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INFINITELY MANY HOMOCLINIC SOLUTIONS FOR A CLASS OF SUPERQUADRATIC FOURTH-ORDER DIFFERENTIAL EQUATIONS

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Abstract. Using a variant fountain theorem, we prove the existence of infinitely many homoclinic solutions of a class of fourth-order differential equations $u^{(4)}(x)+\omega u''(x)+a(x)u(x)=f(x,u(x))$, $\forall x\in\mathbb{R}$, where $a\in C(\mathbb{R},\mathbb{R})$ may be negative on a bounded interval and $F(x,u)=\int_0^u f(x,t)dt$ is superquadratic at infinity in the second variable but does not need to satisfy the known Ambrosetti-Rabinowitz superquadratic growth condition.

Keywords. Fourth-order differential equation; Homoclinic solution; Variational method; Variant fountain theorem.

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1. Introduction

We consider the nonperiodic fourth-order differential equation

$$u^{(4)}(x)+\omega u''(x)+a(x)u(x)=f(x,u(x)),\ \forall x\in\mathbb{R}, \tag{1.1}$$

where ω is a constant, $a\in C(\mathbb{R},\mathbb{R})$ and $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. It is known that the mathematical modeling of many important problems in different research fields such as mechanical engineering, control systems, and economics leads naturally to solutions of nonlinear differential equations. In particular, fourth order differential equations such as (1.1) have been put forward as a mathematical model for the study of the pattern formation in physics and mechanics, for example, the well-known extended Fisher-Kolmogorov equation proposed by Coullet *et al.* [1] in the study of phase transitions, the fourth-order elastic beam equation in describing a large class of elastic deflection [2], the Swift-Hohenberg equation which is a general model for the pattern-forming process derived in [3] to describe random thermal fluctuations in the Boussinesque equation and in the propagation of lasers [4]. As usual, we say that a solution u of equation (1.1) is homoclinic (to 0) if $u(x)\to 0$ as $x\longrightarrow\pm\infty$. In addiction, if $u\neq 0$, then u is called a nontrivial homoclinic solution. During the previous years, the existence and multiplicity of homoclinic solutions for (1.1) have been extensively investigated via the critical point theory and variational methods. From the beginning, most of them treated the case where $a(x)$ and $f(x,u)$ are either independent of x or

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periodic in x, see [5, 6, 7, 8, 9] and the references cited therein. In this kind of problems, the function a plays an important role. Compared with the case of $a(x)$ and $f(x,u)$ being periodic in x, there are less literatures available for the case where $a(x)$ and $f(x,u)$ are nonperiodic in x, see [10, 11, 12, 13, 14, 15, 16, 17, 18]. We notice that, for the case that equation (1.1) is not periodic, the following coercive condition on a is often needed to obtain the existence of homoclinic solutions:

$(\mathscr{A}_0)$ $a:\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function, and there exists a constant a_0 such that

$$0<a_0\leq a(x)\longrightarrow+\infty \; as \; |x|\longrightarrow\infty$$

and

$$\omega\leq 2\sqrt{a_0},$$

which is used to establish the corresponding compact embedding lemmas on suitable functional spaces, see [10, 11, 15, 16]. Most of these known results were obtained for the case where $F(x,u)=\int_0^u f(x,t)dt$ is superquadratic at infinity in u and satisfies the usual assumption

$$\lim_{|u|\longrightarrow 0}\frac{F(t,u)}{|u|^2}=0 \; uniformly \; for \; t\in\mathbb{R}. \tag{1.2}$$

In this case, the well-known Ambrosetti-Rabinowitz superquadratic condition was usually assumed on F, see [8, 9, 14, 18] and the references therein.

In this paper, we study the existence of infinitely many homoclinic solutions for (1.1) in the case where a is unnecessarily required to be positive, and F satisfies some weak superquadratic conditions at infinity with respect to u. More precisely, we make the following assumptions:

$(\mathscr{A})$
$$\lim_{|x|\longrightarrow\infty}a(x)=+\infty;$$

(H_1) There exist constants $c>0$ and $\nu>2$ such that

$$|f(x,u)|\leq c(|u|+|u|^{\nu-1}),\; \forall(x,u)\in\mathbb{R}\times\mathbb{R};$$

(H_2)
$$\lim_{|u|\longrightarrow\infty}\frac{F(x,u)}{|u|^2}=+\infty,\; uniformly \; for \; x\in\mathbb{R};$$

(H_3) There exists a constant $\sigma\geq 1$ such that

$$\sigma\widehat{F}(x,u)\geq\widehat{F}(x,su),\; \forall(s,x,u)\in[0,1]\times\mathbb{R}\times\mathbb{R},$$

where $\widehat{F}(x,u)=f(x,u)u-2F(x,u)$;

(H_4)
$$F(x,-u)=F(x,u),\; \forall(x,u)\in\mathbb{R}\times\mathbb{R}.$$

Our main result reads as follows.

