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INVERSE STURM-LIOUVILLE PROBLEM WITH A BOUNDARY CONDITION LINEAR IN THE SPECTRAL PARAMETER

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Abstract. The direct problem, i.e., the description of the spectrum of the Sturm-Liouville boundary value problem with a spectral parameter dependent boundary condition is investigated based on quadratic operator pencils. The corresponding inverse problem is solved. The necessary and sufficient conditions for a sequence of numbers to be the spectrum of such problem are given. An algorithm of recovering the parameters of the problem using its spectrum is proposed and the solution is unique.

Keywords. Eigenvalue; Spectrum; Paley-Wiener class; Marchenko equation; Quadratic operator pencil.

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1. Introduction

In this paper, we solve the direct and inverse spectral problems generated by the Sturm-Liouville equation

$$-y''+q(x)y=\lambda^2 y \tag{1.1}$$

with the boundary conditions

$$y(0)=0, \tag{1.2}$$

$$y'(a)+(\lambda\alpha+\beta)y(a)=0, \tag{1.3}$$

where λ is the spectral parameter, the parameters $\alpha>0$ and $\beta\in\mathbf{C}$ and the potential q is real-valued and belongs to $L_2(0,a)$. Condition (1.3) describes the gyroscopic forces (see [1]). By the direct problem, we mean the description of the spectrum of problem (1.1)–(1.3) and by inverse problem recovering q, α and β using the spectrum $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ of problem (1.1)–(1.3). To describe the spectrum of problem (1.1)–(1.3), we consider the corresponding quadratic operator pencil of the form $L(\lambda)=\lambda^2 M-\lambda G-A$. Such an operator pencil with a selfadjoint operator A bounded below describing potential energy, a bounded symmetric operator $M\geq 0$ describing the inertia of the system and an operator G bounded or subordinate to A occur in different physical problems, where, in most of cases, they have spectra consisting of normal

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(or isolated Fredholm) eigenvalues (see Definition 1.1.3 in [2]). Usually, the operator G is symmetric (see, e.g. [3], [4] and Chapter 4 in [2]) or antisymmetric (see [5] and Chapter 2 in [2]). In the first case G describes the gyroscopic effect while in the latter case damping forces. The problems in which gyroscopic forces occur can be found in [6], [7], [8], [9], [10], [11], [12]. The operator G corresponding to problem (1.1)–(1.3) is symmetric and of rank one and such situation is described in [13].

We will use the results presented in Section 2 to obtain general results on location of eigenvalues of problem (1.1)–(1.3). We show that the real eigenvalues compose a 'self-interlacing' sequence, what in the simplest case means that positive eigenvalues interlace with the moduli of the negative eigenvalues. For the finite dimensional case such self-interlacing sequences are described in [14]. Such self-interlacing is also known in case of pure imaginary eigenvalues of the so-called generalized Regge problem (see e.g. [15]). In Section 3, we describe asymptotic behavior of the eigenvalues of problem (1.1)–(1.3). In Section 4, we solve the inverse problem: 1) propose conditions necessary and sufficient for a sequence of numbers to be the spectrum of problem (1.1)-(1.3) with a real $q \in L_2(0,a)$, and the numbers $\alpha > 0$ and $\beta \in \mathbf{R}$, 2) give an algorithm of recovering q, α and β, 3) prove the uniqueness of the solution.

2. Direct Problems

We consider Sturm-Liouville problem (1.1)–(1.3) with the gyroscopic condition at the right end of the interval.

Let us introduce the operators A, G and M acting in the Hilbert space $H = L_2(0,a) \oplus \mathbf{C}$ according to the formulae:

$$A\begin{pmatrix} v(x) \\ c \end{pmatrix} = \begin{pmatrix} -v''(x) + q(x)v(x) \\ v'(a) + \beta v(a) \end{pmatrix},$$

$$D(A) = \left\{ \begin{pmatrix} v(x) \\ c \end{pmatrix} : \ v(x) \in W_2^2(0,a), \quad v(0) = 0, \ c = v(a) \right\},$$

$$M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \ G = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}.$$

The operator A is selfadjoint and bounded below, i.e., there exists a number $\gamma \in \mathbf{R}$ such that $A \geq \gamma I$. Then there exists $\gamma_1 < \gamma$ such that $(A - \gamma_1 I)^{-1}$ is a compact operator (see [2], Proposition 2.1.1).

