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A SPLITTING ALGORITHM IN A UNIFORMLY CONVEX AND 2-UNIFORMLY SMOOTH BANACH SPACE

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Abstract. In this paper, a splitting algorithm is investigated for zero and fixed points of nonlinear operators. A weak convergence theorem is established in a uniformly convex and 2-uniformly smooth Banach space. An application to an equilibrium problem is provided to support our main results.

1. INTRODUCTION

Throughout this paper, we always assume that B is a reflexive Banach space. Let C be a nonempty closed and convex subset of B and let B^* be the topological dual space of B. Let $Proj_C^B : B \to C$ be a mapping. $Proj_C^B$ is said to be

   (1) sunny if for each $x \in C$ and $t \in (0,1)$, we have $Proj_C^B\big((1-t)Proj_C^B x + tx\big) = Proj_C^B x$;

   (2) contraction if $(Proj_C^B)^2 = Proj_C^B$;

   (3) sunny nonexpansive retractction if $Proj_C^B$ is contractive, sunny and nonexpansive.

Let $T : C \to C$ be a mapping. Throughout this paper, we always use $Fix(T)$ to denote the fixed-point set of T. Recall that T is said to be contractive iff there exists a constant $\kappa \in (0,1)$ such that

$$\|Tx - Ty\| \le \kappa \|x - y\|, \quad \forall x, y \in C.$$

From the celebrated Banach fixed-point theorem, we know that $Fix(T)$ is not empty. The classical iterative method to deal with contractive mappings is the Picard iterative method. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

The class of nonexpansive mappings is more general than the class of contractive mappings. For nonexpansve mappings, their fixed-point sets may be empty. Let $B = l^1$: all sequences $\{x_n\}$ such that $\sum |x_n| < \infty$

$\|x_n\|_1 = \sum |x_n|$. Let $T : l^1 \to l^1$ be the shift operator: $Tx_n = (0, x_1, x_2, \cdots)$

$$C := \{\{x_n\} : x_n \geq 0, \|x_n\|_1 = 1\}.$$

Then $T : C \to C$ is a nonexpansive mapping without a fixed point. Another example, in the framework of infinite dimensional spaces, is from Alspach [1]. Let $B = L^1[0,1]$, $\|f\|_1 = \int_0^1 |f(t)|d_t$, $C := \{f \in L^1[0,1] : \int_0^1 f(x)dx = 1, 0 \leq f \leq 2\}$. Then C is weakly compact and convex.

$$(Tf)(t) = \begin{cases} \min\{2f(2t), 2\}, 0 \leq t \leq \frac{1}{2}; \\ \min\{2f(2t-1) - 2, 0\}, \frac{1}{2} < t \leq 1. \end{cases}$$

Then $T : C \to C$ is a nonexpansive mapping without a fixed point. In 1965, Browder [2] proved that if C is also bounded, closed and convex and B is a uniformly convex, then fixed-point sets of nonexpansive mappings are not empty; see also Kirk [3]. Mann-like iterative methods are popular and efficient to deal with fixed points of nonexpansive mappings; see [4, 5, 6, 7, 8] and the references therein.

Next, we denote the value of $x^* \in B^*$ at $x \in B$ by $\langle x, x^* \rangle$. For a sequence $\{x_n\}$ in B, we denote strong convergence of $\{x_n\}$ to $x \in B$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let q be a given real number with $q > 1$. The generalized duality mapping $J_q : B \to 2^{B^*}$ is defined by

$$J_q(x) := \{x^* \in B^* : \langle x^*, x \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \forall x \in B.$$

In particular, J_2 is called the normalized duality mapping. Next, we use J to denote the normalized duality mapping. It is known that

$$J_q(x) = \|x\|^{q-2}J(x).$$

It is known that normalized duality mapping J has the following properties [9]:

(1) if E is reflexive, then J is surjective;
(2) if E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$, where $x^* \in J(x)$ and $y^* \in J(y)$, holds for all $x \neq y$;
(3) if E is smooth, then J is single-valued;
(4) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of B.

From now on, we use j to denote the single-valued normalized duality mapping.