Theorem 1.1. *Assume that $(\mathscr{A})$ and $(H_1)-(H_4)$ are satisfied. Then system (1.1) possesses a sequence of nontrivial homoclinic orbits (u_k) satisfying*

$$\frac{1}{2}\int_{\mathbb{R}}[u_k''(x)^2-\omega u_k'(x)^2+a(x)u_k(x)^2]dx-\int_{\mathbb{R}}F(x,u_k(x))dx\longrightarrow+\infty \; as \; k\longrightarrow\infty.$$

Remark 1.2. In our assumptions, a is unnecessarily positive. in addition, the well-known Ambrosetti-Rabinowitz superquadratic condition is not required in our result. There are functions a and F which satisfy all the conditions in Theorem 1.1 but do not satisfy the corresponding conditions in the references mentioned above for the superquadratic case. For example, let

$$a(x) = |x| - 1,$$
$$f(x,u) = 2uln(1+|u|) + \frac{u|u|}{1+|u|}.$$

Then a simple computation shows that they satisfy $(\mathscr{A})$ and $(H_1)-(H_4)$. However, a does not satisfy the positivity condition and the Ambrosetti-Rabinowitz superquadratic condition does not hold for F.

2. Variational setting and preliminaries

To prove our main result via the critical point theory, we need to establish the variational setting for (1.1). In the following, we shall use $\|.\|_s$ to denote the norm of $L^s(\mathbb{R})$ for any $s \in [2,\infty]$. Let $H^2(\mathbb{R})$ be the Sobolev space with inner product and norm given respectively by

$$< u,v >_{H^2} = \int_{\mathbb{R}} [u''(x)v''(x) + u'(x)v'(x) + u(x)v(x)]dx$$

and

$$\|u\|_{H^2} = < u,u >_{H^2}^{\frac{1}{2}}$$

for all $u,v \in H^2(\mathbb{R})$.

Lemma 2.1 ([7], Lemma 8)**.** *Assume that a satisfies* $(\mathscr{A}_0)$. *Then there exists a constant* $b > 0$ *such that*

$$\int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2]dx \geq b\|u\|_{H^2}^2, \quad \forall u \in H^2(\mathbb{R}).$$

By Lemma 2.1, we define

$$E = \left\{ u \in H^2(\mathbb{R}) / \int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2]dx < \infty \right\}$$

with the inner product

$$< u,v > = \int_{\mathbb{R}} [u''(x)v''(x) - \omega u'(x)v'(x) + a(x)u(x)v(x)]dx$$

and the corresponding norm

$$\|u\| = (\int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2]dx)^{\frac{1}{2}}.$$

It is easy to verify that E is a Hilbert space.

In order to prove our result, the following compactness result is necessary.

Lemma 2.2 ([15], Lemma 2.2)**.** *Assume that a satisfies* $(\mathscr{A}_0)$. *Then E is compactly embedded in* $L^s(\mathbb{R})$ *for all* $s \in [2,\infty]$. *Moreover, for all* $s \in [2,\infty]$, *there exists* $\eta_s > 0$ *such that*

$$\|u\|_{L^s(\mathbb{R})} \leq \eta_s \|u\|, \quad \forall u \in E. \tag{2.1}$$

Next, we shall use the following variant fountain theorem established by Zou [19].

Let E be a Banach space with norm $\|.\|$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $dimX_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider a family of functionals $\Phi_\lambda \in C^1(E,\mathbb{R})$ of the type

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \; u \in E, \; \lambda \in [1,2].$$

Lemma 2.3. *(Variant fountain theorem)* [19] *Assume that the functionals* Φ_λ *satisfy*
(a) Φ_λ *maps bounded sets into bounded sets for all* $\lambda \in [1,2]$ *and*

$$\Phi_\lambda(-u) = \Phi_\lambda(u) \text{ for all } (\lambda, u) \in [1,2] \times E;$$

(b) $B(u) \geq 0$ *for all* $u \in E$ *and* $A(u) \longrightarrow +\infty$ *or* $B(u) \longrightarrow +\infty$ *as* $\|u\| \longrightarrow \infty$*;*
(c) There exist $\rho_k > r_k > 0$ *such that for all* $\lambda \in [1,2]$

$$\alpha_k(\lambda) = \inf_{u \in Z_k, \|u\| = r_k} \Phi_\lambda(u) > \beta_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} \Phi_\lambda(u).$$

Then

$$\alpha_k(\lambda) \leq \xi_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \ \forall \lambda \in [1,2],$$

where

$$B_k = \{u \in Y_k / \|u\| \leq \rho_k\} \text{ and } \Gamma_k = \left\{\gamma \in C(B_k, E) / \gamma \text{ is odd}, \ \gamma_{|\partial B_k} = id\right\}.$$

Moreover, for almost every $\lambda \in [1,2]$*, there exists a sequence* $(u_m^k(\lambda))_{m \in \mathbb{N}}$ *such that*

$$\sup_{m \in \mathbb{N}} \left\|u_m^k(\lambda)\right\| < \infty, \ \Phi'_\lambda(u_m^k(\lambda)) \longrightarrow 0, \ \Phi_\lambda(u_m^k(\lambda)) \longrightarrow \xi_k(\lambda) \text{ as } m \longrightarrow \infty.$$

