big]\\ &\geq -\Big[\frac{4(x_1-y_1)^2+9(x_2-y_2)^2}{4}\Big]\\ &= (-1)\|B(x)-B(y)\|^2,\end{aligned}$$

i.e.,

$$\langle H(u,B(x))-H(u,B(y)),x-y\rangle \geq (-1)\|B(x)-B(y)\|^2.$$

Hence, $H(A,B)$ is 1-relaxed cocoercive with respect to B. Thus, $H(A,B)$ is symmetric cocoercive with respect to A and B. Now, we show the symmetric accretivity of $M(f,g)$. Let $f,g:\mathbb{R}^2\to\mathbb{R}^2$ be the single-valued mappings such that

$$f(x_1,x_2)=\Big(\frac{1}{2}x_1-x_2,x_1+\frac{1}{3}x_2\Big),\ \forall(x_1,x_2)\in\mathbb{R}^2,$$

$$g(x_1,x_2)=\Big(\frac{1}{3}x_1+\frac{1}{3}x_2,-\frac{1}{3}x_1+\frac{1}{4}x_2\Big),\ \forall(x_1,x_2)\in\mathbb{R}^2.$$

Let $M:\mathbb{R}^2\times\mathbb{R}^2\times\rightarrow\mathbb{R}^2$ be a mapping defined by $M(f(x),g(x))=f(x)-g(x)$. Now, for any $w\in\mathbb{R}^2$,

$$\begin{aligned}\langle M(f(x),w)-M(f(y),w),x-y\rangle &= \langle f(x)-f(y),x-y\rangle\\ &= \Big\langle\Big(\frac{1}{2}(x_1-y_1)-(x_2-y_2),(x_1-y_1)+\frac{1}{3}(x_2-y_2)\Big),\\ &(x_1-y_1,x_2-y_2)\Big\rangle\\ &= \Big[\frac{1}{2}(x_1-y_1)^2+\frac{1}{3}(x_2-y_2)^2\Big],\end{aligned}$$

and $\|x-y\|^2=\langle(x_1-y_1,x_2-y_2),(x_1-y_1,x_2-y_2)\rangle=(x_1-y_1)^2+(x_2-y_2)^2$, which imply that

$$\begin{aligned}\langle M(f(x),w)-M(f(y),w),x-y\rangle &= \Big[\frac{1}{2}(x_1-y_1)^2+\frac{1}{3}(x_2-y_2)^2\Big]\\ &\geq \frac{1}{3}\Big[(x_1-y_1)^2+(x_2-y_2)^2\Big]\\ &= \frac{1}{3}\|x-y\|^2,\end{aligned}$$

i.e.,

$$\langle u-v,x-y\rangle\geq\frac{1}{3}\|x-y\|^2,\ \forall x,\ y\in\ \mathbb{R}^2, u\in M(f(x),w)\ \text{ and } v\in M(f(y),w).$$

Hence, $M(f,g)$ is $\frac{1}{3}$-strongly accretive with respect to f and

$$\begin{aligned}\langle M(w,g(x))-M(w,g(y)),x-y\rangle &= -\langle g(x)-g(y),x-y\rangle\\ &= -\Big\langle\Big(\frac{1}{3}(x_1-y_1)+\frac{1}{3}(x_2-y_2),-\frac{1}{3}(x_1-y_1)\\ &+\frac{1}{4}(x_2-y_2)\Big),(x_1-y_1,x_2-y_2)\Big\rangle\\ &= -\Big[\frac{1}{3}(x_1-y_1)^2+\frac{1}{4}(x_2-y_2)^2\Big],\end{aligned}$$

and

$$\|x-y\|^2=\langle(x_1-y_1,x_2-y_2),(x_1-y_1,x_2-y_2)\rangle=(x_1-y_1)^2+(x_2-y_2)^2,$$

which implies that

$$\begin{aligned}\langle M(w,g(x))-M(w,g(y)),x-y\rangle &= -\Big[\frac{1}{3}(x_1-y_1)^2+\frac{1}{4}(x_2-y_2)^2\Big]\\ &\geq -\frac{1}{3}\Big[(x_1-y_1)^2+(x_2-y_2)^2\Big]\\ &= -\frac{1}{3}\|x-y\|^2,\end{aligned}$$

i.e.,

$$\langle u-v,x-y\rangle\geq-\frac{1}{3}\|x-y\|^2,\ \forall x,\ y\in\ \mathbb{R}^2, u\in M(w,g(x))\ \text{ and } v\in M(w,g(y)).$$

Hence, $M(f,g)$ is $\frac{1}{3}$-relaxed accretive with respect to g. Thus, $M(f,g)$ is symmetric accretive with respect to f and g. Also for any $x\in\mathbb{R}^2$, we have

$$\begin{aligned}[H(A,B)+\lambda M(f,g)](x) &= H(A(x),B(x))+\lambda M(f(x),g(x))\\ &= (A(x)+B(x))+\lambda(f(x)-g(x))\\ &= \Big(-\frac{2}{3}x_1,-\frac{5}{4}x_2\Big)+\lambda\Big(\frac{1}{6}x_1-\frac{4}{3}x_2,\frac{4}{3}x_1+\frac{1}{12}x_2\Big)\\ &= \Big[\Big(\frac{\lambda}{6}-\frac{2}{3}\Big)x_1-\frac{4\lambda}{3}x_2,\ \frac{4\lambda}{3}x_1+\Big(\frac{\lambda}{12}-\frac{5}{4}\Big)x_2\Big].\end{aligned}$$

It can be easily verify that the vector on right hand side generates the whole $\mathbb{R}^2$, i.e.,

$$[H(A,B)+\lambda M(f,g)](\mathbb{R}^2)=\mathbb{R}^2,\ \forall\lambda>0.$$

Hence, M is $H(\cdot,\cdot)$-co-accretive with respect to A,B,f and g.