Recall that T is said to be a κ-strict pseudocontraction iff there exist a constant $\kappa \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \kappa\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C.$$

Rewriting the above inequality, we arrive at

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geq \kappa\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C.$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [10] in a real Hilbert space. Recall that a mapping $A : C \to B$ is said to be accretive iff there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0, \quad \forall x, y \in C.$$

Recall that a mapping $A : C \to B$ is said to be inverse-strongly accretive iff there exist $j(x-y) \in J(x-y)$ and a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let $M : Dom(M) \subset B \to 2^B$ be a set-valued operator with domain $Dom(M) = \{z \in B : Mz \neq \emptyset\}$ and range $Ran(M) = \cup\{Mz : z \in Dom(M)\}$. Recall that M is said to be accretive iff, for all $\lambda > 0$ and for each $x, y \in Dom(M)$,

$$\|\lambda(x' - y') + (x - y)\| \geq \|x - y\|, \quad \forall x' \in Mx, y' \in My.$$

The accretive operators were introduced independently in 1967 by Browder [11] and Kato [12]. From Kato [12], A is accretive iff, for $x, y \in Dom(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \geq 0,$$

where $u \in Mx$ and $v \in My$. An accretive operator M is said to be m-accretive if $Ran(I + \lambda A) = E$ for all $\lambda > 0$. In a Hilbert space, it is maximal monotone.

In this paper, we consider the following problem: finding a point x, which is both a fixed point of a strict pseudocontraction T and a zero point of the sum of an m-accretive mapping M and inverse-strongly accretive mapping S, that is,

$$x \in Fix(T) \cap (S + M)^{-1}(0).$$

This problem includes fixed point problem and operator splitting problems as special cases. In the framework of Hilbert spaces, the accretivity coincides the monotonicity. Recently, Iterative methods have been introduced and investigated for solving the above problem in Hilbert spaces; see [13, 14, 15, 16] and the references therein. In the framework of Banach spaces, metric projections and resolvents may not be nonexpansive. There are relatively few results in the framework of Banach spaces, we refer the reader to [17, 18, 19, 21, 22] and the references therein. The aim of this paper is to present a new weakly convergent method in a uniformly convex and 2-uniformly smooth Banach space. The rest of this paper is organized as follows. In Section 2, necessary preliminaries are provided. In Section 3, the weakly convergent method is presented and an application is also provided.

2. PRELIMINARIES

Recall that a Banach space B is said to be uniformly convex if for each $\varepsilon > 0$ there is a $\delta > 0$ such that, $x, y \in B$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds.

Let $N_B = \{x \in B : \|x\| = 1\}$. The norm of B is said to be Gâteaux differentiable if the limit $\lim_{t \to 0}(\|\frac{x}{t} + y\| - \|\frac{x}{t}\|)$ exists for each $x, y \in N_B$. In this case, B is said to be smooth. The norm of B is said to be uniformly Gâteaux differentiable if for each $y \in N_B$, the limit is attained uniformly for all $x \in N_B$. The norm of B is said to be Fréchet differentiable if for each $x \in N_B$, the limit is attained uniformly for all $y \in N_B$. The norm of B is said to be uniformly Fréchet differentiable if the limit is attained uniformly for all $x, y \in N_B$.

Recall that a mapping $T : C \to C$ is said to be closed at y if $x_n \to x$ and $T x_n \to y$ imply that $Tx = y$. Recall that $T : C \to C$ is said to be demiclosed at y if $x_n \rightharpoonup x$ and $T x_n \to y$ imply that $Tx = y$.

Recall that a Banach space B is said to have the Kadec-Klee property if a sequence $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if B is uniformly convex, then it has the Kadec-Klee property; see [23] for more details.

Next, we list some necessary lemmas.

Lemma 2.1 ([24]). *Let C be a nonempty closed convex and bounded subset of a real uniformly convex Banach space B. Then there is a strictly increasing and continuous convex function $\psi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that, for each nonexpansive mapping T on C, $\forall x,y \in C$ and $\forall t \in [0,1]$,*

$$\psi^{-1}\big(\|y-x\| - \|Ty-Tx\|\big) \geq \|(tTx + (1-t)Ty) - T(tx + (1-t)y)\|.$$

Lemma 2.2 ([9]). *A Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\|ax + (1-a)y\|^2 \leq (1-a)\|y\|^2 + a\|x\|^2 - (1-a)a\varphi(\|y-x\|),$$

for all $x,y \in \{x \in B : \|x\| \leq r\}$, where $r > 0$ is some real number, and $a \in [0,1]$.