3. Proof of Theorem 1.1

First, note that $(\mathscr{A})$, (H_1) and (H_2) imply that there exists a constant $a_0 > 0$ such that $\widetilde{a}(x) = a(x) + 2a_0 \geq a_0$ for all $x \in \mathbb{R}$, $\omega \leq 2\sqrt{a_0}$ and $\widetilde{F}(x,u) = F(x,u) + a_0 |u|^2 \geq 0$ for all $(x,u) \in \mathbb{R} \times \mathbb{R}$. Consider the following fourth-order system

$$u^{(4)}(x) + \omega u''(x) + \widetilde{a}(x)u(x) = \widetilde{f}(x, u(x)), \ \forall x \in \mathbb{R}. \tag{3.1}$$

Then (3.1) is equivalent to (1.1). Moreover, it is easy to check that hypotheses $(H_1) - (H_4)$ still hold for $\widetilde{F}(x,u)$ provided that those hold for $F(x,u)$ and function $\widetilde{a}$ satisfies $(\mathscr{A}_0)$. In what follows, we always assume without loss of generality that a satisfies $(\mathscr{A}_0)$, $F(t,x) \geq 0$ for all $(x,u) \in \mathbb{R} \times \mathbb{R}$ and $F(x,u)$ satisfies $(H_1) - (H_4)$. Here $\widetilde{f}(x,u) = f(x,u) + 2a_0 u$.

Consider the variational functional Φ associated to system (1.1):

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2] dx - \int_{\mathbb{R}} F(x, u(x)) dx$$

defined on the Hilbert space E introduced in Section 2. Set

$$\psi(u) = \int_{\mathbb{R}} F(x,u) dx, \ u \in E.$$

Lemma 3.1. *Assume* $(\mathscr{A}_0)$ *and* (H_1) *are satisfied. Then* $\psi \in C^1(E, \mathbb{R})$ *and* $\psi' : E \longrightarrow E^*$ *is compact, and* $\Phi \in C^1(E, \mathbb{R})$*. Moreover, for all* $u, v \in E$

$$\psi'(u)v = \int_{\mathbb{R}} f(x,u) v dx, \tag{3.2}$$

and

$$\Phi'(u)v = < u, v > - \int_{\mathbb{R}} f(x,u) v dx. \tag{3.3}$$

Proof. By (H_1), for any $s \in [0,1]$ and $u, v \in E$, we have

$$\begin{aligned} |f(x,u+sv)v| &\le c(|u+sv| + |u+sv|^{\nu-1})\,|v| \\ &\le c[|u|+|v|+2^{\nu-2}(|u|^{\nu-1}+|v|^{\nu-1})]\,|v| \\ &\le c2^{\nu-2}[|u|\,|v|+|v|^2+|u|^{\nu-1}\,|v|+|v|^{\nu}]. \end{aligned} \tag{3.4}$$

The H$\ddot{o}$lder's inequality implies

$$\int_{\mathbb{R}}(|u|\,|v|+|u|^{\nu-1}\,|v|)dt \le \|u\|_2\,\|v\|_2 + \|u\|_\nu^{\nu-1}\,\|v\|_\nu. \tag{3.5}$$

Using (3.4), (3.5), the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we get, for all $u, v \in E$, that

$$\begin{aligned} \lim_{s\longrightarrow 0}\frac{\psi(u+sv)-\psi(u)}{s} &= \lim_{s\longrightarrow 0}\int_{\mathbb{R}}\int_0^1 f(x,u+sv)vdsdx \\ &= \int_{\mathbb{R}} f(x,u)vdx = J(u,v). \end{aligned} \tag{3.6}$$

Moreover, it follows from (2.1), (H_1) and (3.5) that

$$\begin{aligned} |J(u,v)| &\le \int_{\mathbb{R}} |f(x,u)|\,|v|\,dx \\ &\le c\int_{\mathbb{R}}(|u|\,|v|+|u|^{\nu-1}\,|v|)dx \\ &\le c(\|u\|_2\,\|v\|_2+\|u\|_\nu^{\nu-1}\,\|v\|_2) \\ &\le c(\eta_2^2\,\|u\|+\eta_\nu^\nu\,\|u\|^{\nu-1})\,\|v\|. \end{aligned}$$

Therefore, $J(u,.)$ is linear and bounded, and $J(u,.)$ is the G$\hat{a}$teaux derivative of ψ at u.