Now, we show $M_n\underset{\to}{G}M$. Let

$$f(x_n)=\Big(\frac{1}{2}x_1-x_2+\frac{1}{n},x_1+\frac{1}{3}x_2+\frac{2}{n}\Big),$$

$$g(x_n)=\Big(\frac{1}{3}x_1+\frac{1}{3}x_2+\frac{1}{n^2},-\frac{1}{3}x_1+\frac{1}{4}x_2+\frac{2}{n^2}\Big).$$

Then

$$y(x_n)=M_n(f(x_n),g(x_n))=f(x_n)-g(x_n)=\Big(\frac{1}{6}x_1-\frac{4}{3}x_2+\frac{1}{n}-\frac{1}{n^2},\frac{4}{3}x_1+\frac{1}{12}x_2+\frac{2}{n}+\frac{2}{n^2}\Big).$$

Since,

$$\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}\Big[\frac{1}{2}x_1-x_2+\frac{1}{n},x_1+\frac{1}{3}x_2+\frac{2}{n}\Big]=\Big(\frac{1}{2}x_1-x_2,x_1+\frac{1}{3}x_2\Big),$$

$$\lim_{n\to\infty}g(x_n)=\lim_{n\to\infty}\Big[\frac{1}{3}x_1+\frac{1}{3}x_2+\frac{1}{n^2},-\frac{1}{3}x_1+\frac{1}{4}x_2+\frac{2}{n^2}\Big]=\Big(\frac{1}{3}x_1+\frac{1}{3}x_2,-\frac{1}{3}x_1+\frac{1}{4}x_2\Big).$$

we have $\lim_{n\to\infty}f(x_n)=f(x),\ \lim_{n\to\infty}g(x_n)=g(x)$ as $n\to\infty$. Also,

$$\begin{aligned}\lim_{n\to\infty}y_n &= \lim_{n\to\infty}\Big(\frac{1}{6}x_1-\frac{4}{3}x_2+\frac{1}{n}-\frac{1}{n^2},\frac{4}{3}x_1+\frac{1}{12}x_2+\frac{2}{n}-\frac{2}{n^2}\Big)\\ &= \Big(\frac{1}{6}x_1-\frac{4}{3}x_2,\frac{4}{3}x_1+\frac{1}{12}x_2\Big)=f(x)-g(x),\end{aligned}$$

which shows that $\lim_{n\to\infty}f(x_n)=f(x),\ \lim_{n\to\infty}g(x_n)=g(x)$ and $\lim_{n\to\infty}y_n\to y$ as $n\to\infty$ and hence, $M_n\underset{\to}{G}M$ as $n\to\infty$.

Finally, we show that $J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot),\cdot)}\to J^{H(\cdot,\cdot)}_{\lambda,M(\cdot),\cdot)}$ as $M_n\underset{\to}{G}M$. Now for any $\lambda=1$, the resolvent operators are given by

$$\begin{aligned}R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot),\cdot)}(x) &= [H(A,B)+\lambda M_n(f,g)]^{-1}(x)\\ &= [(A(x)+B(x))+(f(x)-g(x))]^{-1}(x),\\ &= \Big[\Big(-\frac{1}{2}x_1-\frac{4}{3}x_2+\frac{1}{n}-\frac{1}{n^2},\frac{4}{3}x_1-\frac{7}{6}x_2+\frac{2}{n}-\frac{2}{n^2}\Big)\Big]^{-1}\\ &= \frac{1}{85}\Big[\Big(-42x_1+48x_2-\frac{54}{n}+\frac{54}{n^2}\Big),\Big(-48x_1-18x_2+\frac{84}{n}-\frac{84}{n^2}\Big)\Big]\end{aligned}$$

and

$$\begin{aligned} R^{H(\cdot,\cdot)}_{\lambda,M(\cdot),\cdot)}(x) &= [H(A,B)+\lambda M(f,g)]^{-1}(x) \\ &= [(Ax+Bx)+(fx-gx)]^{-1}(x), \\ &= \Big[\Big(-\frac{1}{2}x_1-\frac{4}{3}x_2,\frac{4}{3}x_1-\frac{7}{6}x_2\Big)\Big]^{-1} \\ &= \frac{1}{85}\Big[\Big(-42x_1+48x_2\Big),\Big(-48x_1-18x_2\Big)\Big]. \end{aligned}$$

