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ON A COUPLED SYSTEM OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH RIEMANN–LIOUVILLE TYPE BOUNDARY CONDITIONS

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Abstract. This paper establishes the existence of at least one positive solution for a coupled system of fractional order differential equations supplemented with Riemann–Liouvillle type fractional order boundary conditions. An illustrative example is also presented.

Keywords. Boundary value problem; Fractional derivative; Green's function; Positive solution.

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1. INTRODUCTION

The theory of fractional order differential equations is growing rapidly. Recently, much attention has been paid to the study of the existence of positive solutions of fractional order differential equations satisfying initial and boundary conditions; see [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. The multi-point boundary value problems for ordinary differential equations appear in a variety of areas of applied mathematics, physics and engineering. For instance, many problems in the theory of elastic stability can be handled by multi-point problems; see [9] and the references therein.

In [10], Dou, Li and Liu studied the existence of solutions to the four-point fractional order boundary value problem

$$D^{\alpha}u(t) = f\big(t, u(t), D^{\mu}u(t)\big),\ t \in (0,1),$$
$$u(0) = u'(0) = 0,$$
$$u(1) = au(\eta_1) + bu(\eta_2),$$

where $\alpha \in (2,3)$, $\mu > 0$, $\alpha - \mu \geq 1$, $0 < a, b < 1$, $0 < \eta_1 \leq \eta_2 < 1$, and $a\eta_1^{\alpha-1} + b\eta_2^{\alpha-1} < 1$.

Recently, Prasad *et al.* [11] investigated the existence of multiple positive solutions for a system of fractional order differential equations. Inspired and motivated by those previous works, in this paper, we

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are concerned with the existence of at least one positive solution for a coupled system of fractional order differential equations

$$D_{0^+}^{m_1}x(t)+f_1\Big(t,x(t),y(t)\Big)=0,\ t\in(0,1), \tag{1.1}$$

$$D_{0^+}^{m_2}y(t)+f_2\Big(t,x(t),y(t)\Big)=0,\ t\in(0,1), \tag{1.2}$$

satisfying the boundary conditions

$$x(0)=0,\ \zeta_1 D_{0^+}^{m_3}x(1)=\lambda_1+\vartheta_1 D_{0^+}^{m_3}x(\xi_1), \tag{1.3}$$

$$y(0)=0,\ \zeta_2 D_{0^+}^{m_3}y(1)=\lambda_2+\vartheta_2 D_{0^+}^{m_3}y(\xi_2), \tag{1.4}$$

where $D_{0^+}^{m_j}$ and $D_{0^+}^{m_3}$ are the standard Riemann–Liouville fractional order derivatives, $m_3\in(0,1)$, $m_j\in(1,2]$, $m_j-m_3-1>0$, $0<\xi_1<\xi_2<1$ and λ_j is a parameter for $j=1,2$.

Under sufficient conditions on functions f_1 and f_2, we establish the existence of at least one positive solution to a coupled system of fractional order boundary value problem (1.1)-(1.4) by utilizing fixed point theorems of the cone expansion and the compression of functional type due to Avery, Henderson and O'Regan [12]. By a positive solution of (1.1)-(1.4), we mean $\Big(x(t),y(t)\Big)\in\Big(C[0,1]\times C[0,1]\Big)$ satisfying (1.1)-(1.4) with $x(t),\ y(t)\geq 0$, for all $t\in[0,1]$ and $\Big(x(t),y(t)\Big)\neq(0,0)$. Before stating our main results, we make precise assumptions throughout the paper:

- (A_1) $f_1,f_2:[0,1]\times\mathbb{R}^{+2}\to\mathbb{R}^+$ are continuous,
- (A_2) ζ_j,ϑ_j are positive constants such that $\zeta_1\geq\dfrac{\vartheta_1}{\xi_1^{1+m_3-m_1}}$ and $\zeta_2\geq\dfrac{\vartheta_2}{\xi_2^{1+m_3-m_2}}$,
- (A_3) $\Phi_1,\Phi_2,\Phi_1^*,\Phi_2^*$ are positive constants such that $\dfrac{1}{\Phi_1}+\dfrac{1}{\Phi_2}+\dfrac{1}{\Phi_1^*}+\dfrac{1}{\Phi_2^*}\leq 1$.

The rest of the paper is organized as follows. In Section 2, we present some definitions and background results. For sake of convenience, we also state a fixed point theorem for our main results. In Section 3, we construct the Green functions for the associated fractional order boundary value problems and estimate the bounds for these Green functions. In Section 4, we establish the existence results of at least one positive solution of the system of fractional order boundary value problem (1.1)-(1.4). Finally, in Section 5, as an application, we give an example to illustrate our result.

2. Preliminaries

In this section, we recall some definitions and properties of the fractional calculus. We also state a fixed point theorem of cone expansion and compression of functional type due to Avery, Henderson, O'Regan [12].

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P\subset E$ is called a cone if it satisfies the following conditions:

$$(i)\ x\in P,\ \lambda\geq 0\Rightarrow\lambda x\in P;$$

$$(ii)\ x\in P,\ -x\in P\Rightarrow x=0.$$

Every cone $P\subset E$ induces an ordering in E given by $x\leq y$ if and only if $y-x\in P$.

Definition 2.2. Let P be a cone in the real Banach space B. A map $\alpha : P \to [0,\infty)$ is said to be nonnegative continuous concave functional on P if α is continuous and

$$\alpha\Big(\lambda x + (1-\lambda)y\Big) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$.

