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POSITIVE SOLUTIONS FOR NONLINEAR ϕ-LAPLACIAN THIRD-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS ON INFINITE INTERVALS

ZAKIA BENBAZIZ[1,2], SMAÏL DJEBALI[1,3,*]

[1]Laboratory of Fixed Point Theory and Applications, ENS, Algiers, Algeria
[2]Faculty of Computer Sciences, USTHB, Algiers, Algeria
[3]Department of Mathematics, Faculty of Sciences, Al Imam Mohammad Ibn Saud Islamic University (IMSIU), PB 90950, Riyadh 11623, Saudi Arabia

Abstract. This paper deals with the existence of positive solutions for a ϕ-Laplacian third-order impulsive fully nonlinear boundary value problem posed on the half-line. Our existence result is based on a fixed point theorem and a compactness argument. An example is included to illustrate the existence result.

Keywords. Boundary value problem; Positive solution; Fixed point theorem; Laplacian operator; Impulsive differential equation.

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1. Introduction

Real boundary value problems (bvps for short) on the half line arise in many physical applications such as the modeling of gas flow through semi-infinite porous media, electrical potential in isolated atom of some materials, propagation of laminar flames in long tubes; see, e.g., [1] and references therein. Due to some physical laws such that Newton's laws, a large class of such problems are governed by second-order differential equations. This justifies the large amount of research work for second-order bvps that are available in the very recent literature; see, e.g., [2, 3, 4, 5, 6] and references therein. For basic mathematical methods to deal with such problems, we refer the reader to [1, 7].

In the last couple of years, some authors investigated the existence of positive solutions for higher order differential equations on finite intervals as well as on infinite intervals of the real line; see, e.g., [8, 9, 10, 11] and the references therein. Notice that positive solutions may refer physically to a position, a temperature, density,... Another interesting direction of the mathematical extension is the natural generalization to problems associated to the p-Laplacian operator $s|s|^{p-2}$ $(p > 1)$ and more generally

[*]Corresponding author.
E-mail addresses: zbenbaziz@usthb.dz (Z. Benbaziz), djebali@hotmail.com (S. Djebali).
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to the $s \mapsto \phi$-Laplacian operator; see [3, 4, 5, 6, 8, 12, 13, 14] and the references. We also mention that many dynamical processes in physics, population dynamics, mechanics, and natural sciences may change state abruptly or be subject to short-term perturbations. Physically, these perturbations may be seen as impulses. Milman and Myshkis [16] investigated basic model of problems involving differential equations with impulses; see [15, 16].

This paper is concerned with the following boundary value problem posed on $J = (0, +\infty)$:

$$\begin{cases} (\phi(u''(t)))' &= -q(t)f(t, u(t), u'(t), u''(t)), \quad t \in J', \\ u(0) &= A, \\ u'(0) &= B, \\ u''(+\infty) &= C, \\ \triangle u(t_k) &= I_{1k}(u(t_k), u'(t_k)), \qquad k = 1, 2, 3, \dots \\ \triangle u'(t_k) &= I_{2k}(u(t_k), u'(t_k), u''(t_k)), \qquad k = 1, 2, 3, \dots \\ -\triangle \phi(u''(t_k)) &= I_{3k}(u(t_k), u'(t_k), u''(t_k)), \qquad k = 1, 2, 3, \dots, \end{cases} \tag{1.1}$$

where $A \geq 0, B \geq 0, C \geq 0$, $0 = t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to +\infty$, as $k \to +\infty$, $\triangle u^{(i)}(t_k) = u^{(i)}(t_k^+) - u^{(i)}(t_k^-)$, for $i = 0, 1, 2$, and $J' = [0, +\infty) \setminus \{t_k,\ k = 1, 2, 3, \dots\}$. We set $J_k = (t_k, t_{k+1}]$, $k = 1, 2, 3, \dots$.

The nonlinearity $f : (0, \infty) \times [0, \infty) \times [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is continuous, ϕ is an increasing homeomorphism satisfying $\phi(0) = 0$ and $\phi^{-1}(uv) \geq \phi^{-1}(u)\phi^{-1}(v), \forall u, v \in \mathbb{R}$. $q \in L^1((0, \infty), [0, \infty))$, $I_{1k} \in C(\mathbb{R}^2, [0, \infty))$, and $I_{ik} \in C(\mathbb{R}^3, [0, \infty))$, for $i = 2, 3$ and $k = 1, 2, 3, \dots$

We suppose that the following conditions hold, for $k = 1, 2, 3, \dots$

$$(\mathscr{H}_0) \quad \begin{cases} f(t, (1+t^2)u, (1+t)v, w) \text{ is bounded when } u, v, w \text{ are bounded}, \\ I_{1k}(u, v) \leq a_{1k}u + b_{1k}v + c_{1k}, \ \forall u, v \geq 0, \\ I_{2k}(u, v, w) \leq a_{2k}u + b_{2k}v + c_{2k}w + d_{2k}, \ \forall u, v, w \geq 0, \\ I_{3k}(u, v, w) \leq a_{3k}u + b_{3k}v + c_{3k}w + d_{3k}, \ \forall u, v, w \geq 0, \\ a_i = \sum_{k=1}^{\infty} a_{ik}(1 + t_k^2) < \infty \quad (i = 1, 2, 3), \\ b_i = \sum_{k=1}^{\infty} b_{ik}(1 + t_k) < \infty \quad (i = 1, 2, 3), \\ c_i = \sum_{k=1}^{\infty} c_{ik} < \infty \quad (i = 1, 2, 3), \\ d_i = \sum_{k=1}^{\infty} d_{3k} < \infty \quad (i = 2, 3). \end{cases} \tag{1.2}$$