We associate the quadratic operator pencil

$$L(\lambda) = \lambda^2 M - \lambda G - A$$

with problem (1.1)–(1.3). We identify the spectrum of problem (1.1) - (1.3) with the spectrum of the pencil $L(\lambda)$. It is clear that the operators $M \geq 0$, $G \geq 0$ are bounded. We see that $M + G \geq \varepsilon I$ ($\varepsilon > 0$) and that the spectrum of the pencil $\{\lambda_k\}_{-\infty, k\neq 0}^{\infty}$ consists of normal eigenvalues (see the proof of Proposition 2.1.1 in [2]).

The following result is a direct consequence of Theorem 3.8 of [13].

Theorem 2.1. *1. Problem (1.1)–(1.3) may have nonreal eigenvalues located symmetrically with respect to the real axis. Denote their number with account of multiplicities by* $2\kappa \geq 0$.
2. All the nonreal eigenvalues lie in the open right half-plane.
3. The negative eigenvalues $\{\lambda_k\}_{-\infty}^{\kappa_0}$ *(*$\kappa_0 < 0$*) can be indexed such that* $\lambda_{k-1} < \lambda_k$*. Then* $|\lambda_k| \notin \{\lambda_k\}_{-\infty, k\neq 0}^{\infty}$ *for all* $-\infty < k \leq \kappa_0$.

4. The number n_k of eigenvalues counting with multiplicities is odd in each interval $(|\lambda_k|,|\lambda_{k-1}|)$ $(-\infty < k \le \kappa_0)$.
5. If $0 \notin \{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$, then the number n_0 of eigenvalues in the interval $(0,|\lambda_{\kappa_0}|)$ is even. If $0 \in \{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ then the number of eigenvalues in the interval $(0,|\lambda_{\kappa_0}|)$ is odd and 0 is a simple eigenvalue.
6. Denote by

$$\tilde{\kappa} = \frac{1}{2}\begin{cases} n_0 + \sum\limits_{k=-\infty}^{\kappa_0}(n_k-1) \text{ if } 0 \notin \{\lambda_k\}_{-\infty,k\neq 0}^{\infty}, \\ n_0 - 1 + \sum\limits_{k=-\infty}^{\kappa_0}(n_k-1), \text{ if } 0 \in \{\lambda_k\}_{-\infty,k\neq 0}^{\infty} \end{cases}.$$

Then $\tilde{\kappa}+\kappa = |\kappa_0|-1 = \kappa_A$, where κ_A is the number of negative eigenvalues of the linear operator pencil $\lambda M - A$ or, what is the same, the number of negative eigenvalues of problem

$$-y''+q(x)y = \lambda y,$$
$$y(0) = y'(a)+\beta y(a) = 0.$$

3. Asymptotics of Eigenvalues

In this section, we consider the asymptotic behavior of the eigenvalues of problem (1.1)–(1.3). Denote by $s(\lambda,x)$ the solution of equation (1.1) satisfying the conditions $s(\lambda,0) = s'(\lambda,0)-1 = 0$. Then the spectrum of problem (1.1)-(1.3) is the set of zeros of the characteristic function

$$\phi(\lambda) = s'(\lambda,a)+(\alpha\lambda+\beta)s(\lambda,a). \tag{3.1}$$

It is known (see e.g. eq. (1.2.10) in [16] or Theorem 12.2.9 in [2]) that

$$s(\lambda,x) = \frac{\sin\lambda x}{\lambda} - K(x,x)\frac{\cos\lambda x}{\lambda^2} + \int_0^x K_t(x,t)\frac{\cos\lambda t}{\lambda^2}dt, \tag{3.2}$$

where

$$K(x,t) = \tilde{K}(x,t)-\tilde{K}(x,-t),\ \ K_t(x,t) = \frac{\partial K(x,t)}{\partial t}$$

and $\tilde{K}(x,t)$ is the solution of the integral equation

$$\tilde{K}(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(s)ds + \int_0^{\frac{x+t}{2}} d\alpha \int_0^{\frac{x-t}{2}} q(\alpha+\beta)\tilde{K}(\alpha+\beta,\alpha-\beta)d\beta.$$