Definition 2.3. Let P be a cone in the real Banach space B. A map $\beta : P \to [0,\infty)$ is said to be nonnegative continuous convex functional on P if β is continuous and

$$\beta\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda\beta(x) + (1-\lambda)\beta(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$.

Property 2.4. [12] *Let P be a cone in a Banach space E and let Ω be a bounded open subset of E with $0 \in \Omega$. A continuous functional $\alpha : P \to [0,\infty)$ is said to satisfy Property* **P1** *if one of the following conditions hold:*

(a) *α is convex, $\alpha(0) = 0$, $\alpha(x) \neq 0$ if $x \neq 0$ and $\inf_{x\in P\cap\partial\Omega}\alpha(x) > 0$,*
(b) *α is sublinear, $\alpha(0) = 0$, $\alpha(x) \neq 0$ if $x \neq 0$ and $\inf_{x\in P\cap\partial\Omega}\alpha(x) > 0$,*
(c) *α is concave and unbounded.*

Property 2.5. [12] *Let P be a cone in a Banach space E and let Ω be a bounded open subset of E with $0 \in \Omega$. A continuous functional $\beta : P \to [0,\infty)$ is said to satisfy Property* **P2** *if one of the following conditions hold:*

(a) *β is convex, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$,*
(b) *β is sublinear, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$,*
(c) *$\beta(x+y) \geq \beta(x) + \beta(y)$ for all $x, y \in P$, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$.*

The approach utilized on the existence results in this paper is the following fixed point theorem of the cone expansion and the compression of functional type due to Avery, Henderson and O'Regan's [12], which generalized the functional compression fixed point theorems of Anderson and Avery [13] and Sun and Zhang [14].

Theorem 2.6. [12] *Let Ω_1, Ω_2 be two bounded open sets in a Banach Space E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ in E. Suppose that $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator, α and β are non-negative continuous functional on P, and one of the two conditions holds:*

(i) *α satisfies Property 2.4 with $\alpha(Tx) \geq \alpha(x)$, for all $x \in P \cap \partial\Omega_1$ and β satisfies Property 2.5 with $\beta(Tx) \leq \beta(x)$, for all $x \in P \cap \partial\Omega_2$,*
(ii) *β satisfies Property 2.5 with $\beta(Tx) \leq \beta(x)$, for all $x \in P \cap \partial\Omega_1$ and α satisfies Property 2.4 with $\alpha(Tx) \geq \alpha(x)$, for all $x \in P \cap \partial\Omega_2$, is satisfied.*

Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Green functions and bounds

In this section, we construct the Green functions for the associated fractional order boundary value problems and estimate the bounds for these Green functions, which are needed to establish ou main results.

Lemma 3.1. *Let $\Delta_1 = \Gamma(m_1)\mathscr{M}_1 \neq 0$. If $h_1(t) \in C[0,1]$, then the fractional order boundary value problem*

$$D_{0^+}^{m_1} x(t) + h_1(t) = 0,\ t \in (0,1), \tag{3.1}$$

satisfying boundary conditions (1.3), has a unique solution

$$x(t) = \int_0^1 G_1(t,s)h_1(s)ds + \frac{\lambda_1 \Gamma(m_1 - m_3)t^{m_1-1}}{\Delta_1},$$

where $G_1(t,s)$ is the Green's function for boundary value problem (3.1), and (1.3) and is given by

$$G_1(t,s) = \begin{cases} \underset{t\in[0,\xi_1]}{G_1(t,s)} = \begin{cases} G_{11}(t,s), & 0 \le t \le s \le \xi_1 < 1, \\ G_{12}(t,s), & 0 \le s \le \min\{t,\xi_1\} < 1, \end{cases} \\ \\ \underset{t\in[\xi_1,1]}{G_1(t,s)} = \begin{cases} G_{13}(t,s), & 0 \le \max\{t,\xi_1\} \le s \le 1, \\ G_{14}(t,s), & 0 < \xi_1 \le s \le t \le 1, \end{cases} \end{cases} \tag{3.2}$$

$$\begin{aligned} G_{11}(t,s) =& \frac{1}{\Delta_1}\Big[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \vartheta_1 t^{m_1-1}(\xi_1 - s)^{m_1-m_3-1}\Big], \\ G_{12}(t,s) =& \frac{1}{\Delta_1}\Big[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-1} - \vartheta_1 t^{m_1-1}(\xi_1 - s)^{m_1-m_3-1}\Big], \\ G_{13}(t,s) =& \frac{1}{\Delta_1}\Big[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1}\Big], \\ G_{14}(t,s) =& \frac{1}{\Delta_1}\Big[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-1}\Big], \\ \mathscr{M}_1 =& \zeta_1 - \vartheta_1 \xi_1^{m_1-m_3-1}. \end{aligned}$$

Proof. Let $x(t) \in C[0,1]$ be the solution of fractional order boundary value problem (3.1) and (1.3). An equivalent integral equation for (3.1) is given by

$$x(t) = -\int_0^t \frac{(t-s)^{m_1-1}}{\Gamma(m_1)} h_1(s)ds + c_1 t^{m_1-1} + c_2 t^{m_1-2}.$$

Using the boundary conditions (1.3), one can get $c_2 = 0$ and

$$c_1 = \begin{cases} \dfrac{1}{\Delta_1}\Big[\zeta_1 \displaystyle\int_0^1 (1-s)^{m_1-m_3-1}h_1(s)ds - \vartheta_1 \int_0^{\xi_1} (\xi_1 - s)^{m_1-m_3-1}h_1(s)ds\Big] + \\ \dfrac{\lambda_1 \Gamma(m_1-m_3)t^{m_1-1}}{\Delta_1}. \end{cases}$$