Next, we prove that $J(u,.)$ is weakly continuous in u. To this end, we first claim that if $u_n \rightharpoonup u$ in E, then $f(x,u_n) \longrightarrow f(x,u)$ in $L^2(\mathbb{R})$. Arguing indirectly, by Lemma 2.2, we may assume that there exists a subsequence (u_{n_k}) such that

$$u_{n_k} \longrightarrow u \textit{ both in } L^2(\mathbb{R}) \textit{ and } L^{2(\nu-1)}(\mathbb{R}) \textit{ and } u_{n_k} \longrightarrow u \textit{ a.e. in } \mathbb{R} \textit{ as } k \longrightarrow \infty \tag{3.7}$$

and

$$\int_{\mathbb{R}} |f(x,u_{n_k})-f(x,u)|^2\,dx \ge \varepsilon_0,\ \forall k \in \mathbb{N} \tag{3.8}$$

for some positive constant ε_0. By (3.7) and up to a subsequence if necessary, we can assume that $\sum_{k=1}^\infty \|u_{n_k}-u\|_{L^2} < \infty$ and $\sum_{k=1}^\infty \|u_{n_k}-u\|_{L^{2(\nu-1)}} < \infty$. Let $w(x) = \sum_{k=1}^\infty |u_{n_k}(x)-u(x)|$ for all $x \in \mathbb{R}$. Then $w \in L^2(\mathbb{R}) \cap L^{2(\nu-1)}(\mathbb{R})$. By (H_1), there holds for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\begin{aligned} |f(x,u_{n_k})-f(x,u)|^2 &\le (|f(x,u_{n_k})|+|f(x,u)|)^2 \\ &\le 2(|f(x,u_{n_k})|^2+|f(x,u)|^2) \\ &\le 2c[(|u_{n_k}|+|u_{n_k}|^{\nu-1})^2+(|u|+|u|^{\nu-1})^2] \\ &\le 2^2c[|u_{n_k}|^2+|u_{n_k}|^{2(\nu-1)}+|u|^2+|u|^{2(\nu-1)}] \\ &\le 2^2c[(|u_{n_k}-u|+|u|)^2+(|u_{n_k}-u|+|u|)^{2(\nu-1)}+|u|^2+|u|^{2(\nu-1)}] \\ &\le 2^2c[2(|u_{n_k}-u|^2+|u|^2)+2^{2\nu-3}(|u_{n_k}-u|^{2(\nu-1)}+|u|^{2(\nu-1)})+|u|^2+|u|^{2(\nu-1)}] \\ &\le c_1(|w|^2+|u|^2+|w|^{2(\nu-1)}+|u|^{2(\nu-1)}), \end{aligned}$$

where c_1 is a positive constant. Combining this with (3.7), Lebesgue's Dominated Convergence Theorem implies

$$\lim_{k\longrightarrow\infty}\int_{\mathbb{R}}|f(x,u_{n_k})-f(x,u)|^2\,dx=0,$$

which contradicts to (3.8). Hence the claim above is true.

Now, suppose that $u_{n_k}\rightharpoonup u$ in E. Then $f(x,u_{n_k})\longrightarrow f(x,u)$ in $L^2(\mathbb{R})$. By Hölder's inequality and (2.1), we have

$$\begin{aligned}\|J(u_n,.)-J(u,.)\|_{E^*}&=\sup_{\|v\|=1}\int_{\mathbb{R}}(f(x,u_n)-f(x,u))v\,dx\\&\leq\eta_2\Big(\int_{\mathbb{R}}|f(x,u_n)-f(x,u)|^2\,dx\Big)^{\frac{1}{2}}\longrightarrow 0\ as\ n\longrightarrow\infty.\end{aligned}$$

This means that $u\longmapsto J(u,.)$ is weakly continuous and then it is continuous in E. Therefore $\psi\in C^1(E,\mathbb{R})$ and (3.2) is verified. Furthermore, ψ' is compact by the weak continuity of ψ' since E is reflexive. Due to the form of Φ, (3.3) is also verified and $\Phi\in C^1(E,\mathbb{R})$.

Finally, let $u\in E$ be a critical point of Φ. A standard argument shows that $u\in C^4(\mathbb{R},\mathbb{R})$ and satisfies equation (1.1). The proof of Lemma 3.1 is completed. □

In order to apply the variant fountain theorem to our main result, we choose an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of E and let $X_j=span\{e_j\}$ for all $j\in\mathbb{N}$. Define Y_k and Z_k by

$$Y_k=\oplus_{j=1}^{k}X_j,\ Z_k=\overline{\oplus_{j=k}^{\infty}X_j}$$

and the functionals A,B and Φ_λ on our working space E by

$$A(u)=\frac{1}{2}\|u\|^2,\ B(u)=\psi(u)=\int_{\mathbb{R}}F(x,u)dx,\ \Phi_\lambda(u)=A(u)-\lambda B(u),$$

for all $(\lambda,u)\in[1,2]\times E$. Assumption (H_1) implies that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda\in[1,2]$. Note that $F(x,-u)=F(x,u)$. So we have $\Phi_\lambda(-u)=\Phi_\lambda(u)$ for all $(\lambda,u)\in[1,2]\times E$. Thus condition a) of Lemma 2.3 holds. Since $F(x,u)\geq 0$ for all $(x,u)\in\mathbb{R}\times\mathbb{R}$, it is clear that condition b) is also satisfied. To verify condition c), we need to establish the three following lemmas.