The solution $\tilde{K}(x,t)$ possesses partial derivatives of the first order belonging to $L_2(0,a)$ as a function of each of its variables. Moreover, $K(x,0)=0$ and

$$K(x,x) = \frac{1}{2}\int_0^x q(t)dt.$$

It is also clear that

$$s'(\lambda,x) = \cos\lambda a + K(x,x)\frac{\sin\lambda a}{\lambda} + \int_0^x K_x(x,t)\frac{\sin\lambda t}{\lambda}dt. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we obtain

$$\phi(\lambda) = \cos\lambda a + \alpha\sin\lambda a + \frac{K+\beta}{\lambda}\sin\lambda a - \frac{\alpha K}{\lambda}\cos\lambda a + \frac{\psi(\lambda)}{\lambda}, \tag{3.4}$$

where $K := K(a,a)$ and

$$\psi(\lambda) = \int_0^a K_x(a,t)\sin\lambda t dt - \beta K\frac{\cos\lambda a}{\lambda} + \alpha\int_0^a K_t(a,t)\cos\lambda t dt + \beta\int_0^a K_t(a,t)\frac{\cos\lambda t}{\lambda}dt.$$

Definition 3.1. ([17] Section 2.5 or [2] Definition 12.2.2) An entire function ω of exponential type $\leq \sigma$ is said to belong to the Paley-Wiener class $\mathfrak{L}^\sigma$ if its restriction to the real axis belongs to $L_2(-\infty, \infty)$.

Lemma 3.2. $\psi \in \mathfrak{L}^a$.

Proof. In view of Lemma1.4.3 [16] or Lemma 12.2.1 [2], one has

$$\int_0^a K_x(a,t)\sin\lambda t dt \in \mathfrak{L}^a, \quad \int_0^a K_t(a,t)\cos\lambda t dt \in \mathfrak{L}^a.$$

This proves this lemma. $\square$

Definition 3.3. (see [18] Sec. 1 or [2] Definition 11.2.5) An entire function ω of exponential type is said to be a sine type function if

(i) there is $h > 0$ such that all zeros of ω lie in the strip $\{\lambda \in \mathbf{C} : |\mathrm{Im}| < h\}$.

(ii) there are $h_1 \in \mathbf{R}$ and positive numbers $m < M$ such that $m \leq |\omega(\lambda)| \leq M$ holds for $\lambda \in \mathbf{C}$ with $\mathrm{Im}\lambda = h_1$ (for some $h_1 > h$),

(iii) the exponential type of ω in the lower half-plane coincides with the exponential type of ω in the upper half-plane.

Theorem 3.4.

$$\lambda_k \underset{k\to\infty}{=} \mathrm{sign}(k)\frac{\pi\left(|k|-\frac{1}{2}\right)}{a} + \frac{1}{a}\arccos\frac{1}{1+\alpha^2} + \frac{P}{k} + \frac{\beta_k}{k}, \tag{3.5}$$

where

$$P = \frac{1}{2\pi}\left(\int_0^a q(x)dx + \frac{2\beta}{1+\alpha^2}\right) \tag{3.6}$$

and $\{\beta_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$.

Proof. Denote by

$$\phi^{(0)}(\lambda) = \cos\lambda a + \alpha\sin\lambda a.$$

This function is of sine type as well as the function ϕ given by (3.4). Denote by $\{\lambda_k^{(0)}\}_{k=-\infty,k\neq 0}^{\infty}$ the zeros of $\phi^{(0)}(\lambda)$ and let

$$\Lambda_\delta := \{\lambda \in \mathbf{C} : \exists \lambda_k^{(0)} \in \mathbf{C} : \ |\lambda - \lambda_k^{(0)}| < \delta\}.$$

It is clear that

$$\lambda_k^{(0)} = \mathrm{sign}(k)\frac{\pi\left(|k|-\frac{1}{2}\right)}{a} + \frac{1}{a}\arccos\frac{1}{1+\alpha^2}.$$

Using Lemma 11.2.20 in [2], we conclude that for each $\delta > 0$, there exists $k_\delta > 0$ such that $|\phi^{(0)}(\lambda)| > k_\delta e^{|\mathrm{Im}\lambda| a}$ for all $\lambda \in \mathbf{C}\backslash\Lambda_\delta$.

On the other hand, there exists $R_\delta^{(1)}$ such that

$$\left|\frac{K+\beta}{\lambda}\sin\lambda a - \frac{\alpha K}{\lambda}\cos\lambda a\right| < \frac{k_\delta}{2}e^{|\mathrm{Im}\lambda| a}$$

for all λ with $|\lambda| > R_\delta^{(1)}$. Since ψ is a Paley-Wiener function, one sees that there exists $R_\delta^{(2)} > 0$ such that

$$\left|\frac{\psi(\lambda)}{\lambda}\right| < \frac{k_\delta}{2}e^{|\mathrm{Im}\lambda| a}$$

for all λ with $|\lambda| > R_\delta^{(2)}$. Denote by $R_\delta = \max\{R_\delta^{(1)}, R_\delta^{(2)}\}$. Then, for all $\lambda \in \mathbf{C}\backslash(\Lambda_\delta \cup \{\lambda : |\lambda| < R_\delta\})$,

$$|\phi^{(0)}(\lambda)| > \left|\frac{K+\beta}{\lambda}\sin\lambda a - \frac{\alpha K}{\lambda}\cos\lambda a + \frac{\psi(\lambda)}{\lambda}\right|.$$

