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OPTIMALITY CONDITIONS FOR NONSMOOTH MULTIOBJECTIVE OPTIMIZATION PROBLEMS WITH GENERAL INEQUALITY CONSTRAINTS

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Abstract. In this paper, we study a nonsmooth multiobjective optimization problem involving an equality constraint, a general inequality constraint and a set constraint, in which the general inequality constraint is of the form $g(x) \in D$ with D a closed set. If D is a cone, this problem is reduced to a multiobjective optimiztion problem with cone-constraints. Under suitable conditions, necessary optimality conditions for weakly efficient solutions in terms of the Clarke subdifferentials are established. With some assumptions of generalized convexity, sufficient optimality conditions are derived. Weak and strong duality theorems of the Mond-Weir and Wolfe types are also given.

Keywords. Optimality condition; Clarke's subdifferential; Locally Lipschitz function, Mond–Weir type dual problem; Wolfe type dual problem.

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1. Introduction

The Lagrange multipliers rules for nonsmooth optimization problems have been extensively investigated in terms of the different subdifferentials by many authors; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein. The Clarke subdifferential [1] (also called the Clarke generalized gradient) is an important tool to derive optimality conditions for nonsmooth optimization problems. Together with the Clarke's subdifferential, different subdifferentials such as the Michel–Penot subdifferential [9], the Mordukhovich subdifferential [10], convexificator [3] are good tools for establishing optimality conditions in nonsmooth optimization. Jourani [4] and Jourani–Thibault [5] studied nonsmooth scalar optimization problems with general inequality constraints and derived optimality conditions via the Clarke subdifferentials under suitable constraint qualifications.

The purpose of this paper is to develop necessary and sufficient optimality conditions for weak efficiency of nonsmooth multiobjective optimization problems involving an equality constraint, a general

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inequality constraint and a set constraint in terms of the Clarke subdifferentials. Note that the general inequality constraint is of the form $g(x) \in D$ with D a closed set of a finite dimentional space. If D is a cone, this problem is reduced to a multiobjective optimiztion problem with cone-constraints.

The paper is organized as follows. In Section 2, we give some essential preliminaries. Section 3 is devoted to developing necessary optimality conditions for weak efficiency of nonsmooth multiobjective optimization problems involving an equality constraint, a general inequality constraint and a set constraint in terms of the Clarke subdifferentials. Note that weakly efficient solutions of the problem are taken with respect to a pointed convex cone with nonempty interior. Section 4 deals with sufficient optimality conditions for the weak efficiency. Section 5 gives weak and strong duality theorems of the Mond-Weir and the Wolfe types. A concluding remark is provided in Section 6.

2. Preliminaries

Let X be a real Banach space with the topological dual X^*, and let f be a real-valued function defined on X, which is Lipschitz near $\bar{x} \in X$. We recall [1] that the Clarke generalized derivative of f at $\bar{x} \in X$ in a direction $v \in X$ is defined as

$$f^0(\bar{x}; v) := \limsup_{x \to \bar{x}, t \downarrow 0} \frac{f(x+tv) - f(x)}{t}.$$

The Clarke subdifferential of f at x_0 is defined as

$$\partial f(x_0) := \{\xi \in X^* : \langle \xi, v \rangle \leq f^0(x_0, v) \ \forall v \in X\}.$$

Let K be a cone in X. The dual cone of K is defined as

$$K^* := \{\xi \in X^* : \langle \xi, v \rangle \geq 0 \ \forall v \in K\}.$$

Note that K^* is a weakly* closed convex cone.

The Clarke tangent cone to a set $C \subset X$ at $x_0 \in C$ is defined as

$$T(C; x_0) := \{v \in X : \forall t_n \downarrow 0, \forall x_n \to x_0, x_n \in C, \exists v_n \to v \text{ such that } x_n + t_n v_n \in C\}.$$

Note that if C is convex, then $x - x_0 \in T(C - x_0), \forall x \in C$. The Clarke normal cone to a set $C \subset X$ at $x_0 \in C$ is defined as

$$N(C; x_0) := \{\xi \in X^* : \langle \xi, v \rangle \leq 0 \ \forall v \in T(C; x_0)\}.$$

Let f be a mapping from X into $\mathbb{R}^n$, g and h be mappings from X into finite dimensional spaces Y and Z, respectively. Let C and D be nonempty closed subsets of X and Y, respectively, and let K be a pointed convex cone in $\mathbb{R}^n$ with $\text{int}K \neq \emptyset$, where $\text{int}K$ is the interior of K. Let us consider the following multiobjective optimization problem:

$$\begin{aligned} & \min f(x), \\ (MP) \quad \text{s.t.} \quad & g(x) \in D, \\ & h(x) = 0, \\ & x \in C. \end{aligned}$$

Denote by M the feasible set of Problem (MP). Thus,

$$M = g^{-1}(D) \cap h^{-1}(0) \cap C,$$

where $g^{-1}(D) = \{x \in X : g(x) \in D\}$. Recall that a point $x_0 \in M$ is said to be a local weak Pareto minimum of Problem (MP) iff there exists a number $\delta > 0$ such that

$$(f(M \cap B(x_0;\delta)) - f(x_0)) \cap (-\text{int}K) = \emptyset,$$

where $B(x_0;\delta)$ stands for the open ball of radius δ around x_0. In the sequel, we assume that f, g, h are locally Lipschitz at $x_0 \in M$.

We recall a result on the Tammer type scalarization function, which is needed in the following (see, for example, [2]).

Proposition 2.1. *For $e \in intK$, the function*

$$P(y) := \inf\{t \in \mathbb{R} : y \in te - K\}$$

is continuous positively homogeneous subadditive on $\mathbb{R}^n$.

Let $F : M \times M \to \mathbb{R}^n$. Let us consider the vector equilibrium problem (VEP): Find $x \in M$ such that

$$F(x,y) \notin -\text{int}K \ (\forall y \in M). \tag{2.1}$$

A vector $x_0 \in M$ is called a local weakly efficient solution of (VEP) iff there is a number $\delta > 0$ such that (2.1) holds for all $y \in M \cap B(x_0;\delta)$ (see [8]). If $B(x_0;\delta)$ can be taken as a whole of $\mathbb{R}^n$, then we obtain the notion of weakly efficient solution. For $F(x,y) = f(y) - f(x)$, we get the multiobjective optimization problem (MP). Then a local weakly efficient solution of (VEP) becomes a local weakly efficient solution of (MP).