Hence the unique solution of the problem given by (3.1) and (1.3) is

$$\begin{aligned} x(t) =& \frac{1}{\Delta_1}\Big[\zeta_1 \int_0^1 (1-s)^{m_1-m_3-1}h_1(s)ds - \vartheta_1 \int_0^{\xi_1} (\xi_1 - s)^{m_1-m_3-1}h_1(s)ds \\ & - \int_0^t \mathscr{M}_1 (t-s)^{m_1-m_3-1}h_1(s)ds\Big] + \frac{\lambda_1 \Gamma(m_1-m_3)t^{m_1-1}}{\Delta_1} \\ =& \int_0^1 G_1(t,s)h_1(s)ds + \frac{\lambda_1 \Gamma(m_1-m_3)t^{m_1-1}}{\Delta_1}. \end{aligned}$$

□

Lemma 3.2. *Assume that condition (A_2) is satisfied. Then the Green's function $G_1(t,s)$ given in (3.2) is nonnegative, for all $t,s \in [0,1]$.*

Proof. Consider the Green's function $G_1(t,s)$ given by (3.2). Let $0 \le t \le s \le \xi_1 \le 1$. Then

$$
\begin{aligned}
G_{11}(t,s) &= \frac{1}{\Delta_1}\left[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \vartheta_1 t^{m_1-1}(\xi_1-s)^{m_1-m_3-1}\right] \\
&\ge \frac{t^{m_1-1}}{\Delta_1}\left[\zeta_1(1-s)^{m_1-m_3-1} - \vartheta_1(\xi_1-\xi_1 s)^{m_1-m_3-1}\right] \\
&= \frac{t^{m_1-1}}{\Delta_1}\left[\zeta_1(1-s)^{m_1-m_3-1} - \vartheta_1 \xi_1^{m_1-m_3-1}(1-s)^{m_1-m_3-1}\right] \\
&= \frac{t^{m_1-1}}{\Delta_1}\left[\zeta_1 - \vartheta_1 \xi_1^{m_1-m_3-1}\right](1-s)^{m_1-m_3-1} \\
&= \frac{t^{m_1-1}}{\Delta_1}\left[\mathscr{M}_1(1-s)^{m_1-m_3-1}\right] \ge 0.
\end{aligned}
$$

Letting $0 \le s \le \min\{t,\xi_1\} \le 1$, we have

$$
\begin{aligned}
G_{12}(t,s) &= \frac{1}{\Delta_1}\left[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-1} - \vartheta_1 t^{m_1-1}(\xi_1-s)^{m_1-m_3-1}\right] \\
&\ge \frac{1}{\Delta_1}\left[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-1} - \vartheta_1 t^{m_1-1}(\xi_1-\xi_1 s)^{m_1-m_3-1}\right] \\
&= \frac{\mathscr{M}_1}{\Delta_1}\left[t^{m_1-1}(1-s)^{m_1-m_3-1} - (t-s)^{m_1-1}\right] \\
&\ge \frac{\mathscr{M}_1}{\Delta_1}\left[t^{m_1-1}(1-s)^{m_1-m_3-1} - (t-ts)^{m_1-1}\right] \\
&= \frac{\mathscr{M}_1 t^{m_1-1}(1-s)^{m_1-1}}{\Delta_1}\left[(1-s)^{-m_3} - 1\right] \ge 0.
\end{aligned}
$$

Letting $0 \le \max\{t,\xi_1\} \le s \le 1$, we have

$$
G_{13}(t,s) = \frac{1}{\Delta_1}\left[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1}\right] \ge 0.
$$

Letting $0 \le \xi_1 \le s \le t \le 1$, we have

$$
\begin{aligned}
G_{14}(t,s) &= \frac{1}{\Delta_1}\left[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-1}\right] \\
&= \frac{t^{m_1-1}}{\Delta_1}\left[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-1}\right] \\
&\ge \frac{t^{m_1-1}(1-s)^{m_1-1}}{\Delta_1}\left[\zeta_1(1-s)^{-m_3} - \mathscr{M}_1\right] \ge 0.
\end{aligned}
$$

□

Lemma 3.3. *Assume that condition (A_2) is satisfied. Then the Green's function $G_1(t,s)$ given in (3.2) satisfies the inequalities*

- (H1) $G_1(t,s) \le G_1(1,s)$, $\forall t,s \in [0,1]$,
- (H2) $G_1(t,s) \ge \tau^{m_1-1} G_1(1,s)$, $t \in [\tau,1], s \in [0,1]$.

Proof. Consider the Green's function $G_1(t,s)$ given by (3.2). Letting $0 \le t \le s \le \xi_1 \le 1$, we have

$$\begin{aligned}\frac{\partial G_{11}(t,s)}{\partial t} &= \frac{(m_1-1)}{\Delta_1}\Big[\zeta_1 t^{m_1-2}(1-s)^{m_1-m_3-1} - \vartheta_1 t^{m_1-2}(\xi_1-s)^{m_1-m_3-1}\Big]\\ &\ge \frac{(m_1-1)}{\Delta_1}\Big[\zeta_1 t^{m_1-2}(1-s)^{m_1-m_3-1} - \vartheta_1 t^{m_1-2}(\xi_1-\xi_1 s)^{m_1-m_3-1}\Big]\\ &= \frac{(m_1-1)t^{m_1-2}}{\Delta_1}\Big[\mathscr{M}_1(1-s)^{m_1-m_3-1}\Big] \ge 0.\end{aligned}$$