For $\lambda = 1$, the generalized Yosida approximation operators are given by

$$\begin{aligned} J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot),\cdot)}(x) &= \frac{1}{\lambda}\Big[I-R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot),\cdot)}\Big](x) \\ &= \Big[x_1-\frac{1}{85}\Big(-42x_1+48x_2-\frac{54}{n}+\frac{54}{n^2}\Big),x_2-\frac{1}{85}\Big(-48x_1-18x_2+\frac{84}{n}-\frac{84}{n^2}\Big)\Big] \\ &= \frac{1}{85}\Big[\Big(127x_1-48x_2+\frac{54}{n}-\frac{54}{n^2}\Big),\Big(48x_1+103x_2-\frac{84}{n}+\frac{84}{n^2}\Big)\Big] \end{aligned}$$

and

$$\begin{aligned} J^{H(\cdot,\cdot)}_{\lambda,M(\cdot),\cdot)}(x) &= \frac{1}{\lambda}\Big[I-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot),\cdot)}\Big](x) \\ &= \Big[x_1-\frac{1}{85}\Big(-42x_1+48x_2\Big),x_2-\frac{1}{85}\Big(-48x_1-18x_2\Big)\Big] \\ &= \frac{1}{85}\Big[127x_1-48x_2,48x_1+103x_2\Big], \end{aligned}$$

which show that

$$\|J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot),\cdot)}(x)-J^{H(\cdot,\cdot)}_{\lambda,M(\cdot),\cdot)}(x)\| \to 0, \text{ as } n\to\infty.$$

Thus, we have

$$J^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot),\cdot)}(x) \to J^{H(\cdot,\cdot)}_{\lambda,M(\cdot),\cdot)}(x), \text{ as } M_n \underset{\to}{G} M.$$

5. A SYSTEM OF THE GENERALIZED YOSIDA INCLUSIONS AND EXISTENCE RESULT

This section begins with the formulation of a system of the generalized Yosida inclusions and we discuss the existence of the unique solution. Let for each $i=1,2;X_i$ be a q_i-uniformly smooth Banach spaces equipped with norms $\|.\|_i$. Let $A_i,B_i,f_i,g_i,P_i : X_i \to X_i$ and $N_i,F_i : X_1\times X_2 \to X_i;H_i : X_i\times X_i \to X_i$ be the single-valued mappings. Let $Q_i : X_i \to 2^{X_i}$ be the multi-valued mappings. Let $M_1 : X_1\times X_1 \to 2^{X_1}$ be an $H_1(\cdot,\cdot)$-co-accretive mapping with respect to A_1,B_1,f_1 and g_1 and $M_2 : X_2\times X_2 \to 2^{X_2}$ be an $H_2(\cdot,\cdot)$-co-accretive mapping with respect to A_2,B_2,f_2 and g_2. We consider the following system of the generalized Yosida inclusions (in short: SGYI): Find $(x,y)\in X_1\times X_2, u\in Q_1(x), v\in Q_2(y)$ such that

$$\begin{cases} 0 \in J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}(x)+F_1(P_1(x),v)+M_1(f_1(x),g_1(x)), \\ 0 \in J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))}(y)+F_2(u,P_2(y))+M_2(f_2(y),g_2(y)). \end{cases} \tag{5.1}$$

***Special Cases*:**

(1) If $J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}, J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))} \equiv 0$, then SGYI (5.1) coincides with the system of generalized variational inclusions (5.1) of Ahmad *et al.* [24].

(2) If $J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}, J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))} \equiv J^{H(\cdot,\cdot)}_{\lambda,M}, F_1, F_2 \equiv 0, f_1, f_2, g_1, g_2 \equiv I$, the identity mappings and $M_1(\cdot,\cdot), M_2(\cdot,\cdot) = M(\cdot)$, then SGYI (5.1) reduces to the following Yosida inclusion problem.

Find $x \in X$ such that

$$0 \in J^{H(\cdot,\cdot)}_{\lambda,M}(x) + M(x). \tag{5.2}$$

Problem (5.2) was studied in [12].

(3) If $J^{H(\cdot,\cdot)}_{\lambda,M} = T$, where $T : X \to X$ is a mapping, then problem (5.2) further reduces to variational inclusion problem (10) of Li and Huang [25].

Note that for suitable choices of mappings involved in the formulation of SGYI (5.1), one can obtain many problems existing in the literature.

Definition 5.1. A mapping $F : X_1 \times X_2 \to X_i$ is said to be (β,γ)-mixed Lipschitz continuous, if there exist constants $\beta > 0, \gamma > 0$ such that

$$\|F(x_1,y_1) - F(x_2,y_2)\|_i \leq \beta\|x_1 - x_2\|_1 + \gamma\|y_1 - y_2\|_2, \forall\, x_1,x_2 \in X_1, y_1,y_2 \in X_2. \tag{5.3}$$

Lemma 5.2. *For any $(x,y) \in X_1 \times X_2, u \in Q_1(x), v \in Q_2(y)$; (x,y) is a solution of SGYI (5.1), if and only if (x,y) satisfies*

$$x = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x),B_1(x)) - \lambda_1 F_1(P_1(x),v) - \lambda_1 J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}(x)];$$

$$y = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(g_2(\cdot),f_2(\cdot))}[H_2(A_2(y),B_2(y)) - \lambda_2 F_2(u,P_2(y)) - \lambda_2 J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))}(y)],$$

where λ_1 and $\lambda_2 > 0$ are constants.