Lemma 3.2. *Suppose that $(\mathscr{A}_0)$ holds. Then, for any $p\in[2,\infty]$,*

$$l_p(k)=\sup_{u\in Z_k,\|u\|=1}\|u\|_{L^p}\longrightarrow 0\ as\ k\longrightarrow\infty. \tag{3.9}$$

Proof. It is clear that $0<l_p(k+1)\leq l_p(k)$, so that $l_p(k)\longrightarrow\bar{l}_p$ as $k\longrightarrow\infty$. For every $k\geq 1$, there exists $u_k\in Z_k$ such that $\|u_k\|=1$ and $\|u_k\|_{L^p}>\frac{1}{2}l_p(k)$. For any $v\in E$, let $v=\sum_{i=1}^{\infty}v_ie_i$. By the Cauchy-Schartz inequality, one has

$$\begin{aligned}|<u_k,v>|&=\left|<u_k,\sum_{i=1}^{\infty}v_ie_i>\right|\\&=\left|<u_k,\sum_{i=k+1}^{\infty}v_ie_i>\right|\\&\leq\|u_k\|\left\|\sum_{i=k+1}^{\infty}v_ie_i\right\|\\&\leq\sum_{i=k+1}^{\infty}|v_i|\,\|e_i\|\longrightarrow 0\ as\ k\longrightarrow\infty,\end{aligned}$$

which implies that $u_k \rightharpoonup 0$. Without loss of generality, Lemma 2.2 implies that $u_k \longrightarrow 0$ in $L^2(\mathbb{R})$. Thus we have proved that $\bar{l}_p = 0$. The proof of Lemma 3.2 is completed. □

Lemma 3.3. *Assume that* $(\mathscr{A}_0)$ *and* (H_1) *are satisfied. Then there exist a positive integer* k_0 *and a sequence* $r_k \longrightarrow +\infty$ *as* $k \longrightarrow \infty$ *such that*

$$\alpha_k(\lambda) = \inf_{u \in Z_k, \|u\| = r_k} \Phi_\lambda(u) > 0, \ \forall k \geq k_0. \tag{3.10}$$

Proof. From (H_1) and the fact that $F(x,u) \geq 0$, we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - 2c(\|u\|_{L^2}^2 + \|u\|_{L^\nu}^\nu), \forall (\lambda,u) \in [1,2] \times E. \end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11) yields

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2cl_2^2(k)\|u\|^2 - 2cl_\nu^\nu(k)\|u\|^\nu, \ \forall (\lambda,u) \in [1,2] \times E. \tag{3.12}$$

In view of (3.9), there exists an integer k_0 such that

$$2cl_2^2(k) \leq \frac{1}{4}, \forall k \geq k_0. \tag{3.13}$$

For any $k \geq k_0$, let us define

$$r_k = (18cl_\nu^\nu(k))^{\frac{1}{2-\nu}}. \tag{3.14}$$

Since $\nu > 2$, one has $r_k \longrightarrow +\infty$ as $k \longrightarrow \infty$. From (3.12), (3.13) and (3.14), we deduce that, for all $k \geq k_0$,

$$\inf_{u \in Z_k, \|u\|=1} \Phi_\lambda(u) \geq \frac{1}{2}r_k^2 - \frac{1}{4}r_k^2 - \frac{1}{8}r_k^{2-\nu}r_k^\nu = \frac{1}{8}r_k^2 > 0, \tag{3.15}$$

which completes the proof of Lemma 3.3. □

Lemma 3.4. *Assume that* $(\mathscr{A}_0)$*,* (H_1) *and* (H_2) *are satisfied. Then, for any* $k \geq k_0$*, there exists* $\rho_k > r_k$ *such that*

$$\beta_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} \Phi_\lambda(u) < 0,$$

where k_0 *is the positive integer obtained in Lemma 3.3.*

Proof. First, we claim that for any finite-dimensional subspace $F \subset E$, there exists a constant $\varepsilon_0 > 0$ such that

$$meas(\{x \in \mathbb{R} / |u(x)| \geq \varepsilon_0 \|u\|\}) \geq \varepsilon_0, \ \forall u \in F \setminus \{0\}. \tag{3.16}$$

In not, for any $n \in \mathbb{N}$, there exists $u_n \in F \setminus \{0\}$ such that

$$meas(\left\{x \in \mathbb{R} / |u_n(x)| \geq \frac{1}{n}\|u_n\|\right\}) < \frac{1}{n}.$$

Letting $v_n = \frac{u_n}{\|u_n\|} \in F$, one has $\|v_n\| = 1$ and

$$meas(\left\{x \in \mathbb{R} / |v_n(x)| \geq \frac{1}{n}\right\}) < \frac{1}{n}, \forall n \in \mathbb{N}. \tag{3.17}$$

Since F is finite-dimensional, up to a subsequence if necessary, we may assume $v_n \longrightarrow v_0$ in E for some $v_0 \in E$. Evidently, $\|v_0\| = 1$. Since any two norms on F are equivalent, we have

$$\int_{\mathbb{R}} |v_n - v_0|\, dx \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.18}$$

