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EXISTENCE OF POSITIVE SOLUTIONS FOR DIFFERENTIAL EQUATIONS INVOLVING RIEMANN-LIOUVILLE AND CAPUTO FRACTIONAL DERIVATIVES

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Abstract. In this paper, the existence of positive solutions is considered for differential equations involving the Riemann-Liouville fractional derivative and the Caputo fractional derivative with p-Laplacian operators. Based on the monotone iterative technique, we obtain the existence of positive solutions for boundary value problems. Approximation of solutions is investigated. An example is also provided to illustrate the effectiveness of the main results.

Keywords. Positive solution; Riemann-Liouville fractional derivative; Caputo fractional derivative; p-Laplacian opeator; Monotone iterative technique.

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1. INTRODUCTION

Fractional derivatives play an important role in the modeling of different physical and natural phenomena, such as, fluid mechanics, control systems and many other branches of engineering. Recently, there are many results on the Riemann-Liouville fractional differential equations with p-Laplacian (see [1]-[6]) and the Caputo fractional differential equations with p-Laplacian operators (see [7]-[16]).

In [1], Han, Lu and Zhang studied the existence of positive solutions for the following problem with the generalized p-Laplacian operator:

$$D_{0^+}^{\beta}(\phi(D_{0^+}^{\alpha}u(t))) = \lambda f(u(t)), \ 0 < t < 1,$$
$$u(0) = u'(0) = u'(1) = 0, \ \phi(D_{0^+}^{\alpha}u(0)) = (\phi(D_{0^+}^{\alpha}u(1)))' = 0,$$

where $2 < \alpha \le 3$, $1 < \beta \le 2$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann-Liouville fractional derivatives. Based on the properties of the Green's function and the Guo-Krasnosel'skii fixed point theorem on cones, some new existence results of at least one or two positive solutions are obtained for the boundary value problem.

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In [7], Chen, Liu and Hu investigated the existence of solutions by the coincidence degree for the Caputo fractional p-Laplacian equation

$$D^\beta_{0+} \phi_p(D^\alpha_{0+} x(t)) = f(t, x(t), D^\alpha_{0+} x(t)),\ 0 < t < 1,$$

$$D^\alpha_{0+} x(0) = D^\alpha_{0+} x(1) = 0,$$

where D^β_{0+} and D^α_{0+} are the Caputo fractional derivatives, $0 < \alpha, \beta \leq 1$ and $1 < \alpha + \beta \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $f : [0,1] \times R^2 \to R$ is continuous. They used $Lu = D^\beta_{0+} \phi_p(D^\alpha_{0+} x(t))$ with $D^\alpha_{0+} x(0) = D^\alpha_{0+} x(1) = 0$ and obtained $dimKerL = 1$.

Motivated by the above works, we investigate the existence of positive solutions for fractional differential equation:

$$\begin{cases} D^\beta_{0+}[\phi_p(^cD^\alpha_{0+} u(t))] = f(t, u(t)),\ t \in (0,1), \\ [\phi_p(^cD^\alpha_{0+} u(0))]^{(i)} = 0,\ i = 0, 1, \cdots, m-1, \\ \phi_p(^cD^\alpha_{0+} u(1)) = 0, \\ u^{(j)}(0) = 0,\ j = 2, 3, \cdots, n, \\ u'(1) = a \int_0^1 h(t) u'(t) dt,\ u(0) = b \int_0^1 h(t) u(t) dt, \end{cases} \tag{1.1}$$

where $1 < n < \alpha \leq n+1$, $1 < m < \beta \leq m+1$ and $m+n+1 < \alpha+\beta \leq m+n+2$, $\phi_p(u) = |u|^{p-2}u$, $p > 1$. D^β_{0+} is the Riemann-Liouville fractional derivative, $^cD^\alpha_{0+}$ is the Caputo fractional derivative. A function $u(t)$ is a positive solution of the boundary value problem (1.1) if and only if $u(t)$ satisfies the boundary value problem (1.1), and $u(t) \geq 0$ for $t \in [0,1]$. We use the monotone iterative technique to obtain the existence of positive solutions for boundary value problem (1.1). To the best of our knowledge, there are no results on the existence of positive solutions for mixed fractional p-Laplacian differential equations via the monotone iterative technique.