Proof. One can prove the lemma easily by using the definition of resolvent operators. □

Theorem 5.3. *For each $i = 1,2$, let X_i be a q_i-uniformly smooth Banach space equipped with norms $\|.\|_i$. Let $A_i, B_i, f_i, g_i : X_i \to X_i$ be the single-valued mappings such that A_i be η_i-expansive, B_i is σ_i-Lipschitz continuous and f_i is τ_i-Lipschitz continuous. Let $H_i : X_i \times X_i \to X_i$ be a symmetric cocoercive mapping with respect to A_i and B_i with constants μ_i and γ_i, respectively and (v_i,δ_i)-mixed Lipschitz continuous. Let $P_i : X_i \to X_i; N_i : X_1 \times X_2 \to X_i$ be the single-valued mappings and let $Q_i : X_i \to 2^{X_i}$ be $\mathscr{D}$-Lipschitz continuous multi-valued mappings with constants $\lambda_{\mathscr{D}_{Q_i}}$. Let $F_1 : X_1 \times X_2 \to E_1$ be ρ_1-strongly accretive mapping in the first argument, λ_{F_1}-Lipschitz continuous in the second argument and N_{F_1}-Lipschitz continuous with respect to P_1 in the first argument and let $F_2 : X_1 \times X_2 \to X_2$ be ρ_2-strongly accretive mapping in the second argument, λ_{F_2}-Lipschitz continuous in the first argument and S_{F_2}-Lipschitz continuous with respect to P_2 in the second argument. Let $M_1 : X_1 \times X_1 \to 2^{X_1}$ be an $H_1(\cdot,\cdot)$-co-accretive mapping with respect to A_1, B_1, f_1 and g_1 and $M_2 : X_2 \times X_2 \to 2^{X_2}$ be an $H_2(\cdot,\cdot)$-co-accretive mapping with respect to A_2, B_2, f_2 and g_2. Suppose that there exist constants $\lambda_1, \lambda_2 > 0$ satisfying*

$$\begin{cases} k_1 = l_1 + \theta_2\lambda_2\lambda_{F_2}\lambda_{D_{Q_1}} + \vartheta_1\tau_1 < 1; \\ k_2 = l_2 + \theta_1\lambda_1\lambda_{F_1}\lambda_{D_{Q_2}} + \vartheta_2\tau_2 < 1, \end{cases} \tag{5.4}$$

where

$$l_1 = \theta_1[\sqrt[q_1]{1 - q_1(\mu_1\eta_1^{q_1} - \gamma_1\sigma_1^{q_1}) + c_{q_1}(v_1+\delta_1)^{q_1}} + \sqrt[q_1]{1 - \lambda_1 q_1\rho_1 + c_{q_1}\lambda_1^{q_1}N_{F_1}^{q_1}} + \lambda_1 m_1];$$

$$l_2 = \theta_2[\sqrt[q_2]{1 - q_2(\mu_2\eta_2^{q_2} - \gamma_2\sigma_2^{q_2}) + c_{q_2}(v_2+\delta_2)^{q_2}} + \sqrt[q_2]{1 - \lambda_2 q_2\rho_2 + c_{q_2}\lambda_2^{q_2}S_{F_2}^{q_2}} + \lambda_2 m_2];$$

$$\theta_1 = \frac{1}{\lambda_1(\alpha_1 - \beta_1) + (\mu_1\eta_1^q - \gamma_1\sigma_1^q)};\ \theta_2 = \frac{1}{\lambda_2(\alpha_2 - \beta_2) + (\mu_2\eta_2^q - \gamma_2\sigma_2^q)};$$

and $m_1 = \frac{1}{\lambda}(1+\theta_1),\ m_2 = \frac{1}{\lambda}(1+\theta_2)$*. In addition, the following conditions hold*

$$\|J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x) - J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_2))}(x)\| \le \vartheta_2\|f_2(y_1)-f_2(y_2)\|_2, \tag{5.5}$$

$$\|J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_1),g_2(\cdot))}(y) - J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_2),g_2(\cdot))}(y)\| \le \vartheta_1\|f_1(x_1)-f_1(x_2)\|_1 \tag{5.6}$$

Then SGYI (5.1) has a unique solution.

Proof. It is clear from (2.2) and Lemma 3.2 that resolvent operators $R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}$ and $R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}$ are θ_1 and θ_2-Lipschitz continuous, respectively and generalized Yosida approximation operators $J^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}$ and $J^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}$ are κ_1 and κ_2-Lipschitz continuous, respectively, where

$$\theta_1 = \frac{1}{\lambda_1(\alpha_1-\beta_1)+(\mu_1\eta_1^{q_1}-\gamma_1\sigma_1^{q_1})},$$

$$\theta_2 = \frac{1}{\lambda_2(\alpha_2-\beta_2)+(\mu_2\eta_2^{q_2}-\gamma_2\sigma_2^{q_2})}$$

and $\kappa_1 = \frac{1}{\lambda_1}(1+\theta_1), \kappa_2 = \frac{1}{\lambda_2}(1+\theta_2)$.