Lemma 2.3 ([9]). *Let B be a real 2-uniformly smooth Banach space. Then the following inequality holds: $\|x+y\|^2 - \|x\|^2 \leq 2\langle y, j(x+y)\rangle$, $\forall x,y \in B$, and $\|x+y\|^2 - \|x\|^2 \leq 2\langle y, j(x)\rangle + 2\|\Upsilon y\|^2$, $\forall x,y \in B$, where Υ is the smooth constant.*

Lemma 2.4 ([25]). *Let C be a nonempty closed convex subset of a real uniformly convex Banach space B. Let $T : C \to C$ be a strict pseudocontraction. Then $I - T$ is demiclosed at the origin.*

Lemma 2.5 ([10]). *Let B be a real uniformly convex Banach space. Let B^* the dual space of B such that it has the Kadec-Klee property. Suppose that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} \|(1-a)p_1 + ax_n - p_2\|$ exists for all $a \in [0,1]$ and $p_1, p_2 \in \omega_w(x_n)$, where $\omega_w(x_n) := \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$ Then $\omega_w(x_n)$ is a singleton.*

Lemma 2.6 ([10]). *Let B be a real Banach space and let C be a nonempty closed and convex subset of B. Let $S : C \to B$ be a single valued operator and let $M : B \to 2^B$ be an m-accretive operator. Then*

$$Fix((I + \lambda M)^{-1}(I - \lambda S)) = (S + M)^{-1}(0),$$

where λ is any positive real number.

3. MAIN RESULTS

Theorem 3.1. *Let B be a 2-uniformly smooth (with smooth constant Υ) and uniformly convex Banach space. Let C be a nonempty closed convex subset of B and let $Proj_C^B$ be the sunny nonexpansive retractction from B onto C. Let $S : C \to E$ be an inverse-strongly accretive operator with the constant $\alpha > 0$, $M : Dom(M) \subset B \to 2^B$ an m-accretive operator and $T : C \to B$ a κ-strict pseudocontraction such that $(S+M)^{-1}(0) \cap Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by: $x_0 \in C$ and*

$$x_{n+1} = Proj_C^B\Big((1-\alpha_n)(I + \lambda_n M)^{-1}(x_n - \lambda_n S x_n) + \alpha_n\big((1-\beta_n)x_n + \beta_n T x_n\big)\Big),$$

$n \geq 0$, where $\{\lambda_n\}$ is a positive number sequence, and $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$. Assume that $\lambda \leq \lambda_n \leq \lambda' < \frac{\alpha}{\Upsilon^2}$, where λ and λ' are two constants, $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, where α and α' are two constants, and $\beta \leq \beta_n \leq \frac{\kappa}{\Upsilon^2}$, where $\beta > 0$ is a constant. Then $\{x_n\}$ converges weakly to some point in $Fix(T) \cap (S+M)^{-1}(0)$.

Proof. The proof is split into the following three steps.

Step 1. Show that both sequence $\{x_n\}$ and sequence $\{y_n\}$ are bounded.

Since $Fix(T) \cap (S+M)^{-1}(0)$ is assumed to be nonempty, we can fix an element in $Fix(T) \cap (S+M)^{-1}(0)$, say, x^*. Since S is an inverse-strongly accretive operator, we have

$$\langle Sx - Sx^*, j(x-y) \rangle \geq \alpha \|Sx - Sx^*\|^2, \quad \forall x \in C.$$

Set $y_n = (1-\beta_n)x_n + \beta_n Tx_n$ and $\Lambda_n = x_n - \lambda_n Sx_n$. Using Lemma 2.3, we see that

$$
\begin{aligned}
\|(x^* - \lambda_n Sx^*) - \Lambda_n\|^2 &= \|(x^* - x_n) - \lambda_n(Sx^* - Sx_n)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n\langle Sx^* - Sx_n, j(x^* - x_n)\rangle + 2\Upsilon^2\lambda_n^2\|Sx_n - Sx^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n(\alpha - \Upsilon^2\lambda_n)\|Sx^* - Sx_n\|^2.
\end{aligned}
\tag{3.1}
$$

This in turn implies that

$$\|(x^* - \lambda_n Sx^*) - \Lambda_n\| \leq \|x_n - x^*\|. \tag{3.2}$$

Note that

$$\langle Tx_n - Tx^*, j(x_n - x^*)\rangle \leq \|x_n - x^*\| - \kappa\|(I-T)x_n\|.$$

Using Lemma 2.3, we see that

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n Tx_n - x^*\|^2 \\
&= \|x_n - x^* + \beta_n(Tx_n - Tx^* - (x_n - x^*))\|^q \\
&\leq \|x_n - x^*\|^2 + 2\beta_n(\langle Tx_n - Tx^*, j(x_n - x^*)\rangle - \langle(x_n - x^*), j(x_n - x^*)\rangle) \\
&\quad + 2\Upsilon^2\beta_n^2\|Tx_n - Tx^* - (x_n - x^*)\|^2 \\
&\leq \|x_n - x^*\|^2 + 2\beta_n(\|x_n - x^*\|^2 - \kappa\|(I-T)x_n\|^2) - 2\beta_n\|x_n - x^*\|^2 \\
&\quad + 2\Upsilon^2\beta_n^2\|Tx_n - x_n\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\beta_n(\kappa - \Upsilon^2\beta_n)\|Tx_n - x_n\|^2.
\end{aligned}
\tag{3.3}
$$