In this paper, we always suppose the following conditions are satisfied:

(H_1) $h(t) : [0,1] \to [0,\infty)$ with $h \in C[0,1]$ and $f(t,u) : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous;

(H_2) $a > 0$, $b > 0$, $0 < a \int_0^1 h(t) dt < 1$, $0 < b \int_0^1 h(t) dt < 1$, and

$$(\alpha - 1) \int_0^1 t\, h(t) dt > \int_0^1 h(t) dt > 0.$$

2. PRELIMINARIES

To show the main result of this paper, we give the following some basic definitions, which can be found in [18, 19].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I^\alpha_{0+} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$, where

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx.$$

Definition 2.2. For a continuous function $y : (0, +\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $\alpha > 0$ is defined as

$$D^\alpha_{0+} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.3. For a continuous function $y : (0, +\infty) \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$ is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t - s)^{n - \alpha - 1} y(s) ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

The following lemmas play an important role in this paper.

Lemma 2.4. *The boundary value problem (1.1) is equivalent to the following equation:*

$$u(t) = d_0 + d_1 t - \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} v(s) ds, \tag{2.1}$$

where

$$d_0 = d_1 \frac{b \int_0^1 t\, h(t) dt}{1 - b \int_0^1 h(t) dt} - \frac{b \int_0^1 h(t) \int_0^t (t - s)^{\alpha - 1} v(s) ds dt}{[1 - b \int_0^1 h(t) dt] \Gamma(\alpha)}, \tag{2.2}$$

$$d_1 = \frac{\int_0^1 (1 - s)^{\alpha - 2} v(s) ds}{[1 - a \int_0^1 h(t) dt] \Gamma(\alpha - 1)} - \frac{a \int_0^1 h(t) \int_0^t (t - s)^{\alpha - 2} v(s) ds dt}{[1 - a \int_0^1 h(t) dt] \Gamma(\alpha - 1)}, \tag{2.3}$$

$$v(s) = \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right), \tag{2.4}$$

$$H(s, \tau) = \frac{1}{\Gamma(\beta)} \begin{cases} (s - s\tau)^{\beta - 1} - (s - \tau)^{\beta - 1}, & 0 \leq \tau \leq s \leq 1, \\ (s - s\tau)^{\beta - 1}, & 0 \leq s \leq \tau \leq 1. \end{cases} \tag{2.5}$$

$\phi_q(s)$ *is the inverse function of* $\phi_p(s)$*, a.e.,* $\phi_q(s) = |s|^{q-2} s$*,* $\frac{1}{p} + \frac{1}{q} = 1$*,* $u(t) \in C[0, 1]$*.*

Proof. From $D_{0+}^{\beta} [\phi_p({}^c D_{0+}^{\alpha} u(t))] = f(t, u(t))$, we get

$$\phi_p({}^c D_{0+}^{\alpha} u(t)) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta - 1} f(\tau, u(\tau)) d\tau + c_1 t^{\beta - 1} + c_2 t^{\beta - 2} + \cdots + c_{m+1} t^{\beta - m - 1}.$$

In view of $[\phi_p({}^c D_{0+}^{\alpha} u(0))]^{(i)} = 0$, $i = 0, 1, \cdots, m - 1$, we get $c_2 = \cdots = c_{m+1} = 0$, i.e.,

$$\phi_p({}^c D_{0+}^{\alpha} u(t)) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta - 1} f(\tau, u(\tau)) d\tau + c_1 t^{\beta - 1}.$$

Condition $\phi_p({}^c D_{0+}^{\alpha} u(1)) = 0$ implies that

$$c_1 = -\frac{1}{\Gamma(\beta)} \int_0^1 (1 - \tau)^{\beta - 1} f(\tau, u(\tau)) d\tau.$$

It follows that

$$\phi_p({}^c D_{0+}^{\alpha} u(t)) = -\int_0^1 H(t, \tau) f(\tau, u(\tau)) d\tau.$$

Hence,

$${}^c D_{0+}^{\alpha} u(t) = -\phi_q \left(\int_0^1 H(t, \tau) f(\tau, u(\tau)) d\tau \right).$$

Letting

$$v(t) = \phi_q \left(\int_0^1 H(t, \tau) f(\tau, u(\tau)) d\tau \right),$$

we arrive at

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) \mathrm{d}s + d_0 + d_1 t + d_2 t^2 + \cdots + d_n t^n.$$