Using the Rouche's theorem for $|k|$ large enough each disc $|\lambda-\lambda_k^{(0)}|<\delta$ contains exactly one zero of $\phi(\lambda)$. Since δ can be chosen arbitrary (positive) small, we obtain

$$\lambda_k=\operatorname{sign}(k)\frac{\pi\left(|k|-\frac{1}{2}\right)}{a}+\frac{1}{a}arccos\frac{1}{1+\alpha^2}+\Delta_k, \tag{3.7}$$

where $\Delta_k \underset{k\to\infty}{\to} 0$. Substituting (3.7) into (3.4), we obtain (3.5) with (3.6). This completes the proof. □

4. Inverse Problems

Our aim is to recover q, α and β using the spectrum of problem (1.1)–(1.3).

Theorem 4.1. *Let a sequence* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}=\{\lambda_k\}_{-\infty}^{\kappa_0}\cup\{\lambda_k\}_{|\kappa_0|}^{\infty}\cup\{\lambda_k\}_{\kappa_0+1,k\neq 0}^{|\kappa_0|-1}$, *where* $\kappa_0\le -1$, $\lambda_{k+1}>\lambda_k>0$ *for* $k\ge|\kappa_0|$ *and* $0>\lambda_k>\lambda_{k-1}$ *for* $k\le\kappa_0$ *if* $0\notin\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *or* $\{\lambda_k\}_{-\infty}^{\kappa_0}\cup\{\lambda_k\}_{k=|\kappa_0|-1}^{\infty}\cup\{\lambda_{\kappa_0+1}\}\cup\{\lambda_k\}_{\kappa_0+2,k\neq 0}^{|\kappa_0|-2}$ *if* $\lambda_{\kappa_0+1}=0$ *satisfies the conditions:*
1. $\operatorname{Re}\lambda_k>0$ *for all* $k\in\{k>\kappa_0,k\neq 0\}$ *if* $0\notin\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *and for* $k\in\{k>\kappa_0+1\}$ *if* $\lambda_{\kappa_0+1}=0$.
2. If for some k $\operatorname{Im}\lambda_k\neq 0$, *then* $\lambda_{-k}=\overline{\lambda}_k$.
3. $0<|\lambda_{\kappa_0}|<\lambda_{|\kappa_0|}<|\lambda_{\kappa_0-1}|<\lambda_{|\kappa_0|+1}\ldots<|\lambda_{-k}|<\lambda_k<\ldots$.
4. For each $k\le\kappa_0$, *the interval* $(|\lambda_k|,|\lambda_{k-1}|)$ *contains odd number (with account of multiplicities) of elements of* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$.
5. The interval $[0,|\lambda_{\kappa_0}|)$ *contains even or zero number of elements of* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$.
6.

$$\lambda_k=\operatorname{sign}(k)\frac{\pi\left(|k|-\frac{1}{2}\right)}{a}+R+\frac{P}{k}+\frac{\beta_k}{k}, \tag{4.1}$$

where $a>0$, $R\in(0,\frac{\pi}{2a})$, $P\in\mathbf{R}$ *and* $\{\beta_k\}_{-\infty,k\neq 0}^{\infty}\in l_2$.

Then there exists a unique set $\{\alpha>0,\ \beta\in\mathbf{R},\text{real } q(x)\in L_2(0,a)\}$ *such that* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *is the spectrum of problem (1.1)–(1.3).*

Proof. Let us construct the function

$$\phi(\lambda)=\prod_{-\infty,k\neq 0}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right)$$

if $0\notin\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ and

$$\phi(\lambda)=\lambda\prod_{-\infty,k\neq 0,\ k_0+1}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right)$$

if $\lambda_{\kappa_0+1}=0$. We compare $\phi(\lambda)$ with $\phi^{(0)}(\lambda)$ defined by

$$\phi^{(0)}(\lambda):=\cos\lambda a+\alpha\sin\lambda a=\prod_{-\infty,k\neq 0}^{\infty}\left(1-\frac{\lambda}{\lambda_k^{(0)}}\right), \tag{4.2}$$

where

$$\alpha=\sqrt{\frac{1}{\cos aR}-1} \tag{4.3}$$

and

$$\lambda_k^{(0)}=\operatorname{sign}(k)\frac{\pi\left(|k|-\frac{1}{2}\right)}{a}+R. \tag{4.4}$$