We recall a scalarization result in [2].

Proposition 2.2. *The point x_0 is a weakly efficient solution of Problem (VEP) if and only if there exists a continuous positively homogeneous subadditive function P on $\mathbb{R}^n$ satisfying*

$$\textit{if } y_2 - y_1 \in intK, \textit{ then } P(y_1) < P(y_2),$$

and $P(F(x,y)) \geq 0$ ($\forall\, y \in M$).

Remark 2.3. The function P in Propositions 2.2 can be taken as just the function P in Proposition 2.1.

3. Necessary optimality conditions for local weakly efficient solutions

In order to derive the necessary conditions for weakly efficient solutions of (MP), we consider the following constraint qualification (CQ), which is studied in [4]: For every $(y^*, z^*) \in N(D, g(x_0)) \times Z^* \setminus \{(0,0)\}$,

$$0 \notin \partial(y^* \circ g)(x_0) + \partial(z^* \circ h)(x_0) + N(C, x_0).$$

We begin with a Karush–Kuhn–Tucker necessary condition for local weakly efficient solutions.

Theorem 3.1. *Let x_0 be a local weakly efficient solution of Problem (MP). Assume that the constraint qualification (CQ) holds at x_0. Then there exist $(\overline{\mu}, \overline{\nu}) \in N(D, g(x_0)) \times Z^*$, and a continuous positively homogeneous subadditive function P on $\mathbb{R}^n$ such that*

$$0 \in \partial\varphi(x_0) + \partial(\overline{\mu} \circ g)(x_0) + \partial(\overline{\nu} \circ h)(x_0) + N(C, x_0), \tag{3.1}$$

where $\varphi(x) = P(f(x) - f(x_0))$.

Proof. Since x_0 is a local weakly efficient solution of (MP), it is also a local weakly efficient solution of (VEP). Taking account of Proposition 2.2, it follows that there exists a continuous positively homogeneous subadditive function P on $\mathbb{R}^n$ satisfying

$$\text{if } y_2 - y_1 \in \text{int}K, \text{ then } P(y_1) < P(y_2),$$

and $P(F(x,y)) \geq 0$ $(\forall\, y \in M \cap B(x_0;\delta))$. Since $\varphi(x_0) = 0$, we deduce that x_0 is a local solution of the following problem:

$$(P) \quad \begin{aligned} &\min \varphi(x), \\ \text{s.t.} \quad & g(x) \in D, \\ & h(x) = 0, \\ & x \in C. \end{aligned}$$

Applying Theorem 3.2 [4] to Problem (P), we get (3.1). □

Let us study some properties of the function P in Theorem 3.1.

Proposition 3.2. $P^0(0;v) = P(v)$, $\forall v \in \mathbb{R}^n$.

Proof. Since P is positively homogeneous subadditive, it is a convex function. Hence, P is regular in the sense of Clarke. It follows that

$$P^0(0;v) = P'(0;v) = \lim_{t \to 0^+} \frac{P(0+tv) - P(0)}{t} = P(v).$$

□

Proposition 3.3. $\partial P(0) \subset K^*$.

Proof. Assume the contrary that there is $\xi \in \partial P(0)$, but $\xi \notin K^*$. Hence, there exists $v \in K$ such that $\langle \xi, v \rangle < 0$. Hence,

$$\langle \xi, -v \rangle > 0. \tag{3.2}$$

On the other hand, since $\xi \in \partial P(0)$, we find from Proposition 3.2 that

$$\langle \xi, -v \rangle \leq P(-v) - P(0) = P(-v). \tag{3.3}$$

By definition, one has

$$P(-v) = \inf\{t \in \mathbb{R} : -v \in te - K\}.$$

But $-v \in 0.e - K$. Hence, $P(-v) \leq 0$. It follows from (3.3) that $\langle \xi, -v \rangle \leq 0$. This contradicts (3.2). This completes the proof. □

Example 3.4. Let $K = \mathbb{R}^2_+, e = (1,1)$. It can be seen that $P(v_1, v_2) = \max\{v_1, v_2\}$. Then,

$$\partial P(0) = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 : \alpha_1 v_1 + \alpha_2 v_2 \leq \max\{v_1, v_2\}\}.$$

Let us see

$$\partial P(0) = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 : \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1\}.$$

In fact, if $\alpha=(\alpha_1,\alpha_2)\in\partial P(0)$, then $\alpha_1 v_1+\alpha_2 v_2\le\max\{v_1,v_2\}$. Taking $v_1=v_2=1$, we get $\alpha_1+\alpha_2\le 1$. Taking $v_1=v_2=-1$, we obtain $-\alpha_1-\alpha_2\le -1$. Hence, $\alpha_1+\alpha_2\ge 1$. Hence, $\alpha_1+\alpha_2=1$. Moreover, $\partial P(0)\subset K^*=\mathbb{R}^2_+$. It follows that $\alpha_1,\alpha_2\ge 0$. Conversely, we have

$$\alpha_1 v_1+\alpha_2 v_2\le\alpha_1\max\{v_1,v_2\}+\alpha_2\max\{v_1,v_2\}=\max\{v_1,v_2\}.$$

A Karush–Kuhn–Tucker necessary can be stated as follows.