This shows that $\|y_n - x^*\| \leq \|x_n - x^*\|$ for each $n \geq 1$. It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|(1-\alpha_n)(I+\lambda_n M)^{-1}(x_n - \lambda_n Sx_n) + \alpha_n((1-\beta_n)x_n + \beta_n Tx_n) - x^*\| \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\|(I+\lambda_n M)^{-1}(x_n - \lambda_n Sx_n) - (I+\lambda_n M)^{-1}(x^* - \lambda_n Sx^*)\| \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\|(x^* - \lambda_n Sx^*) - \Lambda_n\| \\
&\leq \|x_n - x^*\|.
\end{aligned}
$$

This implies that $\lim_{n\to\infty}\|x_n - p\|$. Hence, $\{x_n\}$ is a bounded sequence. From (3.3), we see that $\{y_n\}$ is also a bounded sequence. This completes that both sequence $\{x_n\}$ and sequence $\{y_n\}$ are bounded.

Step 2. $\omega_w(x_n) \subset (S+M)^{-1}(0) \cap Fix(T)$, where $\omega_w(x_n)$ is the weak limit set of $\{x_n\}$.

Set $\xi_n = (I+\lambda_n M)^{-1}(x_n - \lambda_n Sx_n)$. Since M is an m-accretive operator, we have

$$\left\langle \frac{x_n - (I+\lambda M)^{-1}(I-\lambda S)x_n}{\lambda} - \frac{x_n - \xi_n}{\lambda_n}, j\big((I+\lambda M)^{-1}(I-\lambda S)x_n - \xi_n\big) \right\rangle \geq 0.$$

Hence,

$$\|(I+\lambda M)^{-1}(I-\lambda S)x_n - \xi_n\|^2$$

$$\leq \frac{|\lambda_n - \lambda|}{\lambda_n}\langle x_n - \xi_n, j\big((I+\lambda M)^{-1}(I-\lambda S)x_n - \xi_n\big)\rangle$$

$$\leq \|(I+\lambda M)^{-1}(I-\lambda S)x_n - \xi_n\|\|x_n - \xi_n\|.$$

This shows that

$$\|(I+\lambda M)^{-1}(I-\lambda S)x_n - \xi_n\| \leq \|x_n - \xi_n\|. \tag{3.4}$$

On the other hand, it follows Lemma 2.2 that

$$\|x_{n+1} - x^*\|^2 \leq \|(1-\alpha_n)\xi_n + \alpha_n y_n - x^*\|^2$$

$$\leq \alpha_n\|y_n - x^*\|^2 + (1-\alpha_n)\|\xi_n - x^*\|^2 - (1-\alpha_n)\alpha_n\varphi(\|\xi_n - y_n\|)$$

$$\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|\Lambda_n - (I-\lambda_n S)x^*\|^2 - (1-\alpha_n)\alpha_n\varphi(\|\xi_n - y_n\|)$$

$$\leq \|x_n - x^*\|^2 - (1-\alpha_n)\alpha_n\varphi(\|\xi_n - y_n\|).$$

Hence,

$$(1-\alpha_n)\alpha_n\varphi(\|\xi_n - y_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$$

$$\leq \psi(\|x_n - x^*\| - \|x_{n+1} - x^*\|),$$

where ψ is an appropriate constant. So,

$$\lim_{n\to\infty}\varphi(\|\xi_n - y_n\|) = 0.$$

Using the property of φ, we find that $\lim_{n\to\infty}\|\xi_n - y_n\| = 0$. Since $\|\cdot\|^2$ is a convex function, we also have

$$\|x_{n+1} - x^*\|^2 \leq \|(1-\alpha_n)\xi_n + \alpha_n y_n - x^*\|^2$$

$$\leq (1-\alpha_n)\|\xi_n - x^*\|^2 + \alpha_n\|y_n - x^*\|^2 \tag{3.5}$$

$$\leq (1-\alpha_n)\|\xi_n - x^*\|^2 + \alpha_n\|x_n - x^*\|^2.$$

Using Lemma 2.2, we find that

$$\|\xi_n - x^*\|^2 \leq \left\|\frac{\lambda_n}{2}\Big(\frac{\Lambda_n - \xi_n}{\lambda_n} - \frac{(I-\lambda_n S)x^* - x^*}{\lambda_n}\Big) + (\xi_n - x^*)\right\|^2$$

$$= \left\|\frac{1}{2}\big(\Lambda_n - (I-\lambda_n S)x^*\big) + \frac{1}{2}(\xi_n - x^*)\right\|^2$$

