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A WIDE NEIGHBORHOOD PRIMAL-DUAL INTERIOR-POINT ALGORITHM WITH ARC-SEARCH FOR LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. In this paper, ellipsoidal estimations are used to track the central path of a linear complementarity problem (LCP). A wide neighborhood primal-dual interior-point algorithm is devised to search an ε-approximate solution of the LCP along the ellipse. The algorithm is proved to be polynomial with the complexity bound $O(n\log\frac{(x^0)^T s^0}{\varepsilon})$, which is as good as the linear programming analogue. The numerical results show that the proposed algorithm is efficient and reliable.

Keywords. Interior-point algorithm; Ellipsoidal approximation; Polynomial complexity; Linear complementarity problem.

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1. Introduction

After Karmarkar's landmark paper [1], primal-dual interior-point methods (IPMs) have shown their power in solving linear programming (LP), quadratic programming (QP), linear complementarity problem (LCP) and other optimization problems [2, 3, 4, 5]. It is noted that LCPs [6, 7] are a general class of mathematical problems having important applications in mathematical programming and equilibrium problems. Therefore, we should pay more attention on IPMs for LCPs.

The existing primal-dual IPMs usually search for the optimizers along a straight line related to the first and higher-order derivatives associated with the central path [8, 9]. Although, theoretically, the central path has an important role in primal-dual IPMs, but practically, following the central path is difficult and arduous. To remedy, Yang [10, 11, 12, 13] developed the idea of arc-search in LP, which search for optimizers along an ellipse that is an approximation of the central path. Yang [11] showed that arc-search along ellipse may be a better method than one-dimensional search methods because the algorithm is proved to be polynomial with a better bound than the bounds of all existing higher-order algorithms. Then, the arc-search techniques were applied to primal-dual path-following IPM for convex quadratic programming (CQP) by Yang [14]. The algorithm enjoyed currently the best known polynomial complexity bound $O(\sqrt{n}\log\frac{1}{\varepsilon})$. Later, Yang *et al.* [15, 16] introduced two wide neighborhood infeasible

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IPMs with arc-search for LP and symmetric optimization (SO) with the complexity bounds $O(n^{\frac{5}{4}}\log\frac{1}{\varepsilon})$ and $O(r^{\frac{5}{4}}\log\frac{1}{\varepsilon})$, respectively. Pirhaji *et al.* [17] explored a l_2-neighborhood infeasible IPM for LCP with the iteration bound $O(n\log\frac{1}{\varepsilon})$.

Here, our main purpose is to generalize the arc-search feasible IPM for LP [11, 18] to LCP. Compared with [16, 17], the new algorithm has a weak restrict on step size and works in the negative infinity neighborhood of the central path, which is a popular wide neighborhood where the software packages run. Moreover, the algorithm enjoys the iteration complexity $O(n\log\frac{(x^0)^T s^0}{\varepsilon})$, which is abound at least as good as the best bound of existing wide neighborhood IPMs. The numerical tests show that the proposed algorithm has a tendency to solve problems faster than some other IPMs.

The outline of this paper is organized as follows. In Section 2, we recall some fundamental concepts of the LCP such as the central path and its ellipsoidal estimation. In Section 3, we introduce a wide neighborhood primal-dual IPM with arc-search for the LCP. In Section 4, we give the technical results that are required in the convergent analysis and obtain the complexity bound of algorithm. In Section 5, we provide some preliminary numerical results. Finally, we close the paper by some conclusions in Section 6.

We use the following notations throughout the paper. All vectors are considered to be column vectors. Denote the vector of all ones with appropriate dimension by e, $n\times n$-dimensional matrix space by $\mathbf{R}^{n\times n}$, the non-negative orthant of $\mathbf{R}^n$ by $\mathbb{R}^n_+$, Hadamard product of two vectors $x\in\mathbf{R}^n$ and $s\in\mathbf{R}^n$ by xs, the ith component of x by x_i, the Euclidean norm and the 1-norm of x by $||x||$ and $||x||_1$, the transpose of matrix A by A^T. If $x\in\mathbf{R}^n$, X:=diag(x) represents the diagonal matrix having the elements of x as diagonal entries. Finally, we define an initial point of any algorithm by (x^0,s^0), the point after the kth iteration by (x^k,s^k).

2. The central path and its ellipsoidal approximation

The LCP requires to find a pair of vectors $(x,s)\in\mathbf{R}^{2n}$ such that

$$\begin{cases} s = Mx+q, \\ xs = 0, \\ x,s\geq 0, \end{cases} \tag{2.1}$$

where $q\in\mathbf{R}^n$ and M is a $n\times n$ positive semi-definite matrix, that is, $x^T Mx\geq 0$, for $x\in\mathbf{R}^n$.