Theorem 3.5. *Let x_0 be a local weakly efficient solution of Problem (MP). Assume that the constraint qualification (CQ) holds at x_0. Then, there exist $\overline{\lambda}=(\overline{\lambda}_1,\ldots,\overline{\lambda}_n)\in K^*,\overline{\lambda}\ne 0,(\overline{\mu},\overline{\nu})\in N(D,g(x_0))\times Z^*$ such that*

$$0\in\sum_{i=1}^n\overline{\lambda}_i\partial f_i(x_0)+\partial(\overline{\mu}\circ g)(x_0)+\partial(\overline{\nu}\circ h)(x_0)+N(C,x_0). \tag{3.4}$$

Proof. Since x_0 is a local weakly efficient solution of Problem (MP), we invoke Theorem 3.1 to deduce that there exist $(\overline{\mu},\overline{\nu})\in N(D,g(x_0))\times Z^*$, and a continuous positively homogeneous subadditive function P on $\mathbb{R}^n$ such that

$$0\in\partial\varphi(x_0)+\partial(\overline{\mu}\circ g)(x_0)+\partial(\overline{\nu}\circ h)(x_0)+N(C,x_0), \tag{3.5}$$

where $\varphi(x)=P(f(x)-f(x_0))$. Since P is a convex function, it is regular in the sense of Clarke. Applying Theorem 2.3.9 [1] on a chain rule, we get

$$\partial\varphi(x_0)\subset\text{co}\{\sum_{i=1}^n\alpha_i\xi_i:\alpha=(\alpha_1,\ldots,\alpha_n)\in\partial P(0),\xi_i\in\partial f_i(x_0)\}, \tag{3.6}$$

where co stands for the convex hull. It follows from (3.5) that there exists $\overline{\gamma}\in\partial\varphi(x_0)$ such that

$$0\in\overline{\gamma}+\partial(\overline{\mu}\circ g)(x_0)+\partial(\overline{\nu}\circ h)(x_0)+N(C,x_0). \tag{3.7}$$

By (3.6), we find that

$$\overline{\gamma}\in\text{co}\{\sum_{i=1}^n\alpha_i\xi_i:\alpha\in\partial P(0),\xi_i\in\partial f_i(x_0)\}.$$

Hence, there exists $\eta_1,\ldots,\eta_m\ge 0,\sum_{i=1}^m\eta_k=1$ such that

$$\overline{\gamma}=\sum_{i=1}^m\eta_k\Big(\sum_{i=1}^n\alpha_i^{(k)}\xi_i^{(k)}\Big), \tag{3.8}$$

where $\alpha^{(k)}=(\alpha_1^{(k)},\ldots,\alpha_n^{(k)})\in\partial P(0),\xi_i^{(k)}\in\partial f_i(x_0)$. It follows from (3.8) that

$$\overline{\gamma}\in\sum_{k=1}^m\eta_k\Big(\sum_{i=1}^n\alpha_i^{(k)}\partial f_i(x_0)\Big). \tag{3.9}$$

Moreover,

$$\sum_{k=1}^m\eta_k\Big(\sum_{i=1}^n\alpha_i^{(k)}\partial f_i(x_0)\Big)=\sum_{i=1}^n\Big(\sum_{k=1}^m\eta_k\alpha_i^{(k)}\Big)\partial f_i(x_0). \tag{3.10}$$

Putting $\overline{\lambda}_i=\sum_{k=1}^m\eta_k\alpha_i^{(k)}$, we obtain $\overline{\lambda}:=(\overline{\lambda}_1,\ldots,\overline{\lambda}_n)=\sum_{k=1}^m\eta_k(\alpha_1^{(k)},\ldots,\alpha_n^{(k)})=\sum_{k=1}^m\eta_k\alpha^{(k)}\in\partial P(0)$, as $\partial P(0)$ is convex. Combining (3.7), (3.9), (3.10) yields that

$$0\in\sum_{i=1}^n\overline{\lambda}_i\partial f_i(x_0)+\partial(\overline{\mu}\circ g)(x_0)+\partial(\overline{\nu}\circ h)(x_0)+N(C,x_0).$$

Moreover, by Proposition 3.3, $\partial P(0) \subset K^*$. Hence, $\overline{\lambda} \in K^*$. On the other hand, since $\overline{\lambda} \in \partial P(0)$, it follows that

$$\langle \overline{\lambda}, v \rangle \leq P(v) - P(0) = P(v) \ (\forall v \in \mathbb{R}^n).$$

Taking $v = -e$, one has that $-e \in -1.e - K$. But $P(-e) = \inf\{t \in \mathbb{R} : -e \in te - K\}$. Hence, $P(-e) \leq -1$. So,

$$\langle \overline{\lambda}, -e \rangle \leq P(-e) \leq -1.$$

Hence, $\overline{\lambda} \neq 0$. This completes the proof. □

Remark 3.6. Theorem 3.5 here includes Theorem 3.2 (i) [4] as a special case for scalar optimization problems with general inequality constraint.

In case that $\mathbb{R}^n_+ \subset K$, we get the following.

Corollary 3.7. *Let x_0 be a local weakly efficient solution of Problem (MP). Assume that the constraint qualification (CQ) holds at x_0, and $\mathbb{R}^n_+ \subset K$. Then, there exist $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_n) \in K^*$ with $\overline{\lambda} \neq 0$, $\overline{\lambda}_i \geq 0$ $(\forall i = 1, \ldots, n)$, and $(\overline{\mu}, \overline{\nu}) \in N(D, g(x_0)) \times Z^*$ such that (3.4) holds.*

Proof. We invoke Theorem 3.5 to deduce that there exist $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_n) \in K^*$, $\overline{\lambda} \neq 0$, $(\overline{\mu}, \overline{\nu}) \in N(D, g(x_0)) \times Z^*$ such that (3.4) holds. Since $\mathbb{R}^n_+ \subset K$, it holds that $K^* \subset (\mathbb{R}^n_+)^* = \mathbb{R}^n_+$. By Proposition 3.3, $\partial P(0) \subset K^*$. Hence, $\partial P(0) \subset \mathbb{R}^n_+$. For $\overline{\lambda} \in \partial P(0)$, one has $\overline{\lambda}_i \geq 0$ $(i = 1, \ldots, n)$. This completes the proof. □

Under condition $\mathbb{R}^n_+ \subset \text{int}K \cup \{0\}$, we can establish a Karush-Kuhn-Tucker necessary efficient condition in which all Lagrange multipliers corresponding to components of the objective are positive.