Next, we define a mapping $T : X_1\times X_2 \to X_1\times X_2$ by

$$T(x,y) = (N(x,y), S(x,y)), \forall (x,y)\in X_1\times X_2, \tag{5.7}$$

where $N : X_1\times X_2 \to X_1,\ S : X_1\times X_2 \to X_2$ are the single-valued mappings defined by

$$N(x,y) = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x),B_1(x)) - \lambda_1F_1(P_1(x),v) - \lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}(x)], \lambda_1>0, \tag{5.8}$$

$$S(x,y) = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(g_2(\cdot),f_2(\cdot))}[H_2(A_2(y),B_2(y)) - \lambda_2F_2(u,P_2(y)) - \lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))}(y)], \lambda_2>0. \tag{5.9}$$

Using (5.8) and Lipschitz continuity of resolvent operator $R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}$, we have

$$\begin{aligned}
&\|N(x_1,y_1)-N(x_2,y_2)\|_1\\
&= \|R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_1),B_1(x_1)) - \lambda_1F_1(P_1(x_1),v_1) - \lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_1)]\\
&\quad -R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_2),B_1(x_2)) - \lambda_1F_1(P_1(x_2),v_2) - \lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_2))}(x_2)]\|_1\\
&\le \theta_1\|H_1(A_1(x_1),B_1(x_1)) - H_1(A_1(x_2),B_1(x_2)) - \lambda_1(F_1(P_1(x_1),v_1)-F_1(P_1(x_2),v_1))\|_1\\
&\quad +\theta_1\lambda_1\|F_1(P_1(x_2),v_1)-F_1(P_1(x_2),v_2)\|_1\\
&\quad +\theta_1\lambda_1\|J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_1) - J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_2))}(x_2)\|\\
&\le \theta_1[\|H_1(A_1(x_1),B_1(x_1)) - H_1(A_1(x_2),B_1(x_2)) - (x_1-x_2)\|_1\\
&\quad +\|(x_1-x_2) - \lambda_1(F_1(P_1(x_1),v_1)-F_1(P_1(x_2),v_1))\|_1]\\
&\quad +\theta_1\lambda_1\|F_1(P_1(x_2),v_1)-F_1(P_1(x_2),v_2)\|_1\\
&\quad +\theta_1\lambda_1\|J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_1) - J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_2))}(x_2)\|.
\end{aligned} \tag{5.10}$$

It follows from Lemma 2.1, symmetric cocoercivity and (ν_1,δ_1)-mixed Lipschitz continuity of H_1 that

$$\begin{aligned}
\|H_1(A_1(x_1),B_1(x_1)) - {} & H_1(A_1(x_2),B_1(x_2)) - (x_1-x_2)\|_1^{q_1}\\
&\le \|x_1-x_2\|_1^{q_1} - q_1\langle H_1(A_1(x_1),B_1(x_1)) - H_1(A_1(x_2),B_1(x_2)),\\
&\quad J_{q_1}(x_1-x_2)\rangle_1 + c_{q_1}\|H_1(A_1(x_1),B_1(x_1)) - H_1(A_1(x_2),B_1(x_2))\|_1^{q_1}\\
&\le \|x_1-x_2\|_1^{q_1} - q_1(\mu_1\|A_1(x_1)-A_1(x_2)\|_1^{q_1}\\
&\quad -\gamma_1\|B_1(x_1)-B_1(x_2)\|_1^{q_1}) + c_{q_1}(\nu_1+\delta_1)^{q_1}\|x_1-x_2\|_1^{q_1}.
\end{aligned} \tag{5.11}$$

Since A_1 is η_1-expansive and B_1 is σ_1-Lipschitz continuous, we have

$$\begin{aligned}\|H_1(A_1(x_1),B_1(x_1))-H_1(A_1(x_2),B_1(x_2))-(x_1-x_2)\|_1^{q_1}\\ \leq [1-q_1(\mu_1\eta_1^{q_1}-\gamma_1\sigma_1^{q_1})+c_{q_1}(v_1+\delta_1)^{q_1}]\|x_1-x_2\|_1^{q_1},\end{aligned}$$

which implies that

$$\begin{aligned}\|H_1(A_1(x_1),B_1(x_1)) &-H_1(A_1(x_2),B_1(x_2))-(x_1-x_2)\|_1\\ &\leq \sqrt[q_1]{1-q_1(\mu_1\eta_1^{q_1}-\gamma_1\sigma_1^{q_1})+c_{q_1}(v_1+\delta_1)^{q_1}}\|x_1-x_2\|_1.\end{aligned} \tag{5.12}$$

Since F_1 is ρ_1-strongly accretive and N_{F_1}-Lipschitz continuous with respect to P_1 in the first argument, we find from Lemma 2.1 that

$$\begin{aligned}\|(x_1-x_2)-\lambda_1(F_1(P_1(x_1),v_1) &-F_1(P_1(x_2),v_1))\|_1^{q_1}\\ &\leq \|x_1-x_2\|_1^{q_1}-\lambda_1q_1\langle F_1(P_1(x_1),v_1)-F_1(P_1(x_2),v_1),\\ &J_{q_1}(x_1-x_2)\rangle_1+c_{q_1}\lambda_1^{q_1}\|F_1(P_1(x_1),v_1)-F_1(P_1(x_2),v_1))\|_1^{q_1}\\ &\leq (1-\lambda_1q_1\rho_1+c_{q_1}\lambda_1^{q_1}N_{F_1}^{q_1})\|x_1-x_2\|_1^{q_1},\end{aligned} \tag{5.13}$$

which implies that

$$\|(x_1-x_2)-\lambda_1(F_1(P_1(x_1),v_1)-F_1(P_1(x_2),v_1))\|_1 \leq \sqrt[q_1]{1-\lambda_1q_1\rho_1+c_{q_1}\lambda_1^{q_1}N_{F_1}^{q_1}}\|x_1-x_2\|_1. \tag{5.14}$$