Denote the feasible set of problem (2.1) by

$$\mathscr{F} = \{(x,s)\in\mathbf{R}^n\times\mathbf{R}^n \;:\; s = Mx+q,\ (x,s)\geq 0\}, \tag{2.2}$$

and the interior feasible set

$$\mathscr{F}^0 = \{(x,s)\in\mathbf{R}^n\times\mathbf{R}^n \;:\; s = Mx+q,\ \ (x,s)> 0\}. \tag{2.3}$$

Throughout the paper, we assume that the interior-point condition (IPC) holds [19], i.e., $\mathscr{F}^0$ is not empty. Basic idea of the feasible IPMs is to replace the second equation in (2.1) by the perturbed equation $xs=\mu e$, with $\mu>0$, to get the following parameterized system

$$\begin{cases} s = Mx+q, \\ xs = \mu e, \\ x,s\geq 0. \end{cases} \tag{2.4}$$

If the LCP satisfies the IPC, then system (2.4) has a unique solution for every $\mu > 0$, denoted by $(x(\mu), s(\mu))$. The set of all such solution constructs a homotopy path, which is called the central path of the LCP and used as a guideline to solution of the LCP [20]. Thus, the central path is an arc in $\mathbf{R}^{2n}$ parameterized as a function of μ defined as

$$C = \{(x(\mu), s(\mu)) : \ \mu > 0\}. \tag{2.5}$$

If $\mu \to 0$, then the limit of the central path exists and is an optimal solution for LCP.

Recently, Yang [10, 11] defined an ellipse $\xi(\alpha)$ in a $2n$-dimensional space to approximate the central path C, as following

$$\xi(\alpha) = \{(x(\alpha), s(\alpha)) : \ (x(\alpha), s(\alpha)) = \vec{a}\cos(\alpha) + \vec{b}\sin(\alpha) + \vec{c}\}, \tag{2.6}$$

where $\vec{a}, \vec{b} \in \mathbf{R}^{2n}$ are the axes of the ellipse, orthogonal to each other, and $\vec{c} \in \mathbf{R}^{2n}$ is the center of the ellipse.

Let $z = (x(\mu), s(\mu)) = (x(\alpha_0), s(\alpha_0)) \in \xi(\alpha)$ be close to or on the central path. Using the approach of Yang [11], we define the first and second derivatives at $(x(\alpha_0), s(\alpha_0))$ to have the form as if they were on the central path, which satisfy

$$\begin{bmatrix} M & -I \\ S & X \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} 0 \\ xs - \sigma\mu e \end{bmatrix}, \tag{2.7}$$

$$\begin{bmatrix} M & -I \\ S & X \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{s} \end{bmatrix} = \begin{bmatrix} 0 \\ -2\dot{x}\dot{s} \end{bmatrix}, \tag{2.8}$$

where $\mu = \frac{x^T s}{n}$, $\sigma \in (0, \frac{1}{4})$ is the center parameter.

Let $\alpha \in [0, \frac{\pi}{2}]$ and $(x(\alpha), s(\alpha))$ be updated by (x, s) after the searching along the ellipse. Similar to the proof of Theorem 3.1 in [11], we have the following theorem.

Theorem 2.1. *Let $(x(\alpha), s(\alpha))$ be an arc defined by (2.6) passing through a point $(x, s) \in \xi(\alpha)$, and its first and second derivatives at (x, s) be $(\dot{x}, \dot{s})$ and $(\ddot{x}, \ddot{s})$, which are defined by (2.7) and (2.8). Then, an ellipse approximation of the central path is given by*

$$x(\alpha) = x - \sin(\alpha)\dot{x} + (1 - \cos(\alpha))\ddot{x}, \tag{2.9}$$

$$s(\alpha) = s - \sin(\alpha)\dot{s} + (1 - \cos(\alpha))\ddot{s}. \tag{2.10}$$

For convenience of reference, assume $g(\alpha) := 1 - \cos(\alpha)$. Using Theorem 2.1, (2.7) and (2.8), we get

$$x(\alpha)s(\alpha) = xs - \sin(\alpha)(xs - \sigma\mu e) + \chi(\alpha), \tag{2.11}$$

where

$$\chi(\alpha) := -g^2(\alpha)\dot{x}\dot{s} - \sin(\alpha)g(\alpha)(\dot{x}\ddot{s} + \dot{s}\ddot{x}) + g^2(\alpha)\ddot{x}\ddot{s}. \tag{2.12}$$

We will use negative infinity neighborhood of the central path in the literature, defined as

$$N_\infty^-(\gamma) = \{(x, s) \in \mathscr{F}^0 : \ \min(xs) \geq \gamma\mu\}, \tag{2.13}$$

where $\gamma \in (0, \frac{1}{2})$ is a constant independent of n. This neighborhood is widely used in implementations too.

In order to facilitate the analysis of the algorithm, we denote

$$\sin(\hat{\alpha}) \ := \max\{\sin(\alpha) : \ (x(\alpha), s(\alpha)) \in N_\infty^-(\gamma), \ \forall \sin(\alpha) \in [0, 1], \ \alpha \in [0, \frac{\pi}{2}]\}. \tag{2.14}$$

3. THE ARC-SEARCH INTERIOR-POINT ALGORITHM

The proposed algorithm starts from an initial arbitrary point $(x^0, s^0) \in N_\infty^-(\gamma)$, which is close to or on the central path C. Using an ellipse that passes through the point (x^0, s^0), the algorithm approximates the central path and gets a new iterate $(x(\alpha), s(\alpha))$, which makes a certain decline of the duality gap. The procedure is repeated until find an ε-approximate solution of the LCP. A more formal description of the algorithm is presented next.