Theorem 3.8. *Let x_0 be a local weakly efficient solution of Problem (MP). Assume that the constraint qualification (CQ) holds at x_0, and $\mathbb{R}^n_+ \subset \text{int}K \cup \{0\}$. Then, there exist $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_n) \in K^*$ with $\overline{\lambda}_i > 0$ $(\forall i = 1, \ldots, n)$, and $(\overline{\mu}, \overline{\nu}) \in N(D, g(x_0)) \times Z^*$ such that*

$$0 \in \sum_{i=1}^{n} \overline{\lambda}_i \partial f_i(x_0) + \partial(\overline{\mu} \circ g)(x_0) + \partial(\overline{\nu} \circ h)(x_0) + N(C, x_0). \tag{3.11}$$

Proof. Applying Theorem 3.5 yields the existence of $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_n) \in K^*, \overline{\lambda} \neq 0, (\overline{\mu}, \overline{\nu}) \in N(D, g(x_0)) \times Z^*$ such that (3.11) holds. Let us see $\overline{\lambda}_i > 0$ $(\forall i = 1, \ldots, n)$. We first show that

$$K^* \subset \mathbb{R}^n_{++} \cup \{0\}.$$

Assume the contrary that this was false. Observing that $K^* \subset \mathbb{R}^n_+$, we deduce that there exist $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \in K^*$ and the indexes k, j such that $\overline{\alpha}_k = 0, \overline{\alpha}_j > 0$. Denote by ε_k the kth unit vector. Since $\mathbb{R}^n_+ \subset \text{int}K \cup \{0\}$, it holds that $\varepsilon_k \in \text{int}K$. Hence, there exists $t > 0$ sufficiently small such that $v = \varepsilon_k - t\varepsilon_j \in K$. Then

$$\langle v, \overline{\alpha} \rangle = \langle \varepsilon_k, \overline{\alpha} \rangle - t \langle \varepsilon_j, \overline{\alpha} \rangle = -t\overline{\alpha}_j < 0.$$

This conflicts with the definition of K^*. Thus $K^* \subset \mathbb{R}^n_{++} \cup \{0\}$. Observing that $\overline{\lambda} \in \partial P(0) \subset K^*$ and $\overline{\lambda} \neq 0$, we get $\overline{\lambda} \in \mathbb{R}^n_{++}$, which means that $\overline{\lambda}_i > 0$ $(i = 1, \ldots, n)$. This completes the proof. □

4. Sufficient Optimality Conditions

To derive sufficient conditions for solutions of (MP), we recall that a real-valued function f is called quasiconvex at x_0 on C if $\forall x \in C$,

$$f(x) \leqslant f(x_0) \Longrightarrow f(tx+(1-t)x_0) \leqslant f(x_0)\ (\forall t \in (0,1)).$$

f is called quasiconvex on C if it is quasiconvex at each point of C. In case f is the Fréchet differentiable at x_0 with the Fréchet derivative $\nabla f(x_0)$, if f is quasiconvex at x_0 on C, then

$$f(x) - f(x_0) \leqslant 0 \Longrightarrow \langle \nabla f(x_0), x - x_0 \rangle \leqslant 0\ (\forall x \in C).$$

Following the definition by Reiland [11], a locally Lipschitz real-valued function f defined on X is called ∂-pseudoconvex at x_0 on a subset C of X if

$$(\exists \xi \in \partial f(x_0)) \langle \xi, x - x_0 \rangle \geqslant 0 \Longrightarrow f(x) - f(x_0) \geqslant 0\ (\forall x \in C).$$

f is called ∂-pseudoconvex on C if it is ∂-pseudoconvex at each point of C.

Let K be a closed convex cone in $\mathbb{R}^n$. Adapting the definition by Sach [12], the locally Lipschitz function $f : C \to \mathbb{R}^n$ is called scalar K-pseudoconvex at x_0 on C if for all $\lambda \in K^*$, $\lambda \circ f$ is ∂-pseudoconvex at x_0 on C .

Let us introduce the following notion.

Definition 4.1. A real-valued function f defined on X is said to be Clarke's semi-regular at x_0 if the Clarke derivative $f^0(x_0; v)$ at x_0 in any direction v exists, and $f^0(x_0; v) = f^+(x_0; v)$, where

$$f^+(x_0; v) = \limsup_{t \to 0+} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

Remark 4.2. (a) If f is Clarke's regular at x_0, then it is Clarke's semi-regular at x_0.
(b) A locally Lipschitz function may be not Clarke's semi-regular. For example, $f(x) = -|x|$. It can be seen that f is locally Lipschitz at $x_0 = 0$. But f is not Clarke' semi-regular at $x_0 = 0$, since $f^0(0;1) = 1, f^+(0;1) = -1$.

Proposition 4.3. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. Assume that f is Clarke's semi-regular at x_0 and quasiconvex at x_0. Then,*

$$f(x) \leq f(x_0) \Longrightarrow \langle \xi, x - x_0 \rangle \leq 0\ (\forall \xi \in \partial f(x_0), \forall x \in C)$$

Proof. Taking $x \in C$ satisfying $f(x) \leq f(x_0)$, by definition, we have

$$f(tx + (1-t)x_0) \leq f(x_0)\ (\forall t \in (0,1)).$$

Setting $v = x - x_0$, we get

$$\frac{f(x_0 + tv) - f(x_0)}{t} \leq 0\ (\forall t \in (0,1)).$$

Hence,

$$\limsup_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t} \leq 0.$$

Making use of the Clarke semi-regularity of f, we deduce that for every $\xi \in \partial f(x_0)$,

$$\langle \xi, x - x_0 \rangle \leq f^0(x_0, v) \leq 0.$$

Hence, the conclusion follows. ∎

Corollary 4.4. *Let $h : X \to \mathbb{R}$ be a locally Lipschitz function. Assume that $\pm h$ are quasiconvex and Clarke's semi-regular at x_0. Then, if $h(x) = h(x_0)$, then $\langle \eta, x - x_0 \rangle = 0$ $(\forall \eta \in \partial h(x_0))$.*

Proof. Since h is quasiconvex and Clarke's semi-regular at x_0, the inequality $h(x) \leq h(x_0)$ implies that

$$\langle \eta, x - x_0 \rangle \leq 0 \ (\forall \eta \in \partial h(x_0)). \tag{4.1}$$

On the other hand, $-h$ is also quasiconvex and Clarke's semi-regular at x_0. Then the inequality $-h(x) \leq -h(x_0)$ implies that

$$\langle \eta, x - x_0 \rangle \leq 0 \ (\forall \eta \in \partial(-h(x_0))). \tag{4.2}$$

It follows from (4.2) that

$$\langle \eta, x - x_0 \rangle \geq 0 \ (\forall \eta \in \partial h(x_0)). \tag{4.3}$$

Combining (4.1) and (4.3) yields that

$$\langle \eta, x - x_0 \rangle = 0 \ (\forall \eta \in \partial h(x_0)).$$

This completes the proof. □

A sufficient optimality condition for weakly efficient solutions can be stated as follows.