Conditions $u^{(j)}(0) = 0,\ j = 2, \cdots, n$, imply that $d_2 = \cdots = d_n = 0$, i.e.,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) \mathrm{d}s + d_0 + d_1 t,$$

which in turn yields that

$$u'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s + d_1.$$

Condition $u'(1) = a \int_0^1 h(t) u'(t) \mathrm{d}t$ implies that

$$d_1 = \frac{\int_0^1 (1-s)^{\alpha-2} v(s) \mathrm{d}s}{[1 - a \int_0^1 h(t) \mathrm{d}t] \Gamma(\alpha-1)} - \frac{a \int_0^1 h(t) \int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s \mathrm{d}t}{[1 - a \int_0^1 h(t) \mathrm{d}t] \Gamma(\alpha-1)}.$$

From $u(0) = b \int_0^1 h(t) u(t) \mathrm{d}t$, we get

$$d_0 = d_1 \frac{b \int_0^1 t\, h(t) \mathrm{d}t}{1 - b \int_0^1 h(t) \mathrm{d}t} - \frac{b \int_0^1 h(t) \int_0^t (t-s)^{\alpha-1} v(s) \mathrm{d}s \mathrm{d}t}{[1 - b \int_0^1 h(t) \mathrm{d}t] \Gamma(\alpha)}.$$

The proof is complete. $\square$

Lemma 2.5. *The function $u(t) \in C[0,1]$ defined by (2.1) is non-decreasing and non-negative.*

Proof. Since

$$u(t) = d_0 + d_1 t - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) \mathrm{d}s,$$

we find that

$$
\begin{aligned}
u'(t) &= d_1 - \frac{\int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s}{\Gamma(\alpha-1)} \\
&= \frac{\int_0^1 (1-s)^{\alpha-2} v(s) \mathrm{d}s}{[1 - a \int_0^1 h(t) \mathrm{d}t] \Gamma(\alpha-1)} - \frac{a \int_0^1 h(t) \int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s \mathrm{d}t}{[1 - a \int_0^1 h(t) \mathrm{d}t] \Gamma(\alpha-1)} - \frac{\int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s}{\Gamma(\alpha-1)} \\
&\geq \frac{\int_0^1 (1-s)^{\alpha-2} v(s) \mathrm{d}s}{\Gamma(\alpha-1)} - \frac{\int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s}{\Gamma(\alpha-1)} \geq 0.
\end{aligned}
$$

Hence, we obtain that $u(t)$ is non-decreasing. It follows that $\min_{t \in [0,1]} u(t) = u(0)$. Since

$$
\begin{aligned}
d_1 &= \frac{\int_0^1 (1-s)^{\alpha-2} v(s) \mathrm{d}s}{[1 - a \int_0^1 h(t) \mathrm{d}t] \Gamma(\alpha-1)} - \frac{a \int_0^1 h(t) \int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s \mathrm{d}t}{[1 - a \int_0^1 h(t) \mathrm{d}t] \Gamma(\alpha-1)} \\
&\geq \frac{\int_0^1 (1-s)^{\alpha-2} v(s) \mathrm{d}s}{\Gamma(\alpha-1)},
\end{aligned}
$$

we find that

$$
\begin{aligned}
u(0) \;=\; d_0 = d_1 & \frac{b \int_0^1 t\, h(t)\mathrm{d}t}{1 - b \int_0^1 h(t)\mathrm{d}t} - \frac{b \int_0^1 h(t) \int_0^t (t-s)^{\alpha-1} v(s)\mathrm{d}s\mathrm{d}t}{[1 - b \int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha)} \\[2mm]
\geq\; & \frac{b \int_0^1 t\, h(t)\mathrm{d}t \;\int_0^1 (1-s)^{\alpha-2} v(s)\mathrm{d}s}{[1 - b \int_0^1 h(t)\mathrm{d}t]\,\Gamma(\alpha-1)} - \frac{b \int_0^1 h(t)\mathrm{d}t \;\int_0^1 (1-s)^{\alpha-1} v(s)\mathrm{d}s}{[1 - b \int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha)} \\[2mm]
\geq\; & \frac{b\left[(\alpha-1)\int_0^1 t\, h(t)\mathrm{d}t - \int_0^1 h(t)\mathrm{d}t\right]}{[1 - b \int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)\mathrm{d}s \;\geq\; 0. \qquad (2.6)
\end{aligned}
$$