$$\leq \frac{1}{2}\|\Lambda_n - (I-\lambda_n S)x^*\|^2 + \frac{1}{2}\|\xi_n - x^*\|^2$$

$$\qquad - \frac{1}{4}\varphi\Big(\|(\xi_n - x^*) - \big(\Lambda_n - (I-\lambda_n S)x^*\big)\|\Big) \tag{3.6}$$

$$\leq \|\Lambda_n - (I-\lambda_n S)x^*\|^2 - \frac{1}{4}\varphi\Big(\|(\xi_n - x^*) - \big(\Lambda_n - (I-\lambda_n S)x^*\big)\|\Big)$$

$$\leq \|x_n - x^*\|^2 - \frac{1}{4}\varphi\Big(\|(\xi_n - x^*) - \big(\Lambda_n - (I-\lambda_n S)x^*\big)\|\Big)$$

$$\qquad - 2\lambda_n(\alpha - \Upsilon^2\lambda_n)\|Sx^* - Sx_n\|^2.$$

Substituting (3.6) and (3.5) yields that

$$\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_n)\|\xi_n - x^*\|^2 + \alpha_n\|x_n - x^*\|^2$$

$$\leq \|x_n - x^*\|^2 - (1 - \alpha_n)\frac{1}{4}\varphi\Big(\|(\xi_n - x^*) - \big(\Lambda_n - (I - \lambda_n S)x^*\big)\|\Big) \quad (3.7)$$

$$- 2(1 - \alpha_n)\lambda_n(\alpha - \Upsilon^2\lambda_n)\|Sx^* - Sx_n\|^2).$$

From (3.7), we have

$$2(1 - \alpha_n)\lambda_n(\alpha - \Upsilon^2\lambda_n)\|Sx^* - Sx_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$$

$$\leq \tau(\|x_n - x^*\| - \|x_{n+1} - x^*\|),$$

where τ is an appropriate constant. This implies that

$$\lim_{n\to\infty}\|Sx_n - Sx^*\| = 0.$$

From (3.7), we also have

$$(1 - \alpha_n)\frac{1}{4}\varphi\Big(\|(\xi_n - x^*) - \big(\Lambda_n - (I - \lambda_n S)x^*\big)\|\Big) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \quad (3.8)$$

$$\leq \nu(\|x_n - x^*\| - \|x_{n+1} - x^*\|)$$

where ν is an appropriate constant. This implies that

$$\lim_{n\to\infty}\varphi\Big(\|(\xi_n - x^*) - \big(\Lambda_n - (I - \lambda_n S)x^*\big)\|\Big) = 0.$$

Using the property of φ, we find that

$$\lim_{n\to\infty}\|\xi_n - x_n - \lambda_n(Sx^* - Sx_n)\| = 0.$$

In view of

$$\|\xi_n - x_n\| \leq \|\xi_n - x_n - \lambda_n(Sx^* - Sx_n)\| + \lambda_n\|Sx^* - Sx_n\|,$$

we arrive at

$$\lim_{n\to\infty}\|\xi_n - x_n\| = 0.$$

It follows from (3.4) that

$$\lim_{n\to\infty}\|(I + \lambda M)^{-1}(I - \lambda S)x_n - \xi_n\| = 0,$$

Note that

$$\|(I + \lambda M)^{-1}(I - \lambda S)x_n - x_n\| \leq \|(I + \lambda M)^{-1}(I - \lambda S)x_n - \xi_n\| + \|\xi_n - x_n\|$$

This shows that

$$\lim_{n\to\infty}\|(I + \lambda M)^{-1}(I - \lambda S)x_n - x_n\| = 0. \quad (3.9)$$

Note that

$$\|x_n - Tx_n\| \leq \|x_n - \xi_n\| + \|\xi_n - y_n\| + \|y_n - Tx_n\|$$

$$\leq \|x_n - \xi_n\| + \|\xi_n - y_n\| + (1 - \beta_n)\|x_n - Tx_n\|.$$

This implies that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \quad (3.10)$$

Since $(I + \lambda M)^{-1}(I - \lambda S)$ has the property of nonexpansivity, we find that $\omega_w(x_n) \subset (I + \lambda M)^{-1}(I - \lambda S)$. Since T is a strict pseudocontraction, we find from Lemma 2.4 that $\omega_w(x_n) \subset Fix(T)$. This proves that

$$\omega_w(x_n) \subset Fix(T) \cap (S + M)^{-1}(0).$$

Step 3. $\omega_w(x_n)$ is a singleton, which completes the whole proof.