Theorem 4.5. *Let x_0 be a feasible point of Problem (MP). Assume that the following conditions are fulfilled:*
(i) There exist $\overline{\lambda} \in K^*, \overline{\lambda} \neq 0, \overline{\mu} \in N(D, g(x_0)), \overline{\eta} \in Z^*$ such that

$$0 \in \sum_{i=1}^{n} \overline{\lambda}_i \partial f_i(x_0) + \partial(\overline{\mu} \circ g)(x_0) + \partial(\overline{v} \circ h)(x_0) + N(C, x_0);$$

(ii) f is scalar K-pseudoconvex at x_0 on C; all but at most one of $f_1, \ldots, f_n$ are strictly differentiable at x_0; $\overline{\mu} \circ g$ is quasiconvex and Clarke's semi-regular at x_0; $\pm v \circ h$ is quasiconvex and Clarke's semi-regular at x_0; C and D are convex sets. Then, x_0 is a weakly efficient solution of (MP).

Proof. Since all but at most one of $f_1, \ldots, f_n$ are strictly differentiable at x_0, it follows from Proposition 2.3.3 [1] that

$$\partial(\overline{\lambda} \circ f)(x_0) = \sum_{i=1}^{n} \overline{\lambda}_i \partial f_i(x_0). \tag{4.4}$$

Taking account of condition (i) and (4.4), it follows that there exist $\alpha \in \partial(\overline{\lambda} \circ f)(x_0), \beta \in \partial(\overline{\mu} \circ g)(x_0), \gamma \in \partial(\overline{v} \circ h)(x_0), \sigma \in N(C, x_0)$ such that

$$\alpha + \beta + \gamma + \sigma = 0. \tag{4.5}$$

Since D is convex, for every $x \in M$, we have $g(x) - g(x_0) \in T(D, g(x_0))$, where M is the feasible set of Problem (MP). Moreover, $\overline{\mu} \in N(D, g(x_0))$. Hence, $\langle \overline{\mu}, g(x) - g(x_0) \rangle \leq 0$ and $(\overline{\mu} \circ g)(x) \leq (\overline{\mu} \circ g)(x_0)$. Since $\overline{\mu} \circ g$ is quasiconvex and Clarke's semi-regular at x_0, one finds from Proposition 4.3 that

$$\langle \beta, x - x_0 \rangle \leq 0. \tag{4.6}$$

Since $\pm v \circ h$ is quasiconvex and Clarke's semi-regular at x_0, for $x \in M$, we have $h(x) = h(x_0) = 0$. Taking account of Corollary 4.4, we obtain

$$\langle \gamma, x - x_0 \rangle = 0 \ (\forall x \in M). \tag{4.7}$$

The convexity of C yields that $x-x_0 \in T(C,x_0)$. Therefore,

$$\langle \sigma, x-x_0\rangle \leq 0. \tag{4.8}$$

Combining (4.5)–(4.8) yields that

$$\langle \alpha, x-x_0\rangle \geq 0. \tag{4.9}$$

Making use of the scalar K-pseudoconvexity of f at x_0, one gets

$$(\overline{\lambda}\circ f)(x) \geq (\overline{\lambda}\circ f)(x_0)\ (\forall x\in M).$$

It is easy to see that x_0 is a weakly efficient solution of (MP). □

Theorem 4.5 is illustrated by the following example.

Example 4.6. Let $X=\mathbb{R}^2, Y=\mathbb{R}^2, Z=\mathbb{R}$, $K=\mathbb{R}^2_+, C=[0,1]\times[0,1]$, $D=-\mathbb{R}^2_+, n=2, \bar{x}=(0,0)$. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined as $f=(f_1,f_2)$, where

$$f_1(x) = x_1^3+|x_2|,$$
$$f_2(x) = x_2-x_1,$$

where $(x_1,x_2)\in\mathbb{R}^2$. Let g,h be defined as $g=(g_1,g_2)$,

$$g_1(x) = |x_1|-e^{x_1}+1,$$
$$g_2(x) = -|x_2|+\sin x_1,$$
$$h(x) = 2x_1-x_2.$$

We have $M=\{x\in C:\ g(x)\in D, h(x)=0\}=\{(x_1,x_2)\in\mathbb{R}^2 : x_2=2x_1, 0\leq x_1\leq \frac{1}{2}\}$. The point $\bar{x}=(0,0)$ is a weakly efficient solution of the vector optimization problem: $\min\{f(x):\ x\in M\}$. It can be seen that $g(\bar{x})=0, N(D,g(\bar{x}))=\mathbb{R}^2_+, N(C,\bar{x})=\mathbb{R}^2_-, Z^*=\mathbb{R}$; the functions F is scalar $\mathbb{R}^2_+$-pseudoconvex at $\bar{x}$, f_2 is strictly differentiable at $\bar{x}$. We have $\partial f_1(\bar{x})=\{0\}\times[-1,1]$, $\partial f_2(\bar{x})=\{(-1,1)\}$. For $\mu=(\mu_1,\mu_2), \mu_i\geq 0\,(i=1,2)$, the function μg is quasiconvex and Clarke's semi-regular at $\bar{x}$. For $\nu\in\mathbb{R}$, the function $\pm\nu h$ is quasiconvex and Clarke's semi-regular at $\bar{x}$. Moreover, $\partial h(\bar{x})=\{(2,-1)\}$, and

$$\partial(\mu g)(\bar{x}) = [-2\mu_1+\mu_2,\mu_2]\times[-\mu_2,\mu_2].$$

Condition (i) of Theorem 4.5 is fulfilled with $\overline{\lambda}=(1,1)\in K^*=\mathbb{R}^2_+$, $\overline{\mu}=(1,2)\in N(D,g(\bar{x}))=\mathbb{R}^2_+$, $\overline{\nu}=1\in Z^*=\mathbb{R}$:

$$(0,0)\in 1.\{0\}\times[-1,1]+1.\{(0,1)\}+[0,2]\times[-2,2]+\{(2,-1)\}+\mathbb{R}^2_-.$$

Thus, all hypotheses of Theorem 4.5 are satisfied. Hence, $\bar{x}$ is a weakly efficient solution of the problem mentioned above.