Letting $0 \le s \le \min\{t,\xi_1\} \le 1$, we have

$$\begin{aligned}\frac{\partial G_{12}(t,s)}{\partial t} &= \frac{(m_1-1)}{\Delta_1}\Big[\zeta_1 t^{m_1-2}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-2} - \vartheta_1 t^{m_1-2}(\xi_1-s)^{m_1-m_3-1}\Big]\\ &\ge \frac{(m_1-1)}{\Delta_1}\Big[\zeta_1 t^{m_1-2}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-ts)^{m_1-2} - \vartheta_1 t^{m_1-2}(\xi_1-\xi_1 s)^{m_1-m_3-1}\Big]\\ &= \frac{(m_1-1)t^{m_1-2}\mathscr{M}_1}{\Delta_1}\Big[(1-s)^{m_1-m_3-1} - (1-s)^{m_1-2}\Big] \ge 0.\end{aligned}$$

Letting $0 \le \max\{t,\xi_1\} \le s \le 1$, we have

$$\frac{\partial G_{13}(t,s)}{\partial t} = \frac{(m_1-1)}{\Delta_1}\Big[\zeta_1 t^{m_1-2}(1-s)^{m_1-m_3-1}\Big] \ge 0.$$

Letting $0 < \xi_1 \le s \le t \le 1$, we have

$$\begin{aligned}\frac{\partial G_{14}(t,s)}{\partial t} &= \frac{(m_1-1)}{\Delta_1}\Big[\zeta_1 t^{m_1-2}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-2}\Big]\\ &\ge \frac{(m_1-1)t^{m_1-2}}{\Delta_1}\Big[\zeta_1(1-s)^{m_1-m_3-1} - \mathscr{M}_1(1-s)^{m_1-2}\Big]\\ &= \frac{(m_1-1)t^{m_1-2}}{\Delta_1}\Big[\zeta_1(1-s)^{-m_3} - \mathscr{M}_1\Big](1-s)^{m_1-2} \ge 0.\end{aligned}$$

Therefore $G_1(t,s)$ is increasing in t, which implies $G_1(t,s) \le G_1(1,s)$. Letting $0 \le t \le s \le \xi_1 < 1$ and $t \in [\tau,1]$, we have

$$\begin{aligned}G_{11}(t,s) &= \frac{1}{\Delta_1}\Big[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \vartheta_1 t^{m_1-1}(\xi_1-s)^{m_1-m_3-1}\Big]\\ &= t^{m_1-1}G_{11}(1,s) \ge \tau^{m_1-1}G_{11}(1,s).\end{aligned}$$

Letting $0 \le s \le \min\{t,\xi_1\} < 1$ and $t \in [\tau,1]$, we have

$$\begin{aligned}G_{12}(t,s) &= \frac{1}{\Delta_1}\Big[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-1} - \vartheta_1 t^{m_1-1}(\xi_1-s)^{m_1-m_3-1}\Big]\\ &\ge \frac{t^{m_1-1}}{\Delta_1}\Big[\zeta_1(1-s)^{m_1-m_3-1} - \mathscr{M}_1(1-s)^{m_1-1} - \vartheta_1(\xi_1-s)^{m_1-m_3-1}\Big]\\ &= t^{m_1-1}G_{12}(1,s) \ge \tau^{m_1-1}G_{12}(1,s).\end{aligned}$$

Letting $0 \le \max\{t,\xi_1\} \le s \le 1$ and $t \in [\tau,1]$, we have

$$G_{13}(t,s) = \frac{1}{\Delta_1}\Big[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1}\Big] = t^{m_1-1}G_{13}(1,s) \ge \tau^{m_1-1}G_{13}(1,s).$$

Letting $0 < \xi_1 \le s \le t \le 1$ and $t \in [\tau, 1]$, we have

$$\begin{aligned} G_{14}(t,s) &= \frac{1}{\Delta_1}\left[\zeta_1 t^{m_1-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_1(t-s)^{m_1-1}\right] \\ &\ge t^{m_1-1}G_{14}(1,s) \ge \tau^{m_1-1}G_{14}(1,s), \end{aligned}$$

where $\tau \in (0,1)$ satisfies $\int_\tau^1 G_1(1,s)ds > 0$. □

Lemma 3.4. *Let $\Delta_2 = \Gamma(m_2)\mathscr{M}_2 \neq 0$. If $h_2(t) \in C[0,1]$, then the fractional order boundary value problem*

$$D_{0^+}^{m_2} y(t) + h_2(t) = 0,\ t \in (0,1), \tag{3.3}$$

satisfying the boundary conditions (1.4), has a unique solution

$$y(t) = \int_0^1 G_2(t,s)h_2(s)ds + \frac{\lambda_2 \Gamma(m_2 - m_3)t^{m_2-1}}{\Delta_2},$$

where $G_2(t,s)$ is the Green's function for the boundary value problem (3.3), and (1.4) and is given by

$$G_2(t,s) = \begin{cases} \underset{t\in[0,\xi_2]}{G_2(t,s)} = \begin{cases} G_{21}(t,s), & 0 \le t \le s \le \xi_2 < 1, \\ G_{22}(t,s), & 0 \le s \le \min\{t,\xi_2\} < 1, \end{cases} \\ \\ \underset{t\in[\xi_2,1]}{G_2(t,s)} = \begin{cases} G_{23}(t,s), & 0 \le \max\{t,\xi_2\} \le s \le 1, \\ G_{24}(t,s), & 0 < \xi_2 \le s \le t \le 1, \end{cases} \end{cases} \tag{3.4}$$