Next, putting $T_n = \beta_n T + (1 - \beta_n)I$, we see from Lemma 2.3 that, $\forall x, y \in B$,

$$\|T_n x - T_n y\|^2 \leq \|x - y\|^2 + 2\beta_n \langle Tx - Ty - (x - y), j(x - y) \rangle + 2\Upsilon^2 \beta_n^2 \|Tx - Ty - (x - y)\|^q$$

$$= \|x - y\|^2 + 2\beta_n \langle Tx - Ty, j(x - y) \rangle - 2\beta_n \|x - y\|^2 + 2\Upsilon^2 \beta_n^2 \|Tx - Ty - (x - y)\|^2$$

$$\leq \|x - y\|^2 - 2\beta_n (\kappa - \Upsilon^2 \beta_n) \|Tx - Ty - (x - y)\|^2.$$

$$\leq \|x - y\|^2.$$

This shows that T_n is a nonexpansive mapping for each $n \geq 1$. Define mappings $W_n : C \to C$ by

$$W_n := Proj_C^B \Big(\alpha_n T_n + (1 - \alpha_n)(I + \lambda_n M)^{-1}(I - \lambda_n S) \Big).$$

and set

$$W_{n,m} = W_{n+m-1} W_{n+m-2} \cdots W_n, \quad \forall n, m \geq 1.$$

It follows that $W_{n,m} x_n = x_{n+m}$. Note that

$$\|W_n x - W_n y\| = \|Proj_C^B \Big(\alpha_n T_n + (1 - \alpha_n)(I + \lambda_n M)^{-1}(I - \lambda_n S) \Big) x$$

$$- Proj_C^B \Big(\alpha_n T_n + (1 - \alpha_n)(I + \lambda_n M)^{-1}(I - \lambda_n S) \Big) y\|$$

$$\leq (1 - \alpha_n) \|(I - \lambda_n S)x - (I - \lambda_n S)y\| + \alpha_n \|T_n x - T_n y\|$$

$$\leq \|x - y\|, \quad \forall x, y \in C.$$

This shows that each W_n has the property of nonexpasivity. For all $t \in [0, 1]$ and $n, m \geq 1$, put

$$b_n(t) = \|t x_n - p_2 + (1 - t)p_1\|,$$

and

$$c_{n,m} = \|W_{n,m}(t x_n + (1 - t)p_1) - (t x_{n+m} + (1 - t)p_1)\|,$$

where p_1 and p_2 are in $(S + M)^{-1}(0) \cap Fix(T)$. By virtue of Lemma 2.1, one has

$$c_{n,m} \leq \psi^{-1} \big(\|x_n - p_1\| - \|W_{n,m} p_1 - W_{n,m} x_n\| \big)$$

$$\leq \psi^{-1} \big(\|x_n - p_1\| - (\|p_1 - x_{n+m}\| - \|W_{n,m} p_1 - p_1\|) \big),$$

which shows that $\{c_{n,m}\}$ converges uniformly to zero as $n \to \infty$ for all $m \geq 1$. Note that

$$b_{n+m}(t) \leq \|W_{n,m}(t x_n + (1 - t)p_1) - p_2\| + c_{n,m}$$

$$\leq \|W_{n,m}(t x_n + (1 - t)p_1) - S^{n,m} p_2\| + c_{n,m} + \|p_2 - W_{n,m} p_2\|$$

$$\leq b_n(t) + c_{n,m} + \|p_2 - W_{n,m} p_2\|,$$

from which it is not hard to see for each $t \in [0, 1]$ that $\lim_{n \to \infty} b_n(t)$ exists. Using Lemma 2.5, we find that $\omega_w(x_n)$ is a singleton set. This proves the theorem. $\square$

If T is nonexpansive, that is, $\kappa = \frac{1}{2}$, we find the following result immediately.

Corollary 3.2. *Let B be a 2-uniformly smooth (with smooth constant Υ) and uniformly convex Banach space. Let C be a nonempty closed convex subset of B and let $Proj_C^B$ be the sunny nonexpansive re-tractction from B onto C. Let $S : C \to E$ be an inverse-strongly accretive operator with the constant*

$\alpha > 0$, $M : Dom(M) \subset B \to 2^B$ an m-accretive operator and $T : C \to B$ a nonexpansive mapping such that $(S+M)^{-1}(0) \cap Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by: $x_0 \in C$ and

$$x_{n+1} = Proj_C^B\Big((1-\alpha_n)(I+\lambda_n M)^{-1}(x_n - \lambda_n Sx_n) + \alpha_n\big((1-\beta_n)x_n + \beta_n Tx_n\big)\Big),$$

$n \geq 0$, where $\{\lambda_n\}$ is a positive number sequence, and $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$. Assume that $\lambda \leq \lambda_n \leq \lambda' < \frac{\alpha}{\Upsilon^2}$, where λ and λ' are two constants, $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, where α and α' are two constants, and $\beta \leq \beta_n \leq \frac{1}{2\Upsilon^2}$, where $\beta > 0$ is a constant. Then $\{x_n\}$ converges weakly to some point in $Fix(T) \cap (S+M)^{-1}(0)$.