5. Duality

Denote by cone$D$ the closed convex cone generalized by D. We first introduce the following Mond–Weir type dual problem to (MP):

$$\text{(DMP1)}\quad \begin{aligned} &\max f(u),\\ \text{s.t.}\quad & 0\in\sum_{i=1}^{n}\lambda_i\partial f_i(u)+\partial(\mu\circ g)(u)+\partial(\nu\circ h)(u)+N(C,u),\\ &(\mu\circ g)(u)\geq 0,(\nu\circ h)(u)=0,u\in C,\\ &\lambda\in K^*,\lambda\neq 0,\mu\in-(\text{cone}D)^*,\nu\in Z^*.\end{aligned}$$

The Mond–Weir type dual problem to (MP) is illustrated by the following example.

Example 5.1. Let $X=\mathbb{R}^2, C=[0,1]\times[0,1]$. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined as $f:=(f_1,f_2)$, where

$$f_1(x):=|x_1|+x_2,$$
$$f_2(x):=-\frac{1}{2}\sin x_1+x_2^2$$

$(x=(x_1,x_2)\in\mathbb{R}^2)$. Let $g,h:\mathbb{R}^2\to\mathbb{R}$ be defined as

$$g_1(x):=\begin{cases}-x_1^2, & x_1\leqslant 0,\\ -x_1, & x_1>0.\end{cases}$$
$$g_2(x):=4x_1^3-x_1,$$
$$h(x)=-x_1+2x_2^2.$$

Consider the following multiobjective optimization problem:

$$\min f(x)\,\text{s.t.}\, x\in M:=\{x\in[0,1]\times[0,1]:\, g(x)\leqslant 0, h(x)=0\}.$$

It can be seen that $M=\{(x_1,x_2)\in[0,1]\times[0.1]: x_1=2x_2^2\}$, $T(C,u)=\mathbb{R}^2_+, N(C,u)=\mathbb{R}^2_-$. The dual problem of the Mond–Weir type for this problem is of the form (DMP1), where for $u=(u_1,u_2)\in[0,1]\times[0,1]$,

$$\partial f_1(u)=\begin{cases}\{(1,1)\}, & \text{if } u_1>0,\\ [-1,1]\times\{1\}, & \text{if } u_1=0,\end{cases}$$
$$\partial f_2(u)=\{(-\frac{1}{2}\cos u_1, 2u_2)\},$$
$$\partial g_1(u)=\begin{cases}\{(-1,0)\}, & \text{if } u_1>0,\\ [0,1]\times\{0\}, & \text{if } u_1=0,\end{cases}$$
$$\partial g_2(u)=\{(12u_1^2-1,0)\},$$
$$\partial h(u)=\{(-1,4u_2)\}.$$

Denote by M_1 the feasible of (DMP1). A point $x_0\in M_1$ is called weakly efficient solution of (DMP1) if

$$f(x)-f(x_0)\notin \text{int}K\ (\forall x\in M_1).$$

In the sequel, we give a weak duality theorem for the primal problem (MP) and the dual problem (DMP1).

Theorem 5.2. *(Weak duality) Let x and (u,λ,μ,ν) be the feasible points of (MP) and (DMP1), respectively. Suppose that f is scalar K-pseudoconvex at u on C; all but at most one of $f_1,\dots,f_n$ are strictly differentiable at u; $\overline{\mu}\circ g$ is quasiconvex and Clarke's semi-regular at u; $\pm\nu\circ h$ is quasiconvex and Clarke's semi-regular at u; C and D are convex sets. Then*

$$f(x)-f(u)\notin -intK.$$

Proof. Since (u,λ,μ,ν) is a feasible of (DMP1), there exist $\alpha\in\sum_{i=1}^{n}\lambda_i\partial f_i(u),\beta\in\partial(\mu\circ g)(u),\gamma\in\partial(\nu\circ h)(u),\eta\in N(C,u)$ such that $\alpha+\beta+\gamma+\eta=0$, whence

$$\langle\alpha+\beta+\gamma,x-u\rangle+\langle\eta,x-u\rangle=0. \tag{5.1}$$

Observing that $x\in M$, one has $g(x)\in D$. Moreover, $\mu\in-(\text{cone}D)^*$. Hence, $(\mu\circ g)(x)\le 0$, and so,

$$(\mu\circ g)(x)\le 0\le(\mu\circ g)(u). \tag{5.2}$$

In view of the quasiconvexity and Clarke's semiregularity of $\mu\circ g$ at u, by Proposition 4.3, we get

$$\langle\beta,x-u\rangle\le 0. \tag{5.3}$$

Since $\pm h$ are quasiconvex and Clarke's semi-regular at x_0 and $(\nu\circ h)(x)=0=(\nu\circ h)(u)$, by Corollary 4.4, we deduce that

$$\langle\gamma,x-u\rangle=0\ (\forall\gamma\in\partial(\nu\circ h)(u)). \tag{5.4}$$

The convexity of C implies that $x-u\in T(C,u)$. Hence,

$$\langle\eta,x-u\rangle\le 0. \tag{5.5}$$

Combining (5.1), (5.3)–(5.5) yields that

$$\langle\alpha,x-u\rangle\ge 0. \tag{5.6}$$

Since all but at most one of $f_1,\dots,f_n$ are strictly differentiable at u, it follows from Proposition 2.3.3 [1] that $\partial(\lambda\circ f)(u)=\sum_{i=1}^{n}\lambda_i\partial f_i(u)$. Hence, $\alpha\in\partial(\lambda\circ f)(u)$. Making use of the scalar K-pseudoconvex of f at u, we get

$$(\lambda\circ f)(x)\ge(\lambda\circ f)(u). \tag{5.7}$$

Since $\lambda\in K^*,\lambda\neq 0$, it follows readily from (5.7) that

$$f(x)-f(u)\notin-\text{int}K\ (\forall x\in M).$$

□

Hereafter, we give a strong duality theorem for (MP) and (DMP1).