Comparing with (4.1) with (4.4), we obtain

$$\lambda_k=\lambda_k^{(0)}+\frac{\pi P}{a\lambda_k^{(0)}}+\frac{b_k}{\lambda_k^{(0)}},$$

where $\{b_k\}_{-\infty,k\neq 0}^{\infty}\in l_2$. Using Lemma 5 of [18] or Lemma 11.3.15 of [2], one has

$$\phi(\lambda)=C\phi^{(0)}(\lambda)\left(1+\frac{B}{\lambda}\right)-C\frac{\pi P}{a\lambda}(\phi^{(0)}(\lambda))'+\frac{f(\lambda)}{\lambda}, \tag{4.5}$$

where $f\in\mathfrak{L}^a$. Due to the symmetry of $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ and of $\{\lambda_k^{(0)}\}_{-\infty,k\neq 0}^{\infty}$ with respect to the real axis $B\in\mathbf{R}$, $C\in\mathbf{R}\backslash\{0\}$, substituting (4.2) into (4.5), we obtain

$$\phi(\lambda)=C(\cos\lambda a+\alpha\sin\lambda a)+\frac{C}{\lambda}(B-\pi\alpha P)\cos\lambda a+\frac{C}{\lambda}(B\alpha+\pi P)\sin\lambda a+\frac{f(\lambda)}{\lambda}$$

with $f\in\mathfrak{L}^a$. Thus,

$$\phi\left(\frac{\pi}{a}k\right)=C+\frac{aC}{\pi k}(B-\pi\alpha P)+\frac{a}{\pi k}f\left(\frac{\pi}{a}k\right).$$

By Lemma 1.4.3 of [16] or Lemma 12.2.1 [2], one has

$$f\left(\frac{\pi}{a}k\right)\underset{k\to\infty}{\to}0.$$

It follows that $C=\lim\limits_{k\to\infty}\phi\left(\frac{\pi}{a}k\right)$ and

$$B=\pi\alpha P+\frac{\pi}{a}\lim_{k\to\infty}\left(C^{-1}k\phi\left(\frac{\pi}{a}k\right)-1\right).$$

We define

$$\beta=\frac{B(\alpha^2+1)}{\alpha}. \tag{4.6}$$

Evidently,

$$\frac{\phi(\lambda)+\phi(-\lambda)}{2}=C\cos\lambda a+C\frac{(B\alpha+\pi P)\sin\lambda a}{\lambda}+\frac{f(\lambda)+f(-\lambda)}{2\lambda},$$

we introduce the functions

$$Q(\lambda^2):=\frac{\phi(\lambda)-\phi(-\lambda)}{2\alpha\lambda}=C\frac{\sin\lambda a}{\lambda}+C\left(\frac{B}{\alpha}-\pi P\right)\frac{\cos\lambda a}{\lambda^2}+\frac{f(\lambda)-f(-\lambda)}{2\alpha\lambda^2}, \tag{4.7}$$

and

$$P(\lambda^2):=\frac{\phi(\lambda)+\phi(-\lambda)}{2}-\beta\frac{\phi(\lambda)-\phi(-\lambda)}{2\alpha\lambda}=C\cos\lambda a+C\left(\pi P-\frac{B}{\alpha}\right)\frac{\sin\lambda a}{\lambda}+\frac{\tilde{f}(\lambda)}{\lambda}, \tag{4.8}$$

where $\frac{f(\lambda)-f(-\lambda)}{2\alpha}\in\mathfrak{L}^a$ and $\hat{f}(\lambda)\in\mathfrak{L}^a$. The zeros $\{\mu_k\}_{-\infty,k\neq 0}^{\infty}$ of $P(\lambda^2)$ and the zeros $\{\nu_k\}_{-\infty,k\neq 0}^{\infty}$ of $Q(\lambda^2)$ behave asymptotically as follows (see, e.g. Lemma 3.4.2 in [16] or Lemma 12.3.3 in [2])

$$\mu_k=\frac{\pi(k-\frac{1}{2})}{a}+\frac{\pi P-B\alpha^{-1}}{k}+\frac{\beta_k^{(1)}}{k}, \tag{4.9}$$

and

$$\nu_k=\frac{\pi k}{a}+\frac{\pi P-B\alpha^{-1}}{k}+\frac{\beta_k^{(2)}}{k}, \tag{4.10}$$

where $\{\beta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}\in l_2$ for $j=1,2$.