Furthermore, if T is an identity mapping, we have the following result immediately.

Corollary 3.3. *Let B be a 2-uniformly smooth (with smooth constant Υ) and uniformly convex Banach space. Let C be a nonempty closed convex subset of B and let $Proj_C^B$ be the sunny nonexpansive retractction from B onto C. Let $S : C \to E$ be an inverse-strongly accretive operator with the constant $\alpha > 0$ and let $M : Dom(M) \subset B \to 2^B$ be an m-accretive operator such that $(S+M)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by: $x_0 \in C$ and*

$$x_{n+1} = Proj_C^B\Big((1-\alpha_n)(I+\lambda_n M)^{-1}(x_n - \lambda_n Sx_n) + \alpha_n x_n\Big),$$

$n \geq 0$, where $\{\lambda_n\}$ is a positive number sequence, and $\{\alpha_n\}$ is a real number sequence in $(0,1)$. Assume that $\lambda \leq \lambda_n \leq \lambda' < \frac{\alpha}{\Upsilon^2}$, where λ and λ' are two constants, $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, where α and α' are two constants. Then $\{x_n\}$ converges weakly to some point in $(S+M)^{-1}(0)$.

We remark here that a typical example of both uniformly convex and 2-uniformly smooth Banachs is L^2. In addition, if the domain of M is included in C, then $Proj_C^B$ can be removed from all the above convergence theorems.

Finally, we give an applications to equilibrium problems in a Hilbert space H.

Let C be a nonempty closed and convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C.$$

To study the equilibrium problem, we may assume that F satisfies the following conditions:

(C1) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(C2) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous,

(C3) $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(C4) $F(x,x) = 0$ for all $x \in C$;

If $F(x,y) = \langle Ax, y-x \rangle$, where $A : C \to H$ is some monotone operator, for every $x, y \in C$, we see that the equilibrium problem is reduced to a monotone variational inequality problem.

Lemma 3.4. [28] *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction with conditions (C1)-(C4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y-z, z-x \rangle + rF(z,y) \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : \langle y - z, z - x \rangle + rF(z,y) \geq 0, \quad \forall y \in C\} \tag{3.11}$$

for all $r > 0$ and $x \in H$. Then

(a) *T_r is single-valued firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H;$$

(b) *$Fix(T_r) = EP(F)$, where $EP(F)$ denotes the solution set of the equilibrium problem, is closed and convex.*

Lemma 3.5. [29] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F a bifunction from $C \times C$ to $\mathbb{R}$ with conditions (C1)-(C4) and let A_F be a multivalued mapping on H defined by*

$$A_F x = \begin{cases} \emptyset, & x \notin C, \\ \{z \in H : \langle y - x, z \rangle \leq F(x,y), \quad \forall y \in C\}, & x \in C. \end{cases} \tag{3.12}$$

Then A_F is a maximal monotone operator with the domain $Dom(A_F) \subset C$, $T_\lambda = (I + \lambda A_F)^{-1}$, $\forall \lambda > 0$, and $EP(F) = A_F^{-1}(0)$.

With the above tools, we can find the following result immediately.

Theorem 3.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to B$ be a κ-strict pseudocontraction. Let F a bifunction from $C \times C$ to $\mathbb{R}$ with conditions (C1)-(C4). Assume that $EP(F) \cap Fix(T) \neq \emptyset$, where $EP(F)$ denote the solution set of the equilibrium problem. Let $\{x_n\}$ be a sequence defined by: $x_0 \in C$ and*

$$x_{n+1} = (1 - \alpha_n)z_n + \alpha_n\big((1 - \beta_n)x_n + \beta_n T x_n\big),$$

$n \geq 0$, where $z_n \in C$ such that

$$\langle u - z_n, z_n - x_n \rangle + \lambda_n F(z_n, u) \geq 0, \quad \forall u \in C.$$

$\{\lambda_n\}$ is a positive number sequence, and $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$. Assume that $\lambda \leq \lambda_n \leq \lambda' < \frac{\alpha}{\Upsilon^2}$, where λ and λ' are two constants, $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, where α and α' are two constants, and $\beta \leq \beta_n \leq 2\kappa$, where $\beta > 0$ is a constant. Then $\{x_n\}$ converges weakly to some point in $Fix(T) \cap EP(F)$.