Also, F_1 is λ_{F_1}-Lipschitz continuous in the second argument and Q_2 is $\mathscr{D}$-Lipschitz continuous with constant $\lambda_{D_{Q_2}}$, we have

$$\begin{aligned}\|F_1(P_1(x_2),v_1)-F_1(P_1(x_2),v_2)\|_1\leq \lambda_{F_1}\|v_1-v_2\| &\leq \lambda_{F_1}\mathscr{D}(Q_2(y_1),Q_2(y_2))\\ &\leq \lambda_{F_1}\lambda_{D_{Q_2}}\|y_1-y_2\|_2.\end{aligned} \tag{5.15}$$

Using the Lipschitz continuity of $J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(\cdot,\cdot)}$, condition (5.5) and τ_2-Lipschitz continuity of f_2, we have

$$\begin{aligned}\|J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_1) &-J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_2))}(x_2)\|_1\\ &=\|J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_1)-J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_2)\\ &+\|J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_1)-J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_2))}(x_2)\|\|_1\\ &\leq \|J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_1)-J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_2)\|_1\\ &+\|J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_1))}(x_1)-J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_2))}(x_2)\|\|_1\\ &\leq \kappa_1\|x_1-x_2\|_1+\vartheta_2\|f_2(y_1)-f_2(y_2)\|_2\\ &\leq \kappa_1\|x_1-x_2\|_1+\vartheta_2\tau_2\|y_1-y_2\|_2.\end{aligned} \tag{5.16}$$

Using (5.12),(5.14),(5.15) and (5.16), (5.10) becomes

$$\begin{aligned}\|N(x_1,y_1)-N(x_2,y_2)\|_1 &\leq \theta_1[\sqrt[q_1]{1-q_1(\mu_1\eta_1^{q_1}-\gamma_1\sigma_1^{q_1})+c_{q_1}(v_1+\delta_1)^{q_1}}\\ &+\sqrt[q_1]{1-\lambda_1q_1\rho_1+c_{q_1}\lambda_1^{q_1}N_{F_1}^{q_1}}+\lambda_1\kappa_1]\|x_1-x_2\|_1\\ &+[\theta_1\lambda_1\lambda_{F_1}\lambda_{D_{Q_2}}+\vartheta_2\tau_2]\|y_1-y_2\|_2.\end{aligned} \tag{5.17}$$

Now, using (5.9) and Lipschitz continuity of resolvent operator $R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}$, we have

$$\begin{aligned}
&\|S(x_1,y_1)-S(x_2,y_2)\|_2\\
&=\|R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_1),B_2(y_1))-\lambda_2F_2(u_1,P_2(y_1))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_1),g_2(\cdot))}(y_1)]\\
&-R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_2),B_2(y_2))-\lambda_2F_2(u_2,P_2(y_2))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_2),g_2(\cdot))}(y_2)\|_2\\
&\le\theta_2\|H_2(A_2(y_1),B_2(y_1))-H_2(A_2(y_2),B_2(y_2))-\lambda_2(F_2(u_1,P_2(y_1))-F_2(u_1,P_2(y_2)))\|_2\\
&+\theta_2\lambda_2\|F_2(u_1,P_2(y_2))-F_2(u_2,P_2(y_2))\|_2\\
&+\theta_2\lambda_2\|J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_1),g_2(\cdot))}(y_1)-J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_2),g_2(\cdot))}(y_2)\|_2\\
&\le\theta_2[\|H_2(A_2(y_1),B_2(y_1))-H_2(A_2(y_2),B_2(y_2))-(y_1-y_2)\|_2\\
&+\|(y_1-y_2)-\lambda_2(F_2(u_1,P_2(y_1))-F_2(u_1,P_2(y_2)))\|_2]\\
&+\theta_2\lambda_2\|F_2(u_1,P_2(y_2))-F_2(u_2,P_2(y_2))\|_2\\
&+\theta_2\lambda_2\|J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_1),g_2(\cdot))}(y_1)-J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_2),g_2(\cdot))}(y_2)\|_2
\end{aligned}\tag{5.18}$$

Using the same arguments (5.11)-(5.16) as used for mapping N, we have

$$\begin{aligned}
\|S(x_1,y_1)-S(x_2,y_2)\|_2\ &\le\theta_2[\sqrt[q_2]{1-q_2(\mu_2\eta_2^{q_2}-\gamma_2\sigma_2^{q_2})+c_{q_2}(\nu_2+\delta_2)^{q_2}}\\
&+\sqrt[q_2]{1-\lambda_2q_2\rho_2+c_{q_2}\lambda_2^{q_2}S_{F_2}^{q_2}}+\lambda_2\kappa_2]\|y_1-y_2\|_2\\
&+[\theta_2\lambda_2\lambda_{F_2}\lambda_{D_{Q_1}}+\vartheta_1\tau_1]\|x_1-x_2\|_1.
\end{aligned}\tag{5.19}$$