$$\begin{aligned} G_{21}(t,s) &= \frac{1}{\Delta_2}\left[\zeta_2 t^{m_2-1}(1-s)^{m_2-m_3-1} - \vartheta_2 t^{m_2-1}(\xi_2 - s)^{m_2-m_3-1}\right], \\ G_{22}(t,s) &= \frac{1}{\Delta_2}\left[\zeta_2 t^{m_2-1}(1-s)^{m_2-m_3-1} - \mathscr{M}_2(t-s)^{m_2-1} - \vartheta_2 t^{m_2-1}(\xi_2 - s)^{m_2-m_3-1}\right], \\ G_{23}(t,s) &= \frac{1}{\Delta_2}\left[\zeta_2 t^{m_2-1}(1-s)^{m_2-m_3-1}\right], \\ G_{24}(t,s) &= \frac{1}{\Delta_2}\left[\zeta_2 t^{m_2-1}(1-s)^{m_1-m_3-1} - \mathscr{M}_2(t-s)^{m_2-1}\right], \\ \mathscr{M}_2 &= \zeta_2 - \vartheta_2 \xi_2^{m_2-m_3-1}. \end{aligned}$$

Proof. Since the proof is similar to Lemma 3.1, we omit the proof here. □

Lemma 3.5. *Assume that condition (A_2) is satisfied. Then the Green's function $G_2(t,s)$ given in (3.4) is nonnegative, for all $t,s \in [0,1]$.*

Proof. Since the proof is similar to Lemma 3.2, we omit the proof here. □

Lemma 3.6. *Assume that condition (A_2) is satisfied. Then the Green's function $G_2(t,s)$ given in (3.4) satisfies the inequalities*

$$\begin{aligned} &(H1^*)\ G_2(t,s) \le G_2(1,s),\ for\ all\ t,s \in [0,1], \\ &(H2^*)\ G_2(t,s) \ge \tau^{m_2-1}G_2(1,s),\ for\ all\ t \in [\tau,1], s \in [0,1]. \end{aligned}$$

Proof. Since the proof is similar to Lemma 3.3, we omit the proof here. □

4. MAIN RESULTS

In this section, we establish the existence of at least one positive solution to the system of fractional order boundary value problem (1.1)-(1.4) by using the fixed point theorem of the cone expansion and the compression of functional type due to Avery, Henderson and O'Regan [12], which generalized the functional compression fixed point theorems of Anderson-Avery [13] and Sun-Zhang [14].

We consider the Banach space $\mathscr{B} = \mathscr{E} \times \mathscr{E}$, where $\mathscr{E} = \{x : x \in C[0,1]\}$ equipped with the norm $\|(x,y)\| = \|x\|_0 + \|y\|_0$, for $(x,y) \in \mathscr{B}$ and we denote the norm by $\|x\|_0 = \max_{t\in[0,1]} |x(t)|$. Define a cone $\mathscr{P} \subset \mathscr{B}$ by

$$\mathscr{P} = \Big\{(x,y) \in \mathscr{B} : x(t), y(t) \text{ are nonnegative and increasing on } [0,1] \\ \text{and } \min_{t\in I}\big[x(t)+y(t)\big] \geq \eta\|(x,y)\|\Big\},$$

where $I = [\tau, 1]$ and

$$\eta = \min\big\{\tau^{m_1-1}, \tau^{m_2-1}\big\}. \tag{4.1}$$

Let

$$\mathscr{C} = \min\Big\{\int_0^1 G_1(1,s)ds, \int_0^1 G_2(1,s)ds\Big\}$$

and

$$\mathscr{D} = \max\Big\{\int_\eta^1 \eta G_1(1,s)ds, \int_\eta^1 \eta G_2(1,s)ds\Big\},$$

where η is given in (4.1). Let us define two continuous functionals α and β on the cone $\mathscr{P}$ by

$$\alpha(x,y) = \min_{t\in I}\Big\{|x| + |y|\Big\} \text{ and}$$

$$\beta(x,y) = \max_{t\in[0,1]}\Big\{|x| + |y|\Big\} = x(1) + y(1) = \big\|(x,y)\big\|.$$

It is clear that $\alpha(x,y) \leq \beta(x,y)$ for all $(x,y) \in \mathscr{P}$. We denote the operators $\mathscr{A}_1 : \mathscr{P} \to \mathscr{E}$, $\mathscr{A}_2 : \mathscr{P} \to \mathscr{E}$ and defined by

$$\begin{cases} \mathscr{A}_1(x,y)(t) = \displaystyle\int_0^1 G_1(t,s)f_1\big(s,x(s),y(s)\big)ds + \frac{\lambda_1\Gamma(m_1-m_3)t^{m_1-1}}{\Delta_1}, \\ \mathscr{A}_2(x,y)(t) = \displaystyle\int_0^1 G_2(t,s)f_2\big(s,x(s),y(s)\big)ds + \frac{\lambda_2\Gamma(m_2-m_3)t^{m_2-1}}{\Delta_2}. \end{cases}$$

Theorem 4.1. *Suppose that there exist positive real numbers r, R with $r < \eta R$ and $\lambda_j < \dfrac{r\Delta_j}{\Phi_j^*\Gamma(m_j-m_3)} \leq \dfrac{R\Delta_j}{\Phi_j^*\Gamma(m_j-m_3)}$ such that f_j, $j = 1,2$ satisfies the following conditions:*

(C1) $f_j\big(t,x,y\big) \geq \frac{1}{2}\cdot\frac{r}{\eta\mathscr{D}}$, *$t \in I$ and $(x,y) \in \big[r,R\big]$,*
(C2) $f_j\big(t,x,y\big) \leq \frac{1}{\Phi_j}\cdot\frac{R}{\mathscr{C}}$, *$t \in [0,1]$ and $(x,y) \in \big[0,R\big]$.*

Then the system of fractional order boundary value problem (1.1)-(1.4) has at least one positive and nondecreasing solution, (x^*,y^*) satisfying $r \leq \alpha(x^*,y^*)$ with $\beta(x^*,y^*) \leq R$.