Now let us prove that the zeros $\{\mu_k\}_{-\infty,k\neq 0}^{\infty}$ are interlaced with the zeros $\{\nu_k\}_{-\infty,k\neq 0}^{\infty}$. To this end we consider the auxiliary problem

$$-y''+q_0y=zy, \tag{4.11}$$

$$y(0)=0, \tag{4.12}$$

$$y'(a)+\beta y(a)=0. \tag{4.13}$$

where z is the spectral parameter and q_0 is a real constant. The eigenvalues of this problem are the zeros of the characteristic function

$$\cos\sqrt{z-q_0}a+\beta\frac{\sin\sqrt{z-q_0}a}{\sqrt{z-q_0}}.$$

If $0\notin\{\lambda_k\}_{-\infty,k\neq0}^{\infty}$, we choose $q_0\equiv-\frac{\pi^2}{a^2}(|\kappa_0|-1)^2$. It is clear that problem (4.11)–(4.13) has $|\kappa_0|-1$ negative eigenvalues. If $0\in\{\lambda_k\}_{-\infty,k\neq0}^{\infty}$, then we choose as $\sqrt{-q_0}$ the solution of

$$\cot\sqrt{-q_0}a=-\frac{a\beta}{\sqrt{-q_0}},$$

which lies in the interval $(\frac{\pi}{a}(|\kappa_0|-1)<\sqrt{-q_0}<\frac{\pi}{a}|\kappa_0|)$. Such solution is unique. Under such choice, problem (4.11)–(4.13) has again exactly $|\kappa_0|-1$ negative eigenvalues. Now we consider one more auxiliary problem

$$-y''+q_0y=\lambda^2y, \tag{4.14}$$

$$y(0)=0, \tag{4.15}$$

$$y'(a)+(\alpha\lambda+\beta)y(a)=0. \tag{4.16}$$

Denote by $\{\lambda_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ the eigenvalues of problem (4.14)–(4.16). They satisfy 1.–6. of Theorem 2.1 with the same κ_0 and behave asymptotically (see Theorem 3.4) as follows

$$\lambda_k^{(1)}=\text{sign}(k)\frac{\pi\left(|k|-\frac{1}{2}\right)}{a}+\frac{1}{a}\arccos\frac{1}{1+\alpha^2}+\frac{P^{(1)}}{k}+\frac{\tau_k}{k}, \tag{4.17}$$

where

$$P^{(1)}=\frac{1}{2\pi}\left(aq_0+\frac{2\beta}{1+\alpha^2}\right)$$

and $\{\tau_k\}_{-\infty,k\neq0}^{\infty}\in l_2$. Let $0\notin\{\lambda_k\}_{-\infty,k\neq0}^{\infty}$. Then we introduce the function

$$\Phi(\lambda,\eta)=\prod_{-\infty,k\neq0}^{\infty}\left(1-\frac{\lambda}{\lambda_k^{(1)}+\eta(\lambda_k-\lambda_k^{(1)})}\right). \tag{4.18}$$

Comparing (4.1) with (4.17), we obtain

$$\lambda_k^{(1)}+\eta(\lambda_k-\lambda_k^{(1)})=\lambda_k^{(1)}+\eta\frac{(P-P^{(1)})}{k}+\frac{a_k}{k},$$

where $\{a_k\}_{-\infty,k\neq0}^{\infty}\in l_2$. The product in (4.18) converges uniformly with respect of η belonging to any bounded open domain containing the interval $[0,1]$ to a function which is entire with respect to λ at η fixed and analytic with respect to η at λ fixed. We consider the functions

$$P(\lambda,\eta)=\frac{\Phi(\lambda,\eta)+\Phi(-\lambda,\eta)}{2},$$

and

$$Q(\lambda,\eta)=\frac{\Phi(\lambda,\eta)-\Phi(-\lambda,\eta)}{2\lambda},$$

which are entire functions of λ at η fixed. Denote by $\{\mu_k(\eta)\}_{-\infty,k\neq0}^{\infty}$ the zeros of $P(\lambda,\eta)$ and by $\{\nu_k(\eta)\}_{-\infty,k\neq0}^{\infty}$ the zeros of $Q(\lambda,\eta)$. If $\cos\sqrt{-q_0}a+\beta\frac{\sin\sqrt{-q_0}a}{\sqrt{-q_0}}\neq0$, then

$$\Phi(\lambda,0)=(\cos\sqrt{\lambda^2-q_0}a+(\alpha\lambda+\beta)\frac{\sin\sqrt{\lambda^2-q_0}a}{\sqrt{\lambda^2-q_0}})(\cos\sqrt{-q_0}a+\beta\frac{\sin\sqrt{-q_0}a}{\sqrt{-q_0}})^{-1}.$$