The fact that $\|v_0\| = 1$ implies $\|v_0\|_{L^\infty} > 0$. By the definition of $\|.\|_{L^\infty}$, there exists a constant $\delta_0 > 0$ such that

$$meas(\{x \in \mathbb{R}/\, |v_0(x)| \geq \delta_0\}) \geq \delta_0. \tag{3.19}$$

Otherwise, for each fixed $n \in \mathbb{N}$ and $m > n$, we have

$$meas(\left\{x \in \mathbb{R}/\, |v_0(x)| \geq \frac{1}{n}\right\}) \leq meas(\left\{x \in \mathbb{R}/\, |v_0(x)| \geq \frac{1}{m}\right\}) \leq \frac{1}{m}.$$

Letting $m \longrightarrow \infty$, we obtain $meas(\left\{t \in \mathbb{R}/\, |v_0(x)| \geq \frac{1}{n}\right\}) = 0$. Consequently,

$$\begin{aligned} 0 &\leq meas(\{x \in \mathbb{R}/\, |v_0(x)| \neq 0\}) \\ &= meas(\cup_{n=1}^{\infty} \left\{x \in \mathbb{R}/\, |v_0(x)| \geq \frac{1}{n}\right\}) \\ &\leq \sum_{n=1}^{\infty} meas(\left\{x \in \mathbb{R}/\, |v_0(x)| \geq \frac{1}{n}\right\}) = 0, \end{aligned}$$

which yields $v_0 = 0$ and contradicts $\|v_0\| = 1$. Then (3.19) holds. For any $n \in \mathbb{N}$, let

$$\Lambda_n = \left\{x \in \mathbb{R}/\, |v_n(x)| < \frac{1}{n}\right\}, \Lambda_0 = \{x \in \mathbb{R}/\, |v_0(x)| \geq \delta_0\}.$$

Then for n large enough, by (3.17) and (3.19), we have

$$meas(\Lambda_n \cap \Lambda_0) \geq meas(\Lambda_0) - meas(\Lambda_n^c) \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.$$

Consequently, for n large enough, there holds

$$\begin{aligned} \int_{\mathbb{R}} |v_n - v_0|\, dx &\geq \int_{\Lambda_n \cap \Lambda_0} |v_n - v_0|\, dx \\ &\geq \int_{\Lambda_n \cap \Lambda_0} (|v_0| - |v_n|) dx \quad \geq (\delta_0 - \frac{1}{n}) meas(\Lambda_n \cap \Lambda_0) \geq \frac{\delta_0^2}{4} > 0. \end{aligned}$$

This is in a contradiction to (3.18). Therefore (3.16) holds. Now, note that for any $k \in \mathbb{N}$, Y_k is finite-dimensional. So there exists a constant $\varepsilon_k > 0$ such that

$$meas(\Lambda_u^k) \geq \varepsilon_k,\ \forall u \in Y_k \backslash \{0\}, \tag{3.20}$$

where

$$\Lambda_u^k = \{x \in \mathbb{R}/\, |u(x)| \geq \varepsilon_k \|u\|\}$$

for all $k \in \mathbb{N}$ and $u \in Y_k \backslash \{0\}$. By (H_2), for any $k \in \mathbb{N}$, there exists a constant $R_k > 0$ such that

$$F(x,u) \geq \frac{|u|^2}{\varepsilon_k^3},\ \forall x \in \mathbb{R} \text{ and } |u| \geq R_k. \tag{3.21}$$

Combining (3.20) with (3.21), for any $k \in \mathbb{N}$ and $\lambda \in [1,2]$, we have

$$\begin{aligned}
\Phi_\lambda(u) &\le \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x,u)dx \\
&\le \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \frac{|u|^2}{\varepsilon_k^3}dx \\
&\le \frac{1}{2}\|u\|^2 - \varepsilon_k^2\|u\|^2 meas(\Lambda_u^k)\frac{1}{\varepsilon_k^3} \\
&\le \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2
\end{aligned} \tag{3.22}$$

for all $u \in Y_k$ with $\|u\| \ge \frac{R_k}{\varepsilon_k}$. For any $k \ge k_0$, choose $\rho_k > \max\left\{r_k, \frac{R_k}{\varepsilon_k}\right\}$. Hence, (3.22) implies

$$\rho_k(\lambda) = \max_{u\in Y_k, \|u\|=\rho_k} \Phi_\lambda(u) \le -\frac{1}{2}\rho_k^2.$$