Theorem 5.3. *(Strong duality) Let x_0 be a local weakly efficient solution of (MP) and $g(x_0)=0$. Assume that all hypotheses of Theorem 3.2 are fulfilled. Then there exist $\overline{\lambda}\in K^*,\overline{\lambda}\neq 0,\overline{\mu}\in-(coneD)^*,\overline{\nu}\in Z^*$ such that $(x_0,\overline{\lambda},\overline{\mu},\overline{\nu})$ is a feasible point of (DMP1), and the values of the objective functions of (MP) and (DMP1) at x_0 and $(x_0,\overline{\lambda},\overline{\mu},\overline{\gamma})$, respectively, are equal. Moreover, if all assumptions of Theorem 5.2 hold, then $(x_0,\overline{\lambda},\overline{\mu},\overline{\nu})$ is a weakly efficient solution of (DMP1).*

Proof. We invoke Theorem 3.5 to deduce that there exists $\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_n) \in K^*$, $\overline{\lambda} \neq 0$, $(\overline{\mu}, \overline{\nu}) \in N(D, g(x_0)) \times Z^*$ such that

$$0 \in \sum_{i=1}^{n} \overline{\lambda}_i \partial f_i(x_0) + \partial(\overline{\mu} \circ g)(x_0) + \partial(\overline{\nu} \circ h)(x_0) + N(C, x_0).$$

It can be seen that $N(D, g(x_0)) = -(\text{cone}D)^*$, as D is convex and $g(x_0) = 0$. Hence, $(x_0, \overline{\lambda}, \overline{\mu}, \overline{\nu}) \in M_1$. It is obvious that the values of (MP) and (DMP1) at $\bar{x}$ are equal.

Moreover, if all assumptions of Theorem 5.2 are fulfilled, by this weak duality theorem, $f(x_0) - f(u) \notin -\text{int}K$, $(\forall (u, \lambda, \mu, \nu) \in M_1)$. Hence,

$$f(u) - f(x_0) \notin \text{int}K \ (\forall (u, \lambda, \mu, \nu) \in M_1),$$

which means that $(x_0, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a weak efficient solution of (DMP1). □

Denote $e := (1, \dots, 1)^T \in \mathbb{R}^n$. Now we introduce the dual problem of the Wolfe type for (MP):

$$\text{(DMP2)} \quad \text{s.t.} \quad \begin{aligned} & \max f(u) + (\mu \circ g)(u)e + (\nu \circ h)(u)e, \\ & 0 \in \sum_{i=1}^{n} \lambda_i \partial f_i(u) + \partial(\mu \circ g)(u) + \partial(\nu \circ h)(u) + N(C, u), \\ & u \in C, \lambda \in K^*, \lambda \neq 0, \sum_{i=1}^{n} \lambda_i = 1, \mu \in -(\text{cone}D)^*, \nu \in Z^*. \end{aligned}$$

Denote by M_2 the feasible set of Problem (DMP2).

The dual problem of the Wolfe type (DMP2) to (MP) is illustrated by the following example.

Example 5.4. Let $X = \mathbb{R}^2, C = [0,1] \times [0,1]$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f := (f_1, f_2)$, where

$$\begin{aligned} f_1(x) &:= x_1 + x_2^2, \\ f_2(x) &:= \begin{cases} x_1^2 \sin \frac{1}{x_1} + x_2, & x_1 \neq 0, \\ x_2, & x_1 = 0 \end{cases} \end{aligned}$$

$(x = (x_1, x_2) \in \mathbb{R}^2)$. Let $g, h : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\begin{aligned} g_1(x) &:= \begin{cases} x_1, & x_1 \leqslant 0, \\ -x_1^2, & x_1 > 0. \end{cases} \\ g_2(x) &:= (x_1^3 - 3x_1)x_2, \\ h(x) &= x_2 - \frac{1}{2}x_1^3. \end{aligned}$$

Consider the following multiobjective optimization problem:

$$\min f(x) \, \text{s.t.} \, x \in M := \{x \in [0,1] \times [0,1] : g(x) \leqslant 0, h(x) = 0\}.$$

It can be seen that $M = \{(x_1,x_2) \in [0,1]\times[0.1] : x_2 = \frac{1}{2}x_1^3\}$, $T(C,u) = \mathbb{R}^2_+, N(C,u) = \mathbb{R}^2_-$. The dual problem of the Wolfe type for this problem is of the form (DMP2), where for $u = (u_1,u_2) \in [0,1]\times[0,1]$,

$$\partial f_1(u) = \{(1,2u_2)\},$$

$$\partial f_2(u) = \begin{cases} [-1,1]\times\{1\}, & \text{if } u_1 = 0,\\ \{(2u_1\sin\frac{1}{u_1} - \cos\frac{1}{u_1},1)\}, & \text{if } u_1 > 0, \end{cases}$$

$$\partial g_1(u) = \begin{cases} [0,1]\times\{0\}, & \text{if } u_1 = 0,\\ \{(-2u_1,0)\}, & \text{if } u_1 > 0, \end{cases}$$

$$\partial g_2(u) = \{(3u_1^2u_2 - 3u_2, u_1^3 - 3u_1)\},$$

$$\partial h(u) = \{(-\frac{3}{2}u_1^2,1)\}.$$

A weak duality theorem for (MP) and (DMP2) can be stated as following.