Algorithm 1: (Arc-search IPM for LCP)
Input An accuracy parameter $\varepsilon > 0$, neighborhood parameters $\gamma \in (0, \frac{1}{2})$, center parameter $\sigma \in (0, \frac{1}{4})$, initial point (x^0, s^0) with $\mu^0 = \frac{(x^0)^T s^0}{n}$, and $k := 0$.
If $(x^k)^T s^k \leq \varepsilon$, then stop.
for iteration $k = 1, 2, \ldots$
step 1 Solve the systems (2.7) and (2.8) to get $(\dot{x}, \dot{s})$ and $(\ddot{x}, \ddot{s})$;
step 2 Compute $\sin(\hat{\alpha}^k)$ by (14), and let $(x^{k+1}, s^{k+1}) = (x(\hat{\alpha}^k), s(\hat{\alpha}^k))$;
step 3 Calculate $\mu^{k+1} = \frac{(x^{k+1})^T s^{k+1}}{n}$, $k := k+1$, and go to step 1.
end(for)

Based on the previous discussion, we state several lemmas which are necessary for convergence analysis of the algorithm.

Lemma 3.1. *Let (x, s) be a strictly feasible point of LCP, $(\dot{x}, \dot{s})$ and $(\ddot{x}, \ddot{s})$ meet (2.7) and (2.8) respectively, $(x(\alpha), s(\alpha))$ be calculated using (2.9) and (2.10). Then*

$$Mx(\alpha) - s(\alpha) = q.$$

Proof. Since (x, s) is a strict feasible point, we find from Theorem 2.1, (2.7) and (2.8) that

$$\begin{aligned} Mx(\alpha) - s(\alpha) &= M[x - \sin(\alpha)\dot{x} + (1-\cos(\alpha))\ddot{x}] - [s - \sin(\alpha)\dot{s} + (1-\cos(\alpha))\ddot{s}] \\ &= Mx - \sin(\alpha)M\dot{x} + (1-\cos(\alpha))M\ddot{x} - s + \sin(\alpha)\dot{s} - (1-\cos(\alpha))\ddot{s} \\ &= Mx - s - \sin(\alpha)(M\dot{x} - \dot{s}) + (1-\cos(\alpha))(M\ddot{x} - \ddot{s}) \\ &= q. \end{aligned}$$

This completes the proof. $\square$

Lemma 3.2. *Let $\dot{x}, \dot{s}, \ddot{x}$ and $\ddot{s}$ be defined in (2.7) and (2.8), and let M be a positive semi-define matrix. Then the following relations hold:*

$$\begin{aligned} \dot{x}^T \dot{s} &= \dot{x}^T M \dot{x} \geq 0, \\ \ddot{x}^T \ddot{s} &= \ddot{x}^T M \ddot{x} \geq 0, \\ \ddot{x}^T \dot{s} &= \dot{x}^T \ddot{s} = \dot{x}^T M \ddot{x}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &-(\dot{x}^T \dot{s})(1-\cos(\alpha))^2 - (\ddot{x}^T \ddot{s})\sin^2(\alpha) \\ &\leq (\ddot{x}^T \dot{s} + \dot{x}^T \ddot{s})\sin(\alpha)(1-\cos(\alpha)) \\ &\leq (\dot{x}^T \dot{s})(1-\cos(\alpha))^2 + (\ddot{x}^T \ddot{s})\sin^2(\alpha), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &-(\dot{x}^T \dot{s})\sin^2(\alpha) - (\ddot{x}^T \ddot{s})(1-\cos(\alpha))^2 \\ &\leq (\ddot{x}^T \dot{s} + \dot{x}^T \ddot{s})\sin(\alpha)(1-\cos(\alpha)) \\ &\leq (\dot{x}^T \dot{s})\sin^2(\alpha) + (\ddot{x}^T \ddot{s})(1-\cos(\alpha))^2. \end{aligned} \tag{3.3}$$

Proof. Pre-multiplying $\dot{x}^T$ and $\ddot{x}^T$ to the first rows of (2.7) and (2.8) respectively, we have $\dot{x}^T\dot{s} = \dot{x}^T M\dot{x}$ and $\ddot{x}^T\ddot{s} = \ddot{x}^T M\ddot{x}$. The first two inequalities of (3.1) follow from the fact that M is positive semi-definite. Similarly, we also have $\ddot{x}^T M\dot{x} = \ddot{x}^T\dot{s}$ and $\dot{x}^T M\ddot{x} = \dot{x}^T\ddot{s}$, which mean $\ddot{x}^T\dot{s} = \dot{x}^T\ddot{s} = \dot{x}^T M\ddot{x}$.

Since M is a positive semi-definite matrix, we have

$$\begin{aligned}
&[(1-\cos(\alpha))\dot{x}+\sin(\alpha)\ddot{x}]^T M[(1-\cos(\alpha))\dot{x}+\sin(\alpha)\ddot{x}]\\
&=(\dot{x}^T M\dot{x})(1-\cos(\alpha))^2+2(\dot{x}^T M\ddot{x})\sin(\alpha)(1-\cos(\alpha))+(\ddot{x}^T M\ddot{x})\sin^2(\alpha)\\
&=(\dot{x}^T\dot{s})(1-\cos(\alpha))^2+(\ddot{x}^T\ddot{s})\sin^2(\alpha)+(\ddot{x}^T\dot{s}+\dot{x}^T\ddot{s})\sin(\alpha)(1-\cos(\alpha))\\
&\geq 0.
\end{aligned}$$