It follows from (5.17) and (5.19) that

$$\begin{aligned}
\|N(x_1,y_1)-N(x_2,y_2)\|_1\ &+\|S(x_1,y_1)-S(x_2,y_2)\|_2\\
&\le k_1\|x_1-x_2\|_1+k_2\|y_1-y_2\|_2\\
&\le\ \max\ \{k_1,k_2\}(\|x_1-x_2\|_1+\|y_1-y_2\|_2),
\end{aligned}\tag{5.20}$$

where

$$\begin{cases} k_1=l_1+\theta_2\lambda_2\lambda_{F_2}\lambda_{D_{Q_1}}+\vartheta_1\tau_1;\\ k_2=l_2+\theta_1\lambda_1\lambda_{F_1}\lambda_{D_{Q_2}}+\vartheta_2\tau_2,\end{cases}\tag{5.21}$$

and

$$l_1=\theta_1[\sqrt[q_1]{1-q_1(\mu_1\eta_1^{q_1}-\gamma_1\sigma_1^{q_1})+c_{q_1}(\nu_1+\delta_1)^{q_1}}+\sqrt[q_1]{1-\lambda_1q_1\rho_1+c_{q_1}\lambda_1^{q_1}N_{F_1}^{q_1}}+\lambda_1\kappa_1];$$

$$l_2=\theta_2[\sqrt[q_2]{1-q_2(\mu_2\eta_2^{q_2}-\gamma_2\sigma_2^{q_2})+c_{q_2}(\nu_2+\delta_2)^{q_2}}+\sqrt[q_2]{1-\lambda_2q_2\rho_2+c_{q_2}\lambda_2^{q_2}S_{F_2}^{q_2}}+\lambda_2\kappa_2].$$

Now, we define a norm $\|.\|_*$ on $X_1\times X_2$ by

$$\|(x,y)\|_*=\|x\|_1+\|y\|_2,\ \forall(x,y)\in X_1\times X_2.\tag{5.22}$$

It is easy to see that $(X_1\times X_2,\|.\|_*)$ is a Banach space. Hence, from (5.7), (5.20) and (5.22), we have

$$\|T(x_1,y_1)-T(x_2,y_2)\|_*\le\ \max\ \{k_1,k_2\}\|(x_1,y_1)-(x_2,y_2)\|_*.\tag{5.23}$$

Since $\max\ \{k_1,k_2\}<1$ by condition (5.4), we find from (5.23) that T is a contraction mapping. Hence by Banach Contraction mapping Principle, there exists a unique point $(x,y)\in X_1\times X_2$ such that

$$T(x,y)=(x,y).$$

Thus, we have

$$x=R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x),B_1(x))-\lambda_1F_1(P_1(x),v)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}(x)],$$

$$y=R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y),B_2(y))-\lambda_2F_2(u,P_2(y))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))}(y)].$$

Hence by Lemma 5.2, (x,y) is a unique solution of SGYI (5.1). This completes the proof. □

Next, we suggest the following iterative algorithm to discuss the convergence analysis of SGYI (5.1).

Algorithm 5.4. For any $(x_0,y_0)\in X_1\times X_2$, compute the sequence $\{(x_n,y_n)\}\in X_1\times X_2$ by the following iterative scheme:

$$x_{n+1}=R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1n(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y_n))}(x_n)];\tag{5.24}$$

$$y_{n+1}=R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2n(g_2(\cdot),f_2(\cdot))}[H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x_n),g_2(\cdot))}(y_n)],\tag{5.25}$$

where $n=0,1,2,\cdots;\lambda_1,\lambda_2>0$ are constants.

Theorem 5.5. *Let, for each $i=1,2$, X_i be a q_i-uniformly smooth Banach spaces. Let the mappings A_i, B_i, f_i, g_i, H_i, F_i, P_i, Q_i, N_i and M_i be same as in Theorem 5.3. Let $M_{in}:X_i\times X_i\to 2^{X_i}$ be $H_i(\cdot,\cdot)$-co-accretive mappings and $N_{in}:X_1\times X_2\to X_i$ be the single-valued mappings such that $M_{in}\overrightarrow{G}M_i$ and the condition (5.4) holds. Then the approximate solution (x_n,y_n) generated by Algorithm 5.4 converges strongly to unique solution (x,y) of SGVI (5.1).*

Proof. It follows from Algorithm 5.4 and Theorem 5.3 that

$$\begin{aligned}
&\|x_{n+1}-x\|_1\\
&=\|R^{H_1(\cdot,\cdot)}_{\lambda_1,M_{1n}(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_{1n}(g_1(\cdot),f_2(y_n))}(x_n)]\\
&-R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x),B_1(x))-\lambda_1F_1(P_1(x),v)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}(x)]\|_1\\
&\le\|R^{H_1(\cdot,\cdot)}_{\lambda_1,M_{1n}(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)\ -\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_{1n}(g_1(\cdot),f_2(y_n))}(x_n)]\\
&-R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_{1n}(g_1(\cdot),f_2(y_n))}(x_n)]\|_1\\
&+\|R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}[H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_{1n}(g_1(\cdot),f_2(y_n))}(x_n)]\\
&-R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x),B_1(x))-\lambda_1F_1(P_1(x),v)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}(x)]\|_1\\
&\le a_n+\|R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_{1n}(g_1(\cdot),f_2(y_n))}(x_n)]\\
&-R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x),B_1(x))-\lambda_1F_1(P_1(x),v)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}(x)]\|_1,
\end{aligned}\tag{5.26}$$