The proof of Lemma 3.4 is completed. □

Consequently, Lemmas 3.2, 3.4 show that condition c) of Lemma 2.3 is satisfied for all $k \ge k_0$. By the above, all the conditions of Lemma 2.3 hold for all $k \ge k_0$. Therefore, for any $k \ge k_0$ and $\lambda \in [1,2]$, there exists a sequence $(u_n^k(\lambda))_{n\in\mathbb{N}} \subset E$ such that

$$\sup_{n\in\mathbb{N}}\left\|u_n^k(\lambda)\right\| < \infty,\ \Phi'_\lambda(u_n^k(\lambda)) \longrightarrow 0 \ and\ \Phi_\lambda(u_n^k(\lambda)) \longrightarrow \xi_k(\lambda) \ as\ n \longrightarrow \infty, \tag{3.23}$$

where

$$\xi_k(\lambda) = \inf_{\gamma\in\Gamma_k}\sup_{u\in B_k}\Phi_\lambda(\gamma(u)),\ \forall\lambda \in [1,2]$$

with $B_k = \{u \in Y_k / \|u\| \le \rho_k\}$ and $\Gamma_k = \left\{\gamma \in C(B_k,E)/\gamma \ is\ odd,\ \gamma_{|\partial B_k} = id\right\}$. From (3.14) and Lemma 2.3, we infer that

$$\xi_k(\lambda) \in [\overline{\alpha}_k, \overline{\xi}_k],\ \forall k \ge k_0,\ \lambda \in [1,2], \tag{3.24}$$

where $\overline{\xi}_k = \max_{u\in B_k}\Phi_1(u)$ and $\overline{\alpha}_k = \frac{r_k^2}{8} \longrightarrow \infty$ as $k \longrightarrow \infty$. In view of (3.23), for any $k \ge k_0$, we can choose a sequence $\lambda_n \longrightarrow 1$ and the corresponding sequences $(u_m^k(\lambda_n))$ satisfying

$$\sup_{m\in\mathbb{N}}\left\|u_m^k(\lambda_n)\right\| < \infty \ and\ \Phi'_{\lambda_n}(u_m^k(\lambda_n)) \longrightarrow 0 \ as\ m \longrightarrow \infty. \tag{3.25}$$

Lemma 3.5. *For any $n \in \mathbb{N}$ and $k \ge k_0$, there exists $u_n^k \in E$ such that*

$$\lim_{m\longrightarrow\infty} u_m^k(\lambda_n) = u_n^k \ in\ E. \tag{3.26}$$

Proof. Throughout this proof and for the sake of simplicity, we shall let $u_m = u_m^k(\lambda_n)$ for $m \in \mathbb{N}$. Without loss of generality, we may assume by (3.23) that

$$u_m \rightharpoonup u \ as\ m \longrightarrow \infty \tag{3.27}$$

for some $u \in E$. Using (3.3), we get

$$\begin{aligned}
\|u_m - u\|^2 = {} & \Phi'_{\lambda_n}(u_m)(u_m-u) - \Phi'_{\lambda_n}(u)(u_m-u) \\
& + \lambda_n\int_{\mathbb{R}}(f(x,u_m) - f(x,u))(u_m-u)dx.
\end{aligned} \tag{3.28}$$

By (3.23), one has

$$\Phi'_{\lambda_n}(u_m)(u_m-u) \longrightarrow 0 \ as\ m \longrightarrow \infty. \tag{3.29}$$

Moreover (3.27) yields

$$\Phi'_{\lambda_n}(u)(u_m-u)\longrightarrow 0 \text{ as } m\longrightarrow\infty. \tag{3.30}$$

Now, by (2.1) and Hölder's inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}(f(x,u_m)-f(x,u))(u_m-u)dx\right|\\ &\leq (\int_{\mathbb{R}}|f(x,u_m)-f(x,u)|^2dx)^{\frac{1}{2}}(\int_{\mathbb{R}}|u_m-u|^2dx)^{\frac{1}{2}}\\ &\leq \eta_2(\int_{\mathbb{R}}|f(x,u_m)-f(x,u)|^2dx)^{\frac{1}{2}}\|u_m-u\|.\end{aligned} \tag{3.31}$$

As in the proof of Lemma 3.1, by passing to a subsequence if necessary, we may assume that

$$\int_{\mathbb{R}}|f(x,u_m)-f(x,u)|^2dx\longrightarrow 0 \text{ as } m\longrightarrow\infty.$$

Hence (3.31) implies

$$\int_{\mathbb{R}}(f(x,u_m)-f(x,u))(u_m-u)dx\longrightarrow 0 \text{ as } m\longrightarrow\infty. \tag{3.32}$$

Combining (3.28), (3.29), (3.30) and (3.32) yields $u_m\longrightarrow u$ as $m\longrightarrow\infty$ in E. The proof of Lemma 3.5 is completed. □

Note that (3.23) and (3.24) imply

$$\Phi'_{\lambda_n}(u_n^k)=0,\ \Phi_{\lambda_n}(u_n^k)\in[\overline{\alpha}_k,\overline{\xi}_k],\ \forall n\in\mathbb{N} \text{ and } k\geq k_0. \tag{3.33}$$

Lemma 3.6. *For any $k\geq k_0$, the sequence $(u_n^k)_{n\in\mathbb{N}}$ obtained above is bounded.*

Proof. For notational simplicity, we set $u_n=u_n^k$ for all $n\in\mathbb{N}$. Assume indirectly that (u_n) is unbounded. By going to a subsequence if necessary, we may assume that

$$\|u_n\|\longrightarrow\infty \text{ and } v_n=\frac{u_n}{\|u_n\|}\rightharpoonup v \text{ as } n\longrightarrow\infty. \tag{3.34}$$