From the above result, we can get the first inequality of (3.2). Similarly, we also have

$$\begin{aligned}
&[(1-\cos(\alpha))\dot{x}-\sin(\alpha)\ddot{x}]^T M[(1-\cos(\alpha))\dot{x}-\sin(\alpha)\ddot{x}]\\
&=(\dot{x}^T M\dot{x})(1-\cos(\alpha))^2-2(\dot{x}^T M\ddot{x})\sin(\alpha)(1-\cos(\alpha))+(\ddot{x}^T M\ddot{x})\sin^2(\alpha)\\
&=(\dot{x}^T\dot{s})(1-\cos(\alpha))^2+(\ddot{x}^T\ddot{s})\sin^2(\alpha)-(\ddot{x}^T\dot{s}+\dot{x}^T\ddot{s})\sin(\alpha)(1-\cos(\alpha))\\
&\geq 0,
\end{aligned}$$

which implies the second inequality of (3.2). Substituting $(1-\cos(\alpha))\dot{x}$ and $\sin(\alpha)\ddot{x}$ by $\sin(\alpha)\dot{x}$ and $(1-\cos(\alpha))\ddot{x}$ respectively, following the same way, we obtain (3.3) immediately. □

Remark 3.3. In the current LCP case, (3.1) implies that iterative directions do not have orthogonality, which holds in the LP case. This yields some difficulties in the analysis of the algorithm for LCP. In what follows, we will develop some new results on the iterative directions.

Lemma 3.4. [11] *For* $\alpha \in [0, \frac{\pi}{2}]$,

$$\sin(\alpha) \geq \sin^2(\alpha) = 1-\cos^2(\alpha) \geq 1-\cos(\alpha).$$

To estimate the upper bound for $\mu(\alpha)$, we give the following lemma.

Lemma 3.5. *Let* $(x,s) \in N_\infty^-(\gamma)$, $(\dot{x},\dot{s})$ *and* $(\ddot{x},\ddot{s})$ *be calculated from (2.7) and (2.8). Let* $x(\alpha)$ *and* $s(\alpha)$ *be defined as (2.9) and (2.10) respectively. Then*

$$\begin{aligned}
\mu(\alpha) = \mu[1-(1-\sigma)\sin(\alpha)] &+ \frac{1}{n}[(1-\cos(\alpha))^2(\ddot{x}^T\ddot{s}-\dot{x}^T\dot{s})\\
&-\sin(\alpha)(1-\cos(\alpha))(\dot{x}^T\ddot{s}+\dot{s}^T\ddot{x})]\\
\leq \mu[1-(1-\sigma)\sin(\alpha)] &+ \frac{1}{n}[\ddot{x}^T\ddot{s}(\sin^4(\alpha)+\sin^2(\alpha))].
\end{aligned}$$

Proof. Using (2.9), (2.10), (2.7) and (2.8), we get

$$\begin{aligned}
n\mu(\alpha) &= x(\alpha)^T s(\alpha) \\
&= [x^T - \dot{x}^T \sin(\alpha) + \ddot{x}^T(1-\cos(\alpha))][s - \dot{s}\sin(\alpha) + \ddot{s}(1-\cos(\alpha))] \\
&= x^T s - x^T \dot{s}\sin(\alpha) + x^T \ddot{s}(1-\cos(\alpha)) - \dot{x}^T s \sin(\alpha) + \dot{x}^T \dot{s}\sin^2(\alpha) - \dot{x}^T \ddot{s}\sin(\alpha)(1-\cos(\alpha)) \\
&\quad + \ddot{x}^T s(1-\cos(\alpha)) - \ddot{x}^T \dot{s}\sin(\alpha)(1-\cos(\alpha)) + \ddot{x}^T \ddot{s}(1-\cos(\alpha))^2 \\
&= x^T s - (x^T \dot{s} + s^T \dot{x})\sin(\alpha) + (x^T \ddot{s} + s^T \ddot{x})(1-\cos(\alpha)) \\
&\quad - (\dot{x}^T \ddot{s} + \dot{s}^T \ddot{x})\sin(\alpha)(1-\cos(\alpha)) + \dot{x}^T \dot{s}\sin^2(\alpha) + \ddot{x}^T \ddot{s}(1-\cos(\alpha))^2 \\
&= n\mu[1-(1-\sigma)\sin(\alpha)] + (1-\cos(\alpha))^2(\ddot{x}^T \ddot{s} - \dot{x}^T \dot{s}) - (\dot{x}^T \ddot{s} + \dot{s}^T \ddot{x})\sin(\alpha)(1-\cos(\alpha)) \\
&\le n\mu[1-(1-\sigma)\sin(\alpha)] + (1-\cos(\alpha))^2(\ddot{x}^T \ddot{s} - \dot{x}^T \dot{s}) + (1-\cos(\alpha))^2 \dot{x}^T \dot{s} + \sin^2(\alpha)\ddot{x}^T \ddot{s} \\
&= n\mu[1-(1-\sigma)\sin(\alpha)] + (1-\cos(\alpha))^2 \ddot{x}^T \ddot{s} + \sin^2(\alpha)\ddot{x}^T \ddot{s} \\
&\le n\mu[1-(1-\sigma)\sin(\alpha)] + (\sin^4(\alpha) + \sin^2(\alpha))(\ddot{x}^T \ddot{s}),
\end{aligned}$$

where the first inequality follows from Lemma 3.2, and the second inequality is from Lemma 3.4. This proves the lemma. □

4. The complexity analysis

In the following, we present some technical results which are required for establishing the main results of convergence analysis.