Proof. Define the completely continuous operator $\mathscr{A} : \mathscr{P} \to \mathscr{B}$ by

$$\mathscr{A}(x,y) = \big(\mathscr{A}_1(x,y), \mathscr{A}_2(x,y)\big).$$

It is obvious that a fixed point of $\mathscr{A}$ is the solution of the fractional order boundary value problem (1.1)-(1.4). We seek a fixed point of $\mathscr{A}$. First, we show that $\mathscr{A}:\mathscr{P}\to\mathscr{P}$. Let $(x,y)\in\mathscr{P}$. Clearly, $\mathscr{A}_1(x,y)(t)\geq 0$ and $\mathscr{A}_2(x,y)(t)\geq 0$ for $t\in[0,1]$. Also, for $(x,y)\in\mathscr{P}$,

$$\begin{cases} \|\mathscr{A}_1(x,y)\|_0 \leq \displaystyle\int_0^1 G_1(1,s)f_1\big(s,x(s),y(s)\big)ds+\frac{\lambda_1\Gamma(m_1-m_3)}{\Delta_1}, \\ \|\mathscr{A}_2(x,y)\|_0 \leq \displaystyle\int_0^1 G_2(1,s)f_2\big(s,x(s),y(s)\big)ds+\frac{\lambda_2\Gamma(m_2-m_3)}{\Delta_2}, \end{cases}$$

and

$$\begin{aligned} \min_{t\in I}\mathscr{A}_1(x,y)(t) =& \min_{t\in I}\left[\int_0^1 G_1(t,s)f_1\big(s,x(s),y(s)\big)ds+\frac{\lambda_1\Gamma(m_1-m_3)t^{m_1-1}}{\Delta_1}\right] \\ \geq& \tau^{m_1-1}\int_0^1 G_1(1,s)f_1\big(s,x(s),y(s)\big)ds+\frac{\lambda_1\Gamma(m_1-m_3)\tau^{m_1-1}}{\Delta_1} \\ \geq& \tau^{m_1-1}\left[\int_0^1 G_1(1,s)f_1\big(s,x(s),y(s)\big)ds+\frac{\lambda_1\Gamma(m_1-m_3)}{\Delta_1}\right] \\ \geq& \eta\|\mathscr{A}_1(x,y)\|_0. \end{aligned}$$

Similarly, $\min_{t\in I}\mathscr{A}_2(x,y)(t)\geq\eta\|\mathscr{A}_2(x,y)\|_0$. Therefore

$$\begin{aligned} \min_{t\in I}\Big\{\mathscr{A}_1(x,y)(t)+\mathscr{A}_2(x,y)(t)\Big\} &\geq \eta\|\mathscr{A}_1(x,y)\|_0+\eta\|\mathscr{A}_2(x,y)\|_0 \\ &= \eta\|(\mathscr{A}_1(x,y),\mathscr{A}_2(x,y))\| \\ &= \eta\|\mathscr{A}(x,y)\|. \end{aligned}$$

Hence $\mathscr{A}(x,y)\in\mathscr{P}$ and $\mathscr{A}:\mathscr{P}\to\mathscr{P}$. Moreover, $\mathscr{A}$ is a completely continuous. Let $\Omega_1=\big\{(x,y):\alpha(x,y)<r\big\}$ and $\Omega_2=\big\{(x,y):\beta(x,y)<R\big\}$. It is easy to see that $0\in\Omega_1$, and Ω_1, Ω_2 are bounded open subsets of $\mathscr{E}$. Letting $(x,y)\in\Omega_1$, we have

$$r>\alpha(x,y)=\min_{t\in I}\big\{x(t)+y(t)\big\}\geq\eta\big\{\|x\|+\|y\|\big\}=\eta\beta(x,y).$$

Thus $R>\dfrac{r}{\eta}>\beta(x,y)$, i.e., $(x,y)\in\Omega_2$, so $\Omega_1\subseteq\Omega_2$.

Claim 1. If $(x,y)\in\mathscr{P}\cap\partial\Omega_1$, then $\alpha\big(\mathscr{A}(x,y)\big)\geq\alpha(x,y)$. To see this, let $(x,y)\in\mathscr{P}\cap\partial\Omega_1$. Then $R=\beta(x,y)\geq\big[x(s)+y(s)\big]\geq\alpha(x,y)=r$, for $s\in I$. It follows from (C1), Lemma 3.3 and Lemma 3.6 that

$$\begin{aligned} \alpha\Big(\mathscr{A}(x,y)(t)\Big) &= \min_{t\in I}\sum_{j=1}^{2}\left[\int_0^1 G_j(t,s)f_j\big(s,x(s),y(s)\big)ds+\frac{\lambda_j\Gamma(m_j-m_3)t^{m_j-1}}{\Delta_j}\right] \\ &\geq \sum_{j=1}^{2}\left[\int_\eta^1 \eta G_j(1,s)f_j\big(s,x(s),y(s)\big)ds+\frac{\lambda_j\Gamma(m_j-m_3)}{\Delta_j}\right] \\ &\geq \sum_{j=1}^{2}\left[\int_\eta^1 \eta G_j(1,s)f_j\big(s,x(s),y(s)\big)ds\right] \\ &\geq \frac{1}{2}\cdot\frac{r}{\eta\mathscr{D}}\int_\eta^1\eta G_1(1,s)ds+\frac{1}{2}\cdot\frac{r}{\eta\mathscr{D}}\int_\eta^1\eta G_2(1,s)ds \\ &= \frac{r}{2}+\frac{r}{2}=r=\alpha(x,y). \end{aligned}$$