The proof is complete. $\qquad\square$

Lemma 2.6. [17] *The function $H(s,\tau)$ defined by (2.5) is continuous on $[0,1] \times [0,1]$ and satisfies*

$$
\frac{s^{\beta-1}(1-s)\tau(1-\tau)^{\beta-1}}{\Gamma(\beta)} \leq H(s,\tau) \leq \frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)} \text{ for } s,\tau \in [0,1].
$$

3. MAIN RESULTS

Let the Banach space $E = C[0,1]$ be endowed with the norm $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$. Define the cone P by $P = \{u \in E,\ u(t) \geq 0\}$. And define the operator $T : P \to E$,

$$
Tu(t) = d_0 + d_1 t - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\mathrm{d}s. \qquad (3.1)
$$

where d_0, d_1, $v(s)$ are defined by (2.2)-(2.4). Obviously, $u(t)$ is a solution of problem (1.1) if and only if u is a fixed point of T.

Theorem 3.1. *Assume that there exists $r > 0$ such that*
(C1) $f(t,u_1) \leq f(t,u_2)$, $\forall 0 \leq t \leq 1$, $0 \leq u_1 \leq u_2 \leq r$;
(C2) $\max_{t\in[0,1]} f(t,r) \leq M$, where

$$
M = (rL)^{p-1} \frac{\Gamma(\beta-1)}{B(2,\beta)} - 1 > 0, \qquad (3.2)
$$

$$
L = [1 - a \int_0^1 h(t)\mathrm{d}t][1 - b \int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha).
$$

Then problem (1.1) has two positive solutions w^* and v^* such that

$$
0 \leq w^* < r,\ 0 \leq v^* < r.
$$

$$
w^* = \lim_{k\to+\infty} w_k,\ v^* = \lim_{k\to+\infty} v_k.
$$

$$
w_k = Tw_{k-1},\ v_k = Tv_{k-1},\ k = 1,2,\cdots.
$$

$$
w_0 = \left(\frac{M}{M+1}\right)^{q-1} r,\ v_0 = 0.
$$

Proof. The proof is split into three steps.

Step 1. We prove that $T : P \to P$ is completely continuous.

For any $u \in P$, we find from (2.6) that

$$
Tu(t) \geq Tu(0) = d_0 \geq 0.
$$

It follows that $T : P \to P$. Obviously, T is continuous for the continuity of $f(t,u)$. Let $\Omega \subset P$ be bounded, that is, there exists a positive constant δ, for any $u \in \Omega$, $u(t) \leq \delta$. For any $u \in \Omega$, we take $\lambda = \max_{0 \leq t \leq 1, 0 \leq u \leq \delta} f(t,u) + 1$. It follows that

$$
\begin{aligned}
v(s) &= \phi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \right) \\
&\leq \phi_q \left(\lambda \int_0^1 \frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)} \mathrm{d}\tau \right) = \left(\frac{\lambda B(2,\beta)}{\Gamma(\beta-1)} \right)^{q-1}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
Tu(t) &= d_0 + d_1 t - \frac{\int_0^t (1-s)^{\alpha-1} v(s)\mathrm{d}s}{\Gamma(\alpha)} \\
&= \left(\frac{\int_0^1 (1-s)^{\alpha-2} v(s)\mathrm{d}s}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} - \frac{a\int_0^1 h(t)\int_0^t (t-s)^{\alpha-2} v(s)\mathrm{d}s\mathrm{d}t}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} \right) \left\{ \frac{b\int_0^1 t\, h(t)\mathrm{d}t}{1 - b\int_0^1 h(t)\mathrm{d}t} + t \right\} \\
&\quad - \frac{b\int_0^1 h(t)\int_0^t (t-s)^{\alpha-1} v(s)\mathrm{d}s\mathrm{d}t}{[1 - b\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha)} - \frac{\int_0^t (1-s)^{\alpha-1} v(s)\mathrm{d}s}{\Gamma(\alpha)} \\
&\leq \left(\frac{\int_0^1 (1-s)^{\alpha-2} v(s)\mathrm{d}s}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} \right) \left\{ \frac{b\int_0^1 t\, h(t)\mathrm{d}t}{1 - b\int_0^1 h(t)\mathrm{d}t} + 1 \right\} \\
&\leq \left(\frac{\lambda B(2,\beta)}{\Gamma(\beta-1)} \right)^{q-1} \left(\frac{1}{[1 - a\int_0^1 h(t)\mathrm{d}t][1 - b\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha)} \right).
\end{aligned}
$$

Hence, $T(\Omega)$ is uniformly bounded.