For simplicity in the analysis, we omit the indicator k and set $D = X^{-\frac{1}{2}} S^{\frac{1}{2}}$, where $x, s \in \mathbb{R}^n_+$.

Lemma 4.1. [21] *Let $u, v \in \mathbf{R}^n$. Then*

$$||uv||_1 \le ||Du||\,||D^{-1}v|| \le \frac{1}{2}(||Du||^2 + ||D^{-1}v||^2).$$

In what follows, we give the upper bounds for $||D\dot{x}||$, $||D^{-1}\dot{s}||$, $||D\ddot{x}||$ and $||D^{-1}\ddot{s}||$, which are very important to our subsequent analysis.

Lemma 4.2. *If $(x,s) \in N_\infty^-(\gamma)$, $D = X^{-\frac{1}{2}} S^{\frac{1}{2}}$ and $(\dot{x}, \dot{s})$ is the solution of (2.7). Then*

$$||D\dot{x}||^2 + ||D^{-1}\dot{s}||^2 \le \beta_1 \mu n, \; ||\dot{x}\dot{s}||_1 \le \frac{\beta_1 \mu n}{2},$$

and

$$||D\dot{x}|| \le \sqrt{\beta_1 \mu n}, \; ||D^{-1}\dot{s}|| \le \sqrt{\beta_1 \mu n},$$

where $\beta_1 \ge 1 \ge 1 - 2\sigma + \frac{\sigma^2}{\gamma}$.

Proof. Pre-Multiplying the second equation in (2.7) by $(XS)^{-\frac{1}{2}}$, we get

$$D\dot{x} + D^{-1}\dot{s} = (XS)^{-\frac{1}{2}}(xs - \sigma\mu e).$$

Then, taking squared norm on both sides yields

$$\begin{aligned}||D\dot{x}||^2+2\dot{x}^T\dot{s}+||D^{-1}\dot{s}||^2 &= ||(xs)^{\frac{1}{2}}||^2-2\sigma\mu n+||(XS)^{-\frac{1}{2}}\sigma\mu e||^2\\ &\le x^Ts-2\sigma\mu n+\frac{||\sigma\mu e||^2}{(xs)_{\min}}\\ &\le \mu n-2\sigma\mu n+\frac{\sigma^2\mu n}{\gamma}\\ &\le \beta_1\mu n.\end{aligned}$$

From $\dot{x}^T\dot{s}\ge 0$, we have

$$||D\dot{x}||^2+||D^{-1}\dot{s}||^2\le\beta_1\mu n.$$

Using this result and Lemma 4.1, we obtain

$$||D\dot{x}||\le\sqrt{\beta_1\mu n},\ ||D^{-1}\dot{s}||\le\sqrt{\beta_1\mu n},$$

and

$$||\dot{x}\dot{s}||_1\le\frac{\beta_1\mu n}{2}.$$

This proves this lemma. □

Lemma 4.3. *If $(x,s)\in N_\infty^-(\gamma)$, $D=X^{-\frac{1}{2}}S^{\frac{1}{2}}$ and $(\ddot{x},\ddot{s})$ is the solution of (2.8), then we have*

$$||D\ddot{x}||^2+||D^{-1}\ddot{s}||^2\le\beta_2\mu n^2,\ ||\ddot{x}\ddot{s}||_1\le\frac{\beta_2\mu n^2}{2},$$

and

$$||D\ddot{x}||\le\sqrt{\beta_2\mu}n,\ ||D^{-1}\ddot{s}||\le\sqrt{\beta_2\mu}n,$$

where $\beta_2=\frac{\beta_1^2}{\gamma}\ge 1$.

Proof. Pre-multiplying $(XS)^{-\frac{1}{2}}$ to the second equation in (2.8), taking squared norm on both sides, and using $\ddot{x}^T\ddot{s}\ge 0$, we obtain

$$\begin{aligned}||D\ddot{x}||^2+||D^{-1}\ddot{s}||^2 &\le ||(XS)^{-\frac{1}{2}}(-2\dot{x}\dot{s})||^2\\ &\le\frac{||-2\dot{x}\dot{s}||^2}{(xs)_{\min}}\\ &\le\frac{4||\dot{x}\dot{s}||_1^2}{\gamma\mu}\\ &\le\beta_2\mu n^2,\end{aligned}$$

From the above inequality and Lemma 4.1, we get

$$||D\ddot{x}||\le\sqrt{\beta_2\mu}n,\ ||D^{-1}\ddot{s}||\le\sqrt{\beta_2\mu}n,$$

and

$$||\ddot{x}\ddot{s}||_1\le\frac{\beta_2\mu n^2}{2}.$$

This completes the proof. □

Using Lemma 4.2 and Lemma 4.3, we have the following lemma.

Lemma 4.4. *If $\beta_3=\sqrt{\beta_1\beta_2}\ge 1$, then*

$$||\dot{x}\ddot{s}||_1\le||D\dot{x}||\ ||D^{-1}\ddot{s}||\le\beta_3\mu n^{\frac{3}{2}},\ ||\ddot{x}\dot{s}||_1\le||D\ddot{x}||\ ||D^{-1}\dot{s}||\le\beta_3\mu n^{\frac{3}{2}}.$$

Let $\sin(\hat{\alpha}_0) = \frac{1}{\beta_4 n}$, where $\beta_4 = \frac{\beta_1+\beta_2+4\beta_3}{\sigma(1-\gamma)} \geq 1$. Then we get the next lemmas.