By Lemma 2.2 and (3.34), without loss of generality, we have

$$v_n\longrightarrow v \text{ both in } L^2(\mathbb{R}) \text{ and } L^\nu(\mathbb{R}) \text{ and } v_n(x)\longrightarrow v(x) \text{ a.e. } x\in\mathbb{R} \text{ as } n\longrightarrow\infty. \tag{3.35}$$

Case I. $v=0$. Let (s_n) be a sequence such that

$$\Phi_{\lambda_n}(s_nu_n)=\max_{s\in[0,1]}\Phi_{\lambda_n}(su_n),\ \forall n\in\mathbb{N}. \tag{3.36}$$

For $R>0$, let $w_n=2\sqrt{R}v_n$. By (3.35), we have

$$w_n\longrightarrow 2\sqrt{R}v=0 \text{ both in } L^2(\mathbb{R}) \text{ and } L^\nu(\mathbb{R}), \tag{3.37}$$

which with (H_1) implies

$$\left|\int_{\mathbb{R}}F(x,w_n)dx\right|\leq c\int_{\mathbb{R}}(|w_n|^2+|w_n|^\nu)dx\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{3.38}$$

Note that (3.34) implies that $0 < \frac{2\sqrt{R}}{\|u_n\|} < 1$ for n large enough. This together with (3.36) and (3.38) implies

$$\begin{aligned}\Phi_{\lambda_n}(s_n u_n) &\geq \Phi_{\lambda_n}(w_n)\\ &= \frac{1}{2}\|u_n\|^2 - \lambda_n \int_{\mathbb{R}} F(x, w_n)dx\\ &\geq 2R - 2\int_{\mathbb{R}} F(x, w_n)dx \geq R\end{aligned}$$

for n large enough. Since R is arbitrarily, it follows that

$$\lim_{n\longrightarrow\infty} \Phi_{\lambda_n}(s_n u_n) = +\infty. \tag{3.39}$$

Since $\Phi_{\lambda_n}(0) = 0$ and $\Phi_{\lambda_n}(u_n) \in [\overline{\alpha}_k, \overline{\xi}_k]$, one has $s_n \in]0,1[$ in (3.36) for n large enough. Therefore

$$0 = s_n \frac{d}{ds}(\Phi_{\lambda_n})(su_n)_{|s=s_n} = \Phi'_{\lambda_n}(s_n u_n)s_n u_n. \tag{3.40}$$

Combining (3.33), (3.39) and (H_3) yields

$$\begin{aligned}\Phi_{\lambda_n}(u_n) - \frac{1}{2}\Phi'_{\lambda_n}(u_n)u_n &= \frac{\lambda_n}{2}\int_{\mathbb{R}} \widehat{F}(x, w_n)dx\\ &\geq \frac{\lambda_n}{2\sigma}\int_{\mathbb{R}} \widehat{F}(x, s_n w_n)dx\\ &= \frac{1}{\sigma}[\Phi_{\lambda_n}(s_n u_n) - \frac{1}{2}\Phi'_{\lambda_n}(s_n u_n)s_n u_n]\\ &= \frac{1}{\sigma}\Phi_{\lambda_n}(s_n u_n) \longrightarrow +\infty \ as\ n \longrightarrow \infty,\end{aligned}$$

a contradiction with (3.33).

Case II. $v \neq 0$. The set $\Lambda = \{x \in \mathbb{R}/v(x) \neq 0\}$ has a positive measure. By (3.34), it holds that

$$|u_n(x)| \longrightarrow \infty \ as\ n \longrightarrow \infty,\ \forall x \in \Lambda. \tag{3.41}$$

Combining (3.35), (3.41) and (H_2), Fatou's lemma implies

$$\begin{aligned}\frac{1}{2} - \frac{\Phi_{\lambda_n}}{\|u_n\|^2} &= \lambda_n \int_{\mathbb{R}} \frac{F(x, u_n)}{\|u_n\|^2}dx\\ &\geq \int_{\Lambda} |v_n|^2 \frac{F(x, u_n)}{|u_n|^2}dx \longrightarrow \infty \ as\ n \longrightarrow \infty,\end{aligned}$$

which provides a contradiction with (3.33) and (3.34). The proof of Lemma 3.6 is completed. □

Finally, using the similar arguments in the proof of Lemma 3.5, (3.33) and Lemma 3.6, we can show that for any $k \geq k_0$, the sequence $(u_n^k)_{n\in\mathbb{N}}$ possesses a strong convergent subsequence with the limit u^k being a critical point of $\Phi = \Phi_1$. Since $\overline{\alpha}_k \longrightarrow +\infty$ as $k \longrightarrow \infty$ and $\Phi(u^k) \in [\overline{\alpha}_k, \overline{\xi}_k]$ for all $k \geq k_0$, we find that Φ has infinitely many critical points. Consequently, (1.1) has infinitely many nontrivial homoclinic solutions.

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