where

$$\begin{aligned}
a_n=\ &\|R^{H_1(\cdot,\cdot)}_{\lambda_1,M_{1n}(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_{1n}(g_1(\cdot),f_2(y_n))}(x_n)]\\
&-R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_{1n}(g_1(\cdot),f_2(y_n))}(x_n)]\|_1
\end{aligned}\tag{5.27}$$

and

$$\begin{aligned}
&\|y_{n+1}-y\|_2\\
&=\|R^{H_2(\cdot,\cdot)}_{\lambda_2,M_{2n}(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_{2n}(f_1(x_n),g_2(\cdot))}(y_n)]\\
&-R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y),B_2(y))-\lambda_2F_2(u,P_2(y))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))}(y)]\|_2\\
&\le\|R^{H_2(\cdot,\cdot)}_{\lambda_2,M_{2n}(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_{2n}(f_1(x_n),g_2(\cdot))}(y_n)]\\
&-R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_{2n}(f_1(x_n),g_2(\cdot))}(y_n)]\|_2\\
&+\|R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_{2n}(f_1(x_n),g_2(\cdot))}(y_n)]\\
&-R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y),B_2(y))-\lambda_2F_2(u,P_2(y))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))}(y)]\|_2\\
&\le b_n+\|R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_{2n}(f_1(x_n),g_2(\cdot))}(y_n)]\\
&-R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y),B_2(y))-\lambda_2F_2(u,P_2(y))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))}(y)]\|_2,
\end{aligned}\tag{5.28}$$

where

$$\begin{aligned} b_n = & \|R^{H_2(\cdot,\cdot)}_{\lambda_2,M_{2n}(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_{2n}(f_1(x_n),g_2(\cdot))}(y_n)] \\ & -R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_{2n}(f_1(x_n),g_2(\cdot))}(y_n)]\|_2. \end{aligned} \tag{5.29}$$

By using similar arguments from (5.10)-(5.17), we obtain that

$$\begin{aligned} \|R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))} & [H_1(A_1(x_n),B_1(x_n))-\lambda_1F_1(P_1(x_n),v_n)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_{1n}(g_1(\cdot),f_2(y))}(x_n)] \\ & -R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(g_1(\cdot),f_1(\cdot))}[H_1(A_1(x),B_1(x))-\lambda_1F_1(P_1(x),v)-\lambda_1J^{H_1(\cdot,\cdot)}_{\lambda_1,N_1(g_1(\cdot),f_2(y))}(x)]\|_1, \\ & \leq l_1\|x_n-x\|_1+[\theta_1\lambda_1\lambda_{F_1}\lambda_{D_{Q_2}}+\vartheta_2\tau_2]\|y_n-y\|_2. \end{aligned} \tag{5.30}$$

Also using the same arguments from (5.18)-(5.19), we have

$$\begin{aligned} \|R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))} & [H_2(A_2(y_n),B_2(y_n))-\lambda_2F_2(u_n,P_2(y_n))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_{2n}(f_1(x),g_2(\cdot))}(y_n)] \\ & -R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(f_2(\cdot),g_2(\cdot))}[H_2(A_2(y),B_2(y))-\lambda_2F_2(u,P_2(y))-\lambda_2J^{H_2(\cdot,\cdot)}_{\lambda_2,N_2(f_1(x),g_2(\cdot))}(y)]\|_2 \\ & \leq l_2\|y_n-y\|_2+[\theta_2\lambda_2\lambda_{F_2}\lambda_{D_{Q_1}}+\vartheta_1\tau_1]\|x_n-x\|_1. \end{aligned} \tag{5.31}$$

From (5.26)-(5.31), we have

$$\begin{aligned} \|x_{n+1}-x\|_1+\|y_{n+1}-y\|_2 & \leq k_1\|x_n-x\|_1+k_2\|y_n-y\|_2+a_n+b_n \\ & \leq \max\{k_1,k_2\}(\|x_n-x\|_1+\|y_n-y\|_2)+a_n+b_n. \end{aligned} \tag{5.32}$$

Since $(X_1\times X_2,\|.\|_*)$ is a Banach space as defined by (5.22), we have

$$\begin{aligned} \|(x_{n+1},y_{n+1})-(x,y)\|_* & = \|(x_{n+1}-x),(y_{n+1}-y)\|_* = \|x_{n+1}-x\|_1+\|y_{n+1}-y\|_2 \\ & \leq \max\{k_1,k_2\}(\|(x_n,y_n)-(x,y)\|_*)+a_n+b_n. \end{aligned} \tag{5.33}$$

It follows from (5.27), (5.29) and Theorem 4.2 that

$$a_n,b_n\to 0, \text{ as } n\to\infty. \tag{5.34}$$

Thus, from (5.4), (5.32), (5.33) and (5.34), one has

$$\|(x_{n+1},y_{n+1})-(x,y)\|_*\to 0 \text{ as } n\to\infty.$$

Hence $\{(x_n,y_n)\}$ converges strongly to the unique solution (x,y) of SGYI (5.1). This completes the proof. □

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