Lemma 4.5. *Let $\chi(\alpha)$ be defined as (2.12). For all $\alpha \in (0, \hat{\alpha}_0]$, we have $\chi(\alpha) \geq -\frac{1}{2}\sin(\alpha)\mu(1-\gamma)\sigma e$.*

Proof. Using Lemma 4.2, Lemma 4.3, Lemma 4.4 and $g(\alpha) \leq \sin^2(\alpha)$, we have

$$\begin{aligned}
\chi(\alpha) &= -g^2(\alpha)\dot{x}\dot{s} - \sin(\alpha)g(\alpha)(\dot{x}\ddot{s} + \dot{s}\ddot{x}) + g^2(\alpha)\ddot{x}\ddot{s} \\
&\geq [-\sin^4(\alpha)||\dot{x}\dot{s}||_1 - \sin^3(\alpha)||\dot{x}\ddot{s} + \dot{s}\ddot{x}||_1 - \sin^4(\alpha)||\ddot{x}\ddot{s}||_1]e \\
&\geq -[\frac{1}{2}\sin^4(\alpha)\beta_1\mu n + 2\sin^3(\alpha)\beta_3\mu n^{\frac{3}{2}} + \frac{1}{2}\sin^4(\alpha)\beta_2\mu n^2]e \\
&\geq -\frac{1}{2}\sin(\alpha)\mu[\sin^3(\hat{\alpha}_0)\beta_1 n + 4\sin^2(\hat{\alpha}_0)\beta_3 n^{\frac{3}{2}} + \sin^3(\hat{\alpha}_0)\beta_2 n^2]e \\
&= -\frac{1}{2}\sin(\alpha)\mu[\frac{\beta_1}{\beta_4^3 n^2} + \frac{4\beta_3}{\beta_4^2 n^{\frac{1}{2}}} + \frac{\beta_2}{\beta_4^3 n}]e \\
&\geq -\frac{1}{2}\sin(\alpha)\mu\frac{1}{\beta_4}[\beta_1 + \beta_2 + 4\beta_3]e \\
&= -\frac{1}{2}\sin(\alpha)\mu(1-\gamma)\sigma e.
\end{aligned}$$

This completes the proof. $\square$

Lemma 4.6. *Let $(x,s) \in N_\infty^-(\gamma)$, $\sin(\hat{\alpha})$ be defined by (2.14). Then, for all $\sin(\alpha) \in [0, \sin(\hat{\alpha}_0)]$, we have $(x(\alpha), s(\alpha)) \in N_\infty^-(\gamma)$ and $\sin(\hat{\alpha}) \geq \sin(\hat{\alpha}_0)$.*

Proof. Using (2.11), (2.13), Lemma 4.5 and 3.5, we have

$$\begin{aligned}
\min(x(\alpha)s(\alpha)) &= \min[xs - \sin(\alpha)(xs - \sigma\mu e) + \chi(\alpha)] \\
&\geq \min(xs)(1-\sin(\alpha)) + \sin(\alpha)\sigma\mu + \min(\chi(\alpha)) \\
&\geq (1-\sin(\alpha))\gamma\mu + \sin(\alpha)\sigma\mu - \frac{1}{2}\sin(\alpha)(1-\gamma)\sigma\mu \\
&= \gamma\mu(\alpha) - \sigma\gamma\mu\sin(\alpha) + \sin(\alpha)\sigma\mu - \frac{1}{2}\sin(\alpha)(1-\gamma)\sigma\mu \\
&\quad - \frac{\gamma}{n}[(1-\cos(\alpha))^2(\ddot{x}^T\ddot{s} - \dot{x}^T\dot{s}) - \sin(\alpha)(1-\cos(\alpha))(\dot{x}^T\ddot{s} + \dot{s}^T\ddot{x})] \\
&= \gamma\mu(\alpha) + \sin(\alpha)(1-\gamma)\sigma\mu - \frac{1}{2}\sin(\alpha)(1-\gamma)\sigma\mu - \frac{\gamma}{n}(1-\cos(\alpha))^2(\ddot{x}^T\ddot{s} - \dot{x}^T\dot{s}) \\
&\quad + \frac{\gamma}{n}\sin(\alpha)(1-\cos(\alpha))(\dot{x}^T\ddot{s} + \dot{s}^T\ddot{x}) \\
&\geq \gamma\mu(\alpha) + \frac{1}{2}\sin(\alpha)(1-\gamma)\sigma\mu - \frac{\gamma}{n}(1-\cos(\alpha))^2(\ddot{x}^T\ddot{s} + \dot{x}^T\dot{s}) \\
&\quad + \frac{\gamma}{n}\sin(\alpha)(1-\cos(\alpha))(\dot{x}^T\ddot{s} + \dot{s}^T\ddot{x}) \\
&\geq \gamma\mu(\alpha) + \frac{1}{2}\sin(\alpha)(1-\gamma)\sigma\mu - \frac{\gamma}{n}\sin^2(\alpha)(\dot{x}^T\dot{s}) - \frac{\gamma}{n}(1-\cos(\alpha))^2(\ddot{x}^T\ddot{s}) \\
&\quad + \frac{\gamma}{n}\sin(\alpha)(1-\cos(\alpha))(\dot{x}^T\ddot{s} + \dot{s}^T\ddot{x}) \\
&\geq \gamma\mu(\alpha).
\end{aligned}$$

where the last three inequalities follow from the facts $\dot{x}^T\dot{s} \geq 0$, Lemma 3.4 and (3.3), respectively, which imply $x(\alpha)s(\alpha) > 0$. Since $x(\alpha)$ and $s(\alpha)$ have continuity in $[0, \hat{\alpha}_0]$ and $(x,s) > 0$, hence $x(\alpha) >$