Proof. From Lemma 3.5, we find that the iterative method in Theorem 3.6 can be rewritten as

$$x_{n+1} = (1 - \alpha_n)T_{\lambda_n}x_n + \alpha_n\big((1 - \beta_n)x_n + \beta_n T x_n\big),$$

where $T_{\lambda_n} = (I + rA_F)^{-1}$,

$$A_F x = \begin{cases} \emptyset, & x \notin C, \\ \{z \in H : \langle y - x, z \rangle \leq F(x,y), \quad \forall y \in C\}, & x \in C. \end{cases}$$

On the other hand, smooth constant Υ is $\frac{1}{\sqrt{2}}$ in the framework of Hilbert spaces. From Theorem 3.1, we can obtain the desired conclusion immediately.

$\square$

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REFERENCES

[1] D.E. Alspach, A fixed point free nonexpansive map, 82 (1981), 423-424.

[2] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Natl. Acad. Sci. USA 54 (1965), 1041-1044.

[3] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1965), 1004-1006.

[4] S.Y. Cho, Generalized mixed equilibrium and fixed point problems in a Banach space, J. Nonlinear Sci. Appl. 9 (2016), 1083-1092.

[5] X. Qin, A. Petrusel, J.C. Yao, CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces, J. Nonlinear Convex Anal. 19 (2018), 157-165.

[6] W. Takahashi, Weak and strong convergence theorems for families of nonlinear and nonself mappings in Hilbert spaces, J. Nonlinear Var. Anal. 1 (2017), 1-23.

[7] X. Qin, J.C. Yao, Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators, J. Inequal. Appl. 2016 (2016), Article ID 232.

[8] X. Qin, S.Y. Cho, Convergence analysis of a monotone projection algorithm in reflexive Banach spaces. Acta Math. Sci. 37 (2017), 488-502.

[9] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Space, Noordhoff, Amsterdam, 1976.

[10] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967), 197-228.

[11] F.E. Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. 73 (1967), 875-882.

[12] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Sco. Japan 19 (1967), 508-520.

[13] S.Y. Cho, B.A. Bin Dehaish, X. Qin, Weak convergence of a splitting algorithm in Hilbert spaces, J. Appl. Anal. Comput. 7 (2017), 427-438.

[14] S.Y. Cho, X. Qin, On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems. Appl. Math. Comput. 235 (2014), 430-438.

[15] S.Y. Cho, X. Qin, L. Wang, Strong convergence of a splitting algorithm for treating monotone operators, Fixed Point Theory Appl. 2014 (2014), Article ID 94.

[16] X. Qin, S.Y. Cho, L. Wang, A regularization method for treating zero points of the sum of two monotone operators, Fixed Point Theory Appl. 2014 (2014), Article ID 75.

[17] M. Wen, C. Hu, J. Peng, Iterative approximation of zeros of accretive operators and of solutions to fixed point problems in q-uniformly smooth Banach spaces, J. Fixed Point Theory Appl. 19 (2017), 1439-1452.

[18] K. Aoyama, H. Iiduka, W. Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, Fixed Point Theory Appl. 2006 (2006), Article ID 35390.

[19] S.S. Chang, C.F. Wen, J.C. Yao, Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces, Optimization, 67 (2018), 1183-1196.

[20] X. Qin, S.Y. Cho, L. Wang, Iterative algorithms with errors for zero points of m-accretive operators, Fixed Point Theory Appl. 2013 (2013), Article ID 148.

[21] X. Qin, J.C. Yao, Projection splitting algorithms for nonself operators, J. Nonlinear Convex Anal. 18 (2017), 925-935.

[22] L.C. Ceng, C.F. Wen, Y. Yao, Iteration approaches to hierarchical variational inequalities for infinite nonexpansive mappings and finding zero points of m-accretive operators, J. Nonlinear Var. Anal. 1 (2017), 213-235.

[23] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht, 1990.

[24] R.E. Bruck, The iterative solution of the equation $y \in x + Tx$ for a monotone operator T in Hilbert space, Bull. Amer. Math. Soc. 79 (1973), 1258-1262.

[25] H. Zhou, Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces, Nonlinear Anal. 69 (2008), 3160-3173.

[26] J.G. Falset, W. Kaczor, T. Kuczumow, S. Reich, Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. 43 (2007), 377-401.

[27] X. Qin, S.Y. Cho, L. Wang, Strong convergence of an iterative algorithm involving nonlinear mappings of nonexpansive and accretive type, Optimization, doi: 10.1080/02331934.2018.1491973.

[28] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994), 123-145.

[29] S. Takahashi, W. Takahashi, M. Toyoda, Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces, J. Optim. Theory Appl. 147 (2010), 27-41.