Next, we prove that $T(\Omega)$ is equicontinuous. For each $u \in \Omega$, if $0 \leq t_1 < t_2 \leq 1$, then

$$
\begin{aligned}
&|(Tu)(t_2) - (Tu)(t_1)| \\
&= \left| d_1 t_1 - \frac{\int_0^{t_1} (1-s)^{\alpha-1} v(s)\mathrm{d}s}{\Gamma(\alpha)} - d_1 t_2 + \frac{\int_0^{t_2} (1-s)^{\alpha-1} v(s)\mathrm{d}s}{\Gamma(\alpha)} \right| \\
&\leq d_1 |t_1 - t_2| + \left| \frac{\int_0^{t_1} (1-s)^{\alpha-1} v(s)\mathrm{d}s}{\Gamma(\alpha)} - \frac{\int_0^{t_2} (1-s)^{\alpha-1} v(s)\mathrm{d}s}{\Gamma(\alpha)} \right| \\
&\leq d_1 |t_1 - t_2| + \frac{\int_{t_1}^{t_2} (1-s)^{\alpha-1} v(s)\mathrm{d}s}{\Gamma(\alpha)} \\
&\leq d_1 |t_1 - t_2| + \left(\frac{\lambda B(2,\beta)}{\Gamma(\beta-1)} \right)^{q-1} \frac{|t_1 - t_2|}{\Gamma(\alpha)},
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\int_0^1 (1-s)^{\alpha-2} v(s)\mathrm{d}s}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} - \frac{a\int_0^1 h(t)\int_0^t (t-s)^{\alpha-2} v(s)\mathrm{d}s\mathrm{d}t}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} \\
&\leq \frac{\int_0^1 (1-s)^{\alpha-2} v(s)\mathrm{d}s}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} \\
&\leq \left(\frac{\lambda B(2,\beta)}{\Gamma(\beta-1)} \right)^{q-1} \frac{1}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha)}.
\end{aligned}
$$

Therefore, $T(\Omega)$ is equicontinuous. Applying the *Arzelá − Ascoli* theorem, we conclude that T is a completely continuous operator.

Step 2. Let $\overline{P}_r = \{u \in P : 0 \le \|u\| \le r\}$, then we prove $T : \overline{P}_r \to \overline{P}_r$.

For any $u \in \overline{P}_r$, one has

$$
\begin{aligned}
Tu(t) \;\le\; & Tu(1) = d_0 + d_1 - \frac{\int_0^1 (1-s)^{\alpha-1} v(s) \mathrm{d}s}{\Gamma(\alpha)} \\[2mm]
=\; & \left(\frac{\int_0^1 (1-s)^{\alpha-2} v(s) \mathrm{d}s}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} - \frac{a \int_0^1 h(t) \int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s \mathrm{d}t}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} \right) \left\{ \frac{b\int_0^1 t\, h(t)\mathrm{d}t}{1 - b\int_0^1 h(t)\mathrm{d}t} + 1 \right\} \\[2mm]
& - \frac{b \int_0^1 h(t) \int_0^t (t-s)^{\alpha-1} v(s) \mathrm{d}s \mathrm{d}t}{[1 - b\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha)} - \frac{\int_0^1 (1-s)^{\alpha-1} v(s) \mathrm{d}s}{\Gamma(\alpha)} \\[2mm]
\le\; & \left(\frac{\int_0^1 (1-s)^{\alpha-2} v(s) \mathrm{d}s}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} \right) \left\{ \frac{b\int_0^1 t\, h(t)\mathrm{d}t}{1 - b\int_0^1 h(t)\mathrm{d}t} + 1 \right\} \\[2mm]
\le\; & \left(\frac{MB(2,\beta)}{\Gamma(\beta-1)} \right)^{q-1} \left(\frac{\int_0^1 (1-s)^{\alpha-2} \mathrm{d}s}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha-1)} \right) \left\{ \frac{b\int_0^1 t\, h(t)\mathrm{d}t}{1 - b\int_0^1 h(t)\mathrm{d}t} + 1 \right\} \\[2mm]
\le\; & \left(\frac{MB(2,\beta)}{\Gamma(\beta-1)} \right)^{q-1} \frac{1}{[1 - a\int_0^1 h(t)\mathrm{d}t]\Gamma(\alpha)} \frac{1}{1 - b\int_0^1 h(t)\mathrm{d}t} \\[2mm]
=\; & \left(\frac{B(2,\beta)}{\Gamma(\beta-1)} \right)^{q-1} \frac{M^{q-1}}{L} = \left(\frac{M}{M+1} \right)^{q-1} r < r. \hspace{2cm} (3.3)
\end{aligned}
$$