$0,\ s(\alpha) > 0$. By (2.13) and (2.14), we have $(x(\alpha), s(\alpha)) \in N_\infty^-(\gamma)$. Using (2.14), we obtain $\sin(\hat{\alpha}) \geq \sin(\hat{\alpha}_0)$. The proof is completed. □

Now, we are ready to present the iteration complexity bound of Algorithm 1. To this end, we need to obtain a new upper bound of $\mu(\alpha)$.

From Lemma 3.5 and Lemma 4.3, it follows that for $\alpha \in [0, \hat{\alpha}_0]$. Hence,

$$\begin{aligned}
\mu(\alpha) &\leq \mu[1-(1-\sigma)\sin(\alpha)] + \frac{1}{n}[\ddot{x}^T\ddot{s}(\sin^4(\alpha)+\sin^2(\alpha))] \\
&\leq \mu[1-(1-\sigma)\sin(\alpha)] + \frac{1}{n}[||\ddot{x}\ddot{s}||_1(\sin^4(\alpha)+\sin^2(\alpha))] \\
&\leq \mu[1-(1-\sigma-\frac{\beta_2 n}{2}(\sin^3(\alpha)+\sin(\alpha)))\sin(\alpha)] \\
&\leq \mu[1-(1-\sigma-\frac{\beta_2}{2\beta_4^3 n^2}-\frac{\beta_2}{2\beta_4})\sin(\alpha)] \\
&\leq (1-\frac{1}{4}\sin(\alpha))\mu,
\end{aligned}$$

where the last two inequalities come from $\sin(\hat{\alpha}_0) = \frac{1}{\beta_4 n}$, $\beta_2 < \beta_4$ and $0 < \sigma < \frac{1}{4}$.

Theorem 4.7. *Let $(x^0, s^0) \in N_\infty^-(\gamma)$, $\sin(\hat{\alpha}_0) = \frac{1}{\beta_4 n}$. Then Algorithm 1 terminates in at most $O(n \log \frac{(x^0)^T s^0}{\varepsilon})$ iterations.*

Proof. Using the relation $\sin(\hat{\alpha}) \geq \sin(\hat{\alpha}_0)$, we have

$$\mu(\hat{\alpha}) \leq (1-\frac{1}{4}\sin(\hat{\alpha}))\mu \leq (1-\frac{1}{4}\sin(\hat{\alpha}_0))\mu = (1-\frac{1}{4\beta_4 n})\mu.$$

Then, we obtain

$$(x^k)^T s^k \leq (1-\frac{1}{4\beta_4 n})^k (x^0)^T s^0.$$

Thus the inequality $(x^k)^T s^k \leq \varepsilon$ is satisfied if

$$(1-\frac{1}{4\beta_4 n})^k (x^0)^T s^0 \leq \varepsilon.$$

Taking logarithms, we get

$$k \log(1-\frac{1}{4\beta_4 n}) \leq \log \frac{\varepsilon}{(x^0)^T s^0}.$$

Note that

$$\log(1-\frac{1}{4\beta_4 n}) \leq -\frac{1}{4\beta_4 n} \leq 0,$$

Thus, for $k \geq 4\beta_4 n \log \frac{(x^0)^T s^0}{\varepsilon}$, we have

$$(x^k)^T s^k \leq \varepsilon.$$

This completes the proof. □

5. Numerical results

In this section, we represent some numerical results for our new Algorithm 1 and the large-update algorithm based on trigonometric kernel function ($\psi(t) = \frac{t^2-1}{2} + \frac{6}{\pi} tan(h(t))$,with $h(t) = \frac{\pi(1-t)}{4t+2}$) and classical kernel function ($\psi(t) = \frac{t^2-1}{2} - \log t$) in [22] for solving the same problems. Moreover, the results show that our algorithm is more effective. We list the number of iteration(iter), the duality gap(gap) and the total CPU time(time) when the algorithms terminate. Considering the following test problems

5.1 Text problems

Problem 1

$$M = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}, q = \begin{pmatrix} 1/11 \\ -4 \\ -3/11 \end{pmatrix}, x^0 = \begin{pmatrix} 2.5 \\ 2.5 \\ 1 \end{pmatrix}.$$

Problem 2 [23]

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{pmatrix}, q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Problem 3 [24]

$$M = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \cdots & 0 \\ 0 & -1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix}, q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

5.2 The number of iterations for our algorithm

In the following, we select the different optimization parameters γ, σ and accuracy parameter $\varepsilon = 10^{-6}$ for the Algorithm 1. The number of iterations for test problems are stated in Tables 1-3.