Thus we have

$$P(\lambda,0)=(\cos\sqrt{\lambda^2-q_0}a+\beta\frac{\sin\sqrt{\lambda^2-q_0}a}{\sqrt{\lambda^2-q_0}})(\cos\sqrt{-q_0}a+\beta\frac{\sin\sqrt{-q_0}a}{\sqrt{-q_0}})^{-1}$$

and

$$Q(\lambda,0)=\alpha\frac{\sin\sqrt{\lambda^2-q_0}a}{\sqrt{\lambda^2-q_0}})(\cos\sqrt{-q_0}a+\beta\frac{\sin\sqrt{-q_0}a}{\sqrt{-q_0}})^{-1}.$$

The zeros of $P(\lambda,0)$ and of $Q(\lambda,0)$ are real and pure imaginary. The zeros

$$\{\nu_k(0)\}_{-\infty,k\neq0}^{\infty}=\{\pm\sqrt{(\frac{\pi k}{a})^2+q_0}\}_{-\infty,k\neq0}^{\infty}$$

of $Q(\lambda,0)$ interlace with the zeros $\{\mu_k(0)\}_{-\infty,k\neq0}^{\infty}$ of $P(\lambda,0)$:

$$(\mu_1(0))^2<(\nu_1(0))^2<(\mu_2(0))^2<(\nu_2(0))^2<\ldots$$

Since $\Phi(\overline{\lambda},\eta)=\overline{\Phi(\lambda,\eta)}$ for real η we have

$$P(\overline{\lambda},\eta)=\overline{P(\lambda,\eta)},\;\;Q(\overline{\lambda},\eta)=\overline{Q(\lambda,\eta)}\;\;(\eta\in\mathbf{R}). \tag{4.19}$$

The zeros $\mu_k(\eta)$ and $\nu_k(\eta)$ are piece-wise analytic functions of η which loose their analyticity only when collide (see, e.g. Theorem 9.2.4 in [2]). Due to (4.19) the zeros $\mu_k(\eta)$ and $\mu_{k+1}(\eta)$ can leave the imaginary or real axis while η is growing only if $\mu_k(\eta_1)=\mu_{k+1}(\eta_1)$ for some $\eta_1\in(0,1]$. But $\mu_k(\eta_1)=\nu_k(\eta_1)$. This leads to $P(\mu_k(\eta_1),\eta_1)=Q(\mu_k(\eta_1),\eta_1)=0$ and $\Phi(\mu_k(\eta_1),\eta_1)=0$. If $(\mu_k(\eta_1))^2<0$, then it contradicts statement 2 of Theorem 2.1. If $(\mu_k(\eta_1))>0$, then it contradicts Statement 3 of Theorem 2.1. Finally, $\mu_k(\eta_1)\neq0$ because $\lambda_k(\eta_1)=0$ must be simple zero according to Statement 5 of Theorem 2.1. Thus, we conclude that

$$(\mu_1(\eta))^2<(\nu_1(\eta))^2<(\mu_2(\eta))^2<(\nu_2(\eta))^2<\ldots$$

for all $\eta\in[0,1]$. Since $\mu_k(1)=\mu_k$ and $\nu_k(1)=\nu_k$ for all k, we obtain

$$(\mu_1)^2<(\nu_1)^2<(\mu_2)^2<(\nu_2)^2<\ldots \tag{4.20}$$

The case of $\cos\sqrt{-q_0}a+\beta\frac{\sin\sqrt{-q_0}a}{\sqrt{-q_0}}=0$ corresponds to the case of $\lambda_k^{(1)}=0$ for some k and can be considered similarly. Due to (4.9), (4.10), (4.20) the sequences $\{\mu_k\}_{-\infty,k\neq0}^{\infty}$ and $\{\nu_k\}_{-\infty,k\neq0}^{\infty}$ satisfy the conditions of Theorem 3 4.1 of [16] or Theorem 12.6.2 of [2]. Therefore, there exists a unique real $q\in L_2(0,a)$ which generates problems