Claim 2. If $(x,y)\in\mathscr{P}\cap\partial\Omega_2$, then $\beta\big(\mathscr{A}(x,y)\big)\le\beta(x,y)$. To see this, let $(x,y)\in\mathscr{P}\cap\partial\Omega_2$. Then $\big[x(s)+y(s)\big]\le\beta(x,y)=R$, for $s\in[0,1]$. It follows from (C2), Lemma 3.3 and Lemma 3.6 that

$$\begin{aligned}
\beta\Big(\mathscr{A}(x,y)(t)\Big) &= \max_{t\in[0,1]}\sum_{j=1}^{2}\bigg[\int_0^1 G_j(t,s)f_j\big(s,x(s),y(s)\big)ds+\frac{\lambda_j\Gamma(m_j-m_3)}{\Delta_j}\bigg]\\
&\le \sum_{j=1}^{2}\bigg[\int_0^1 G_j(1,s)f_j\big(s,x(s),y(s)\big)ds+\frac{\lambda_j\Gamma(m_j-m_3)}{\Delta_j}\bigg]\\
&\le \frac{1}{\Phi_1}\cdot\frac{R}{\mathscr{C}}\int_0^1 G_1(1,s)ds+\frac{1}{\Phi_2}\cdot\frac{R}{\mathscr{C}}\int_0^1 G_2(1,s)ds+\frac{R}{\Phi_1^*}+\frac{R}{\Phi_2^*}\\
&= \frac{R}{\Phi_1}+\frac{R}{\Phi_2}+\frac{R}{\Phi_1^*}+\frac{R}{\Phi_2^*}\\
&= R\bigg[\frac{1}{\Phi_1}+\frac{1}{\Phi_2}+\frac{1}{\Phi_1^*}+\frac{1}{\Phi_2^*}\bigg]\le R=\beta(x,y).
\end{aligned}$$

Clearly, α satisfies Property 2.4 (c) and β satisfies Property 2.5 (a). Therefore the condition (i) of Theorem 2.6 is satisfied and hence $\mathscr{A}$ has at least one fixed point $(x^*,y^*)\in\mathscr{P}\cap(\overline{\Omega}_2\backslash\Omega_1)$, i.e., the system of fractional order boundary value problem (1.1)-(1.4) has at least one positive and nondecreasing solution (x^*,y^*) satisfying $r\le\alpha(x^*,y^*)$ with $\beta(x^*,y^*)\le R$. □

Theorem 4.2. *Suppose there exist positive real numbers r, R with $r<R$ and $\lambda_j<\dfrac{r\Delta_j}{\Phi_j^*\Gamma(m_j-m_3)}\le\dfrac{R\Delta_j}{\Phi_j^*\Gamma(m_j-m_3)}$ such that $f_j,\ j=1,2$ satisfies the following conditions:*

- (C3) $f_j\big(t,x,y\big)\le\frac{1}{\Phi_j}\cdot\frac{r}{\mathscr{D}}$, *for all* $t\in[0,1]$ *and* $\big(x,y\big)\in\big[0,r\big]$,
- (C4) $f_j\big(t,x,y\big)\ge\frac{1}{2}\cdot\frac{R}{\eta\mathscr{C}}$, $t\in I$ *and* $\big(x,y\big)\in\bigg[R,\dfrac{R}{\eta}\bigg]$.

Then the system of fractional order boundary value problem (1.1)-(1.4) has at least one positive and nondecreasing solution, (x^*,y^*) satisfying $r\le\beta(x^*,y^*)$ with $\alpha(x^*,y^*)\le R$.

Proof. Let $\Omega_3=\big\{(x,y):\beta(x,y)<r\big\}$ and $\Omega_4=\big\{(x,y):\alpha(x,y)<R\big\}$. We have $0\in\Omega_3$ and $\Omega_3\subseteq\Omega_4$ with Ω_3 and Ω_4 are bounded open subsets of $\mathscr{E}$.

Claim 1. If $(x,y)\in\mathscr{P}\cap\partial\Omega_3$, then $\beta\big(\mathscr{A}(x,y)\big)\le\beta(x,y)$. To see this, let $(x,y)\in\mathscr{P}\cap\partial\Omega_3$. Then $\big[x(s)+y(s)\big]\le\beta(x,y)=r$, for $s\in[0,1]$. It follows from (C3), Lemma 3.3 and Lemma 3.6 that

$$\begin{aligned}
\beta\Big(\mathscr{A}(x,y)(t)\Big) &= \max_{t\in[0,1]}\sum_{j=1}^{2}\bigg[\int_0^1 G_j(t,s)f_j\big(s,x(s),y(s)\big)ds+\frac{\lambda_j\Gamma(m_j-m_3)}{\Delta_j}\bigg]\\
&\le \sum_{j=1}^{2}\bigg[\int_0^1 G_j(1,s)f_j\big(s,x(s),y(s)\big)ds+\frac{\lambda_j\Gamma(m_j-m_3)}{\Delta_j}\bigg]\\
&\le \frac{1}{\Phi_1}\cdot\frac{r}{\mathscr{D}}\int_0^1 G_1(1,s)ds+\frac{1}{\Phi_2}\cdot\frac{r}{\mathscr{D}}\int_0^1 G_2(1,s)ds+\frac{r}{\Phi_1^*}+\frac{r}{\Phi_2^*}\\
&= \frac{r}{\Phi_1}+\frac{r}{\Phi_2}+\frac{r}{\Phi_1^*}+\frac{r}{\Phi_2^*}\\
&= r\bigg[\frac{1}{\Phi_1}+\frac{1}{\Phi_2}+\frac{1}{\Phi_1^*}+\frac{1}{\Phi_2^*}\bigg]\le r=\beta(x,y).
\end{aligned}$$