TABLE 1. The number of iterations for Problem 1

σ	γ	Gap	Iter	Time
1/6	1/12	2.7762478094e−007	9	1.2709
1/8	1/12	1.6573236555e−007	8	0.9082
1/8	1/15	1.6573236555e−007	8	0.9625
1/10	1/20	2.7402251318e−007	7	1.0116

Taking different dimensions and parameters σ, γ, the results are displayed in Tables 1-3. We find that σ has more obvious effects on iteration numbers than γ. For Table 1 and Table 3, the smaller σ is, the better iteration numbers will be. For Table 2, the larger σ is, the better iteration numbers will be.

TABLE 2. The number of iterations for Problem 2

σ	γ	n=10			n=15		
		Iter	Gap	Time	Iter	Gap	Time
1/6	1/12	19	5.0814e-007	1.4537	23	9.3461e-007	1.5209
1/10	1/20	19	5.6208e-007	1.4540	24	4.1131e-007	1.5959
1/15	1/20	20	4.4923e-007	1.6667	26	1.4651e-007	1.7488
1/15	1/30	20	7.4427e-007	1.5874	26	1.7061e-007	1.7962

n=20			n=25			n=30		
Iter	Gap	Time	Iter	Gap	Time	Iter	Gap	Time
27	6.3221e-007	1.6540	31	3.7743e-007	1.8908	33	3.0092e-007	2.0345
28	5.5822e-007	1.6358	32	2.6551e-007	1.8290	35	5.5948e-007	1.8678
31	7.1983e-008	1.8066	35	1.2807e-007	2.0960	39	1.2938e-007	2.2932
30	8.7233e-007	1.9950	35	6.7943e-007	2.1196	38	3.4859e-007	2.4455

TABLE 3. The number of iterations for Problem 3

σ	γ	n=10			n=50			n=100		
		Iter	Gap	Time	Iter	Gap	Time	Iter	Gap	Time
1/6	1/12	11	2.7231e-007	1.4038	13	7.9530e-007	1.5689	15	3.8228e-007	1.7572
1/10	1/20	9	4.3594e-007	1.3646	12	1.1078e-007	1.5790	13	4.9342e-007	1.7395
1/10	1/15	9	4.6934e-007	1.3444	12	1.5584e-007	1.5414	13	5.2948e-007	1.7867
1/15	1/30	9	6.8965e-008	1.3522	11	2.0232e-007	1.4850	12	7.3245e-007	1.7213

n=200			n=500			n=1000		
Iter	Gap	Time	Iter	Gap	Time	Iter	Gap	Time
17	3.6773e-007	2.3122	20	6.8017e-007	5.8220	24	1.7479e-007	21.8176
15	3.3748e-007	1.6358	18	9.1434e-007	5.5928	22	1.5103e-007	20.2268
15	5.0650e-007	2.1649	19	1.0488e-007	5.5357	22	1.7784e-007	19.0403
14	5.9702e-007	2.1715	18	7.7209e-007	5.2088	21	1.2359e-007	19.0295

5.3 The number of iterations for large-update algorithm based on kernel functions in [22]

We take different barrier update parameters θ, threshold parameter $\tau = 2.5$, and accuracy $\varepsilon = 10^{-4}$. The iteration numbers for test problems based on the trigonometric kernel function and classical kernel function are presented in Tables 4-9, respectively.

Comparing the results in Tables 4-9, it suggests that, with higher accuracy, our algorithm has less iteration number. It shows that our algorithm is more efficient in practice and the iteration number is insensitive to the size of problem. We can intuitively see that searching along an ellipse that approximates

TABLE 4. The number of iterations for Problem 1 based on trigonometric kernel function

θ	Gap	Iter
0.3	1.8881905766e−004	46
0.5	1.6761773232e−004	31
0.7	2.4794178804e−004	23

TABLE 5. The number of iterations for Problem 1 based on classical kernel function

θ	Gap	Iter
0.3	1.7165152850e−004	43
0.5	1.6552373682e−004	28
0.7	3.3909102763e−004	20

TABLE 6. The number of iterations for Problem 2 based on trigonometric kernel function

θ	n=10	n=15	n=20	n=25	n=30
0.3	73	83	89	98	105
0.5	49	54	57	60	63
0.7	39	42	46	48	51

TABLE 7. The number of iterations for Problem 2 based on classical kernel function

θ	n=10	n=15	n=20	n=25	n=30
0.3	76	86	95	105	112
0.5	51	56	60	65	68
0.7	41	45	48	49	53

TABLE 8. The number of iterations for Problem 3 based on trigonometric kernel function

θ	n=10	n=50	n=100	n=200	n=500	n=1000
0.3	63	83	98	122	132	176
0.5	42	57	69	84	92	120
0.7	34	44	54	65	66	84

TABLE 9. The number of iterations for Problem 3 based on classical kernel function

θ	n=10	n=50	n=100	n=200	n=500	n=1000
0.3	49	73	78	82	91	119
0.5	33	39	47	58	69	72
0.7	23	33	36	39	42	56

the central path is promising and attractive. Furthermore, the results for our algorithm show very slow growth as n increases, which is precisely what is hoped for interior-point algorithms.

6. CONCLUDING REMARKS

This paper generalized a wide neighborhood IPM with the arc-search from the LP to the LCP, which searches the optimizers along the ellipses that approximate the central path. Moreover, the iteration complexity bound of the proposed algorithm is devised, namely $O(n\log\frac{(x^0)^T s^0}{\varepsilon})$. The numerical results show that the new proposed algorithm is well promising in practice in comparison with some other IPMs. We expect that the same idea will be proved to be efficient for the $P_*(\kappa)$ LCP.

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