Consequently, we get that $T : \overline{P}_r \to \overline{P}_r$.

Step 3. We prove that w^* and v^* are two positive solutions of problem (1.1).

Note that $w_0(t) = \left(\frac{M}{M+1} \right)^{q-1} r$. It is obvious, $w_0(t) \in \overline{P}_r$. Let $w_{k+1}(t) = Tw_k(t)$, $k = 0, 1, 2, \cdots$.

Since $T : \overline{P}_r \to \overline{P}_r$, we get $w_k(t) \in \overline{P}_r$. In view of (3.3), we get $Tw_0(t) \le \left(\frac{M}{M+1} \right)^{q-1} r = w_0(t)$, that is, $w_1(t) \le w_0(t)$, Using the definition of T, we obtain

$$ w_2(t) = Tw_1(t) \le Tw_0(t) = w_1(t). $$

By induction, we obtain

$$ w_{k+1}(t) \le w_k(t), \; k = 0, 1, 2, \cdots. $$

Thus, we get $w^* \in \overline{P}_r$ such that $w_k \to w^*$. Applying the continuity of T and $w_{k+1}(t) = Tw_k(t)$, we get $w^*(t) = Tw^*(t)$. Hence, $w^*(t)$ is a positive solution of problem (1.1). Note that $v_0(t) = 0$. It is obvious, $v_0(t) \in \overline{P}_r$. Let $v_{k+1}(t) = Tv_k(t)$, $k = 0, 1, 2, \cdots$. Since $T : \overline{P}_r \to \overline{P}_r$, we get that $v_k(t) \in \overline{P}_r$. In view of Lemma 2.2, we get $Tv_0(t) \ge 0 = v_0(t)$, that is, $v_1(t) \ge v_0(t)$. From the definition of T, we obtain

$$ v_2(t) = Tv_1(t) \ge Tv_0(t) = v_1(t). $$

By induction, we obtain

$$ v_{k+1}(t) \ge v_k(t), \; k = 0, 1, 2, \cdots. $$

Thus, we get $v^* \in \overline{P}_r$ such that $v_k \to v^*$. Applying the continuity of T and $v_{k+1}(t) = Tv_k(t)$, we get $v^*(t) = Tv^*(t)$. Hence $v^*(t)$ is a positive solution of problem (1.1). $\qquad\square$

Remark 3.2. The positive solutions $w^*(t)$ and $v^*(t)$ of problem (1.1) in Theorem 3.1 may coincide, and in this case, problem (1.1) has a positive solution in $\overline{P}_r$.

Finally, we give a simple example to explain the main theorem.

Example 3.3. For problem (1.1), let $\alpha = 2.6$, $\beta = 2.9$, $a = 10$, $b = 9$, $p = 1.5$, $r = 10$, $h(t) = 0.1t$, $f(t,u) = 5\cos t + \sin t + tu$. From a direct calculation, we get

$$q = 3, \ \int_0^1 h(t)\mathrm{d}s = 0.05, \ L = 0.393147, \ M = 20.568088.$$

$$\max_{t\in[0,1]} f(t,r) < 5+1+10 = 16 < M.$$

Then all the conditions of Theorem 3.1 are satisfied. Hence, we get that the problem has two positive solutions w^* and v^*.

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