Theorem 5.5. *(Weak duality) Let x and (u,λ,μ,ν) be the feasible points of (MP) and (DMP2), respectively. Suppose that all but at most one of $f_1,\ldots,f_n$, $\mu\circ g, \nu\circ h$ are strictly differentiable at u on C; the function $\varphi(x) := \sum_{i=1}^n \lambda_i f_i(x) + (\mu\circ g)(x) + (\nu\circ h)(x)$ is pseudoconvex at u on C; C and D are convex sets. Then*

$$f(x) - \varphi(u,\lambda,\mu,\nu) \notin -intK.$$

Proof. Since (u,λ,μ,ν) is a feasible point of (DMP2), there exist $\hat\alpha \in \sum_{i=1}^n \lambda_i\partial f_i(u)$, $\hat\beta \in \partial(\mu\circ g)(u)$, $\hat\gamma \in \partial(\nu\circ h)(u)$, $\hat\eta \in N(C,u)$ such that

$$\hat\alpha + \hat\beta + \hat\gamma + \hat\eta = 0, \tag{5.8}$$

whence,

$$\langle \hat\alpha + \hat\beta + \hat\gamma, x-u\rangle + \langle \hat\eta, x-u\rangle = 0. \tag{5.9}$$

Since C is convex, one has

$$\langle \hat\eta, x-u\rangle \le 0. \tag{5.10}$$

It follows from (5.9) and (5.10) that

$$\langle \hat\alpha + \hat\beta + \hat\gamma, x-u\rangle \ge 0. \tag{5.11}$$

Since all but at most one of $f_1,\ldots,f_n$ are strictly differentiable at u, it follows from Proposition 2.3.3 [1] that

$$\partial\varphi(u) = \sum_{i=1}^n \overline{\lambda}_i\partial f_i(u) + \partial(\mu\circ g)(u) + \partial(\nu\circ h). \tag{5.12}$$

Thus $\hat\alpha + \hat\beta + \hat\gamma \in \partial\varphi(u)$. In view of the pseudoconvexity of φ at u, one finds from (5.11) that

$$\sum_{i=1}^n \lambda_i f_i(x) + (\mu\circ g)(x) + (\nu\circ h)(x) \ge \sum_{i=1}^n \lambda_i f_i(u) + (\mu\circ g)(u) + (\nu\circ h)(u). \tag{5.13}$$

Since $x \in M$, one has $g(x) \in D, h(x) = 0$. Moreover, $\mu \in -(\text{cone}D)^*$. It follows that $(\mu\circ g)(x) \le 0$. Hence,

$$\sum_{i=1}^n \lambda_i f_i(x) \ge \sum_{i=1}^n \lambda_i f_i(u) + (\mu\circ g)(u) + (\nu\circ h)(u). \tag{5.14}$$

It follows from (5.14) that $\lambda \circ (f(x) - \varphi(u,\lambda,\mu,\nu)) \geq 0$. Since $\lambda \in K^*, \lambda \neq 0$, it can be seen that

$$f(x) - \varphi(u,\lambda,\mu,\nu) \notin -\text{int}K.$$

This completes the proof. □

Finally, we give a strong duality theorem for (MP) and (DMP2).

Theorem 5.6. *(Strong duality) Let x_0 be a local weakly efficient solution of (MP) and $g(x_0) = 0$. Assume that all hypotheses of Theorem 3.5 are fulfilled. Then there exist $\overline{\lambda} \in K^*, \overline{\lambda} \neq 0, \overline{\mu} \in -(coneD)^*, \overline{\nu} \in Z^*$ such that $(x_0, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a feasible point of (DMP2), and the values of the objective functions of (MP) and (DMP2) at x_0 and $(x_0, \overline{\lambda}, \overline{\mu}, \overline{\gamma})$, respectively, are equal. Moreover, if all assumptions of Theorem 5.5 hold, then $(x_0, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a weakly efficient solution of (DMP2).*

Proof. We invoke Theorem 3.5 to deduce that there exists $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_n) \in K^*$, $\overline{\lambda} \neq 0$, $(\overline{\mu}, \overline{\nu}) \in N(D, g(x_0)) \times Z^*$ such that

$$0 \in \sum_{i=1}^{n} \overline{\lambda}_i \partial f_i(x_0) + \partial(\overline{\mu} \circ g)(x_0) + \partial(\overline{\nu} \circ h)(x_0) + N(C, x_0).$$

As also in the proof of Theorem 3.5, we can see that $N(D, g(x_0)) = -(\text{cone}D)^*$, as D is convex and $g(x_0) = 0$. Hence, $(x_0, \overline{\lambda}, \overline{\mu}, \overline{\nu}) \in M_2$. It is obvious that the values of (MP) and (DMP2) at $\bar{x}$ are equal. If all assumptions of Theorem 5.5 are fulfilled, we find from this weak duality theorem that

$$f(x_0) - \varphi(u,\lambda,\mu,\nu)) \notin -\text{int}K \ (\forall (u,\lambda,\mu,\nu) \in M_2).$$

Hence,

$$\varphi(u,\lambda,\mu,\nu) - \varphi(x_0, \overline{\lambda}, \overline{\mu}, \overline{\nu}) \notin \text{int}K \ (\forall (u,\lambda,\mu,\nu) \in M_2),$$

which means that $(x_0, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a weak efficient solution of (DMP2). □

Remark 5.7. Duality theorems obtained here are significant extensions to those known for multiobjective optimization problems involving finitely many inequality constraints.

6. Conclusions

Based on the scalarization method and a result on optimality conditions for nonsmooth scalar optimization problem in [4], we derive necessary optimality conditions for local weakly efficient solutions in terms of the Clarke subdifferentials. Note that the nonsmooth multiobjective optimization problem here involves an equality constraint, a general inequality constraint and a set constraint, in which the general inequality constraint is of the form $g(x) \in D$ with D a closed set in a finite dimensional space. Local weakly efficient solutions of the problem are taken with respect to a pointed convex cone. To prove sufficient optimality conditions, we introduce the notion of the Clarke's semi-regularity function. Under some assumptions of generalized convexity, sufficient optimality conditions are derived. We also study the Mond–Weir type dual problem and the Wolfe type dual problem of the primal problem, and establish some weak and strong duality theorems.

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