$$-y''+q(x)y=\lambda^2y,$$

$$y(0)=y'(a)=0$$

and

$$-y''+q(x)y=\lambda^2y,$$

$$y(0)=y(a)=0$$

with the spectra $\{\mu_k\}_{-\infty,k\neq0}^{\infty}$ and $\{\nu_k\}_{-\infty,k\neq0}^{\infty}$, respectively. We can find q via procedure of [16] described below. Without loss of generality, let us assume that $\mu_1^2>0$, otherwise, we apply a shift of the spectral parameter. The function

$$e(\lambda)=(\omega_2(\lambda)-i\lambda\omega_1(\lambda))e^{-i\lambda a},$$

where

$$\lambda\omega_1(\lambda) := \lambda a \prod_{k=1}^{\infty}\left(\frac{a}{\pi k}\right)^2 (\nu_k^2 - \lambda^2),$$

$$\omega_2(\lambda) := \prod_{k=1}^{\infty}\left(\frac{a}{\pi(k-\frac{1}{2})}\right)^2 (\mu_k^2 - \lambda^2),$$

is the Jost-function of the corresponding prolonged Sturm-Liouville problem on the semi-axis:

$$-y'' + \tilde{q}(x)y = \lambda^2 y, \ \ x \in [0,\infty), \tag{4.21}$$

$$y(0) = 0 \tag{4.22}$$

with

$$\tilde{q}(x) = \begin{cases} q(x), & if\ x \in [0,a], \\ 0, & if\ x \in (a,\infty). \end{cases}$$

Then we construct the S-function of problem (4.21), (4.22):

$$S(\lambda) = \frac{e(-\lambda)}{e(\lambda)}$$

and the function

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1 - S(\lambda))e^{i\lambda x}d\lambda.$$

Solving the Marchenko equation

$$K(x,t) + G(x+t) + \int_x^{\infty} K(x,s)G(s+t)ds = 0,$$

we find $K(x,t)$ and the potential:

$$q(x) = -2\frac{dK(x,x)}{dx}, \ \ x \in [0,a]$$

which is a real function and belongs to $L_2(0,a)$. Our aim is to prove that this q together with α obtained from (4.3) and β obtained from (4.6) generate problem (1.1)–(1.3) whose spectrum is $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$. To this end, we notice that if $s(\lambda,x)$ is the corresponding solutions of (1.1) with the obtained potential $q(x)$ then $s'(\lambda,a)$ and $\omega_2(\lambda)$ have the same set of zeros. Due to Lemma 3.4.2 of [16] or Lemma 12.3.3 [2], one has

$$\omega_2(\lambda) = \cos\lambda a + \left(\pi P - \frac{B}{\alpha}\right)\frac{\sin\lambda a}{\lambda} + \frac{f_2(\lambda)}{\lambda}, \tag{4.23}$$

where $f_2 \in \mathfrak{L}^a$. Thus, comparing (4.23) with (3.3), we obtain $\omega_2(\lambda) = s'(\lambda,a)$. Using the same lemmas, one has

$$\omega_1(\lambda) = \frac{\sin\lambda a}{\lambda} - \left(\pi P - \frac{B}{\alpha}\right)\frac{\cos\lambda a}{\lambda^2} + \frac{f_1(\lambda)}{\lambda^2}, \tag{4.24}$$

where $f_1 \in \mathfrak{L}^a$. Comparing (4.24) with (3.2), we arrive at $\omega_1(\lambda) = s(\lambda,a)$. Comparison of (3.2) with (4.7) shows that $(\alpha C)^{-1}Q(\lambda,1) = (\alpha C)^{-1}Q(\lambda^2) = s(\lambda,a)$. In the same way, comparison (3.3) with (4.8) shows that $C^{-1}(P(\lambda,1) - \beta\alpha^{-1}Q(\lambda,1)) = C^{-1}(P(\lambda^2) - \beta\alpha^{-1}Q(\lambda^2)) = s'(\lambda,a)$. Then $C^{-1}\phi(\lambda) = C^{-1}(P(\lambda) + \alpha\lambda Q(\lambda)) = s'(\lambda,a) + (\alpha\lambda + \beta)s(\lambda,a)$. Therefore the set of zeros of ϕ coincides with the spectrum of problem (1.1)–(1.3) with the obtained q, α and β. Theorem is proved. □

Remark 4.2. Comparing Theorem 4.1 with Theorem 2.1 and Theorem 3.4, we see that the conditions of Theorem 4.1 are necessary and sufficient.

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