Claim 2. If $(x,y) \in \mathscr{P} \cap \partial\Omega_4$, then $\alpha\big(\mathscr{A}(x,y)\big) \geq \alpha(x,y)$. To see this, let $(x,y) \in \mathscr{P} \cap \partial\Omega_4$. Then $\frac{R}{\eta} = \frac{\alpha(x,y)}{\eta} \geq \beta(x,y) \geq \big[x(s)+y(s)\big] \geq \alpha(x,y) = R$, for $s \in I$. It follows from (C4), Lemma 3.3 and Lemma 3.6 that

$$\begin{aligned}
\alpha\Big(\mathscr{A}\big(x,y\big)(t)\Big) &= \min_{t\in I} \sum_{j=1}^{2} \left[\int_0^1 G_j(t,s) f_j\big(s,x(s),y(s)\big)ds + \frac{\lambda_j \Gamma(m_j - m_3) t^{m_j-1}}{\Delta_j}\right] \\
&\geq \sum_{j=1}^{2} \left[\int_\eta^1 \eta G_j(1,s) f_j\big(s,x(s),y(s)\big)ds + \frac{\lambda_j \Gamma(m_j - m_3)}{\Delta_j}\right] \\
&\geq \sum_{j=1}^{2} \left[\int_\eta^1 \eta G_j(1,s) f_j\big(s,x(s),y(s)\big)ds\right] \\
&\geq \frac{1}{2}\cdot\frac{R}{\eta\mathscr{C}} \int_\eta^1 \eta G_1(1,s)ds + \frac{1}{2}\cdot\frac{R}{\eta\mathscr{C}} \int_\eta^1 \eta G_2(1,s)ds \\
&= \frac{R}{2} + \frac{R}{2} = R = \alpha(x,y).
\end{aligned}$$

Clearly α satisfies Property 2.4 (c) and β satisfies Property 2.5 (a). Therefore the condition (ii) of Theorem 4.1 is satisfied. Hence $\mathscr{A}$ has at least one fixed point $(x^*,y^*) \in \mathscr{P} \cap (\overline{\Omega}_4 \backslash \Omega_3)$, i.e., the system of fractional order boundary value problem (1.1)-(1.4) has at least one positive and nondecreasing solution (x^*,y^*) satisfying $r \leq \beta(x^*,y^*)$ with $\alpha(x^*,y^*) \leq R$. □

5. An example

In this section, we demonstrate an example to support our main results. Consider the system of fractional order boundary value problem

$$D_{0^+}^{1.8} x(t) + f_1\big(t,x,y\big) = 0,\ t \in (0,1), \tag{5.1}$$

$$D_{0^+}^{1.9} y(t) + f_2\big(t,x,y\big) = 0,\ t \in (0,1), \tag{5.2}$$

satisfying the boundary conditions

$$\begin{cases} x(0) = 0,\ \frac{11}{78} D_{0^+}^{0.7} x(1) = \lambda_1 + \frac{10}{79} D_{0^+}^{0.7} x\left(\frac{1}{4}\right), \\ y(0) = 0,\ \frac{11}{78} D_{0^+}^{0.7} y(1) = \lambda_2 + \frac{10}{79} D_{0^+}^{0.7} y\left(\frac{1}{3}\right), \end{cases} \tag{5.3}$$

where λ_1, λ_2 are parameters, and

$$\begin{cases} f_1\big(t,x,y\big) = \frac{t(x+y)}{19} + \frac{11t^2}{12} + \frac{77}{79}, \\ f_2\big(t,x,y\big) = \frac{t^2(x+y)}{5} + \frac{11e^{-(x+y)^2}}{12}. \end{cases}$$

Clearly f_1, f_2 are continuous and increasing on $[0,\infty)$. By direct calculations, $\eta = 0.0824$, $\mathscr{C} = 17.8461$ and $\mathscr{D} = 1362.3124$. If we choose $r = 8$, $R = 100$ and $\frac{1}{\Phi_1} = \frac{1}{\Phi_2} = \frac{1}{\Phi_1^*} = \frac{1}{\Phi_2^*} = \frac{1}{4}$, then $r < \eta R$ and f_j, for $j = 1,2$, satisfies

- $f_j\big(t,x,y\big) \geq 0.0356 = \frac{1}{2}\cdot\frac{r}{\eta\mathscr{D}}$, $t \in \big[0.25,1\big]$ and $\big(x,y\big) \in \big[8,100\big]$,
- $f_j\big(t,x,y\big) \leq 1.4009 = \frac{1}{\Phi_j}\cdot\frac{R}{\mathscr{C}}$, $t \in [0,1]$ and $\big(x,y\big) \in \big[0,100\big]$.

Then all the conditions of Theorem 4.1 are satisfied. Thus, for $\lambda_1 \leq 117.6798, \lambda_2 \leq 258.1466$, the system of fractional order boundary value problem (5.1)-(5.3) has at least one positive solution.

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