For $R > 0$, we set

$$M_R = \sup\{f(t, (1+t^2)u, (1+t)v, w) : \ t \in J, \ u, v, w \in [0, R]\}. \tag{1.3}$$

Consider the functional spaces:

$$PC[J, \mathbb{R}] = \left\{ \begin{array}{l} u : J \longrightarrow \mathbb{R} \text{ continuous for } t \neq t_k, \\ u(t_k) = u(t_k^-), \text{ and } u(t_k^+) \text{ exists for } k = 1, 2, \dots \end{array} \right\},$$

$$PC^1[J, \mathbb{R}] = \left\{ \begin{array}{l} u \in PC[J, \mathbb{R}] : u'(t) \text{ exists for } t \neq t_k, \\ u'(t_k) = u'(t_k^-), \text{ and } u'(t_k^+) \text{ exists for } k = 1, 2, \dots \end{array} \right\},$$

and

$$PC^2[J, \mathbb{R}] = \left\{ \begin{array}{l} u \in PC^1[J, \mathbb{R}] : u''(t) \text{ exists for } t \neq t_k, \\ u''(t_k) = u''(t_k^-), \text{ and } u''(t_k^+) \text{ exists for } k = 1, 2, \dots \end{array} \right\}.$$

Here $u^i(t_k^\pm) = \lim_{t\to t_k^\pm} u^i(t)$ $(i = 0,1,2)$. However, the adequate solution space for problem (1.1) is:

$$E = \left\{ u \in PC^2[J,\mathbb{R}] : \lim_{t\to+\infty} u''(t)\, exists \right\}.$$

The following lemma is easy to prove.

Lemma 1.1. *For $u \in E$, $\lim\limits_{t\to+\infty} \frac{u'(t)}{1+t} = \lim\limits_{t\to+\infty} u''(t)$ and $\lim\limits_{t\to+\infty} \frac{u(t)}{1+t^2} = \frac{1}{2} \lim\limits_{t\to+\infty} u''(t)$.*

Thus E is a Banach space with the norm

$$\|u\| = \max\{\|u\|_0,\ \|u\|_1,\ \|u\|_2\},$$

where $\|u\|_0 = \sup\limits_{t\in\mathbb{R}^+} \frac{|u(t)|}{1+t^2}$, $\|u\|_1 = \sup\limits_{t\in\mathbb{R}^+} \frac{|u'(t)|}{1+t}$, and $\|u\|_2 = \sup\limits_{t\in\mathbb{R}^+} |u''(t)|$. Let $P \subset E$ be the positive cone defined by

$$P = \{u \in E : u(t) \geq 0 \text{ and } u'(t) \geq 0,\ t \geq 0\}. \tag{1.4}$$

The basic tool to be used in this work is a fixed point theorem in cones.

Lemma 1.2. [17] *Let X be a Banach space and $P \subset X$ a cone. Assume that Ω_1 and Ω_2 are open subsets of X with $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, where θ is the zero element in X. Let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow P$ be a completely continuous operator such that*

(a) $\|Tu\| \leq \|u\|,\ \forall\, u \in P \cap \partial\Omega_2,$
(b) *there exists a $\Psi \in P$ such that $u \neq Tu + \lambda\Psi,\ \forall\, u \in P \cap \partial\Omega_1, \forall\, \lambda > 0.$*

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

To show the compactness of an operator, we need the following compactness criterion on unbounded intervals of the real line which can be easily derived from Corduneanu's Compactness Criterion [18]:

Lemma 1.3. *Let W be a bounded subset of E. Then W is relatively compact if the following two conditions hold:*
(a) the sets $\{\frac{v(t)}{1+t^2}, v \in W\}$, $\{\frac{v'(t)}{1+t}, v \in W\}$, and $\{v''(t), v \in W\}$ are equicontinuous on any finite subinterval $J_k \cap [0,T]$, $(k = 1,2,\dots)$ for any $T > 0$, i.e., $\forall k = 1,2,\dots$, $\forall \varepsilon > 0, \exists \delta > 0, \forall t_1,t_2 \in J_k \cap [0,T]$,

$$|t_1 - t_2| < \delta \Rightarrow \left|\frac{u(t_1)}{1+t_1^2} - \frac{u(t_2)}{1+t_2^2}\right| < \varepsilon,\ \left|\frac{u'(t_1)}{1+t_1} - \frac{u'(t_2)}{1+t_2}\right| < \varepsilon,$$

and $\left|u''(t_1) - u''(t_2)\right| < \varepsilon$, $\forall u \in W$,
(b) $\forall\, \varepsilon > 0, \exists T = T(\varepsilon) > 0$ such that, $\forall t \geq T$,

$$\left|\frac{u(t)}{1+t^2} - \lim_{t\to+\infty} \frac{u(t)}{1+t^2}\right| < \varepsilon, \left|\frac{u'(t)}{1+t} - \lim_{t\to+\infty} \frac{u'(t)}{1+t}\right| < \varepsilon,$$

and $\left|u''(t) - \lim\limits_{t\to+\infty} u''(t)\right| < \varepsilon$, $\forall u \in W$.

Definition 1.4. A function $u \in E$ such that $\phi(u'') \in PC^1(J',\mathbb{R})$ is called a positive solution if $u \in P$ and u satisfies problem (1.1).

2. Fixed point formulations

Lemma 2.1. *Let $v \in L^1([0,+\infty),(0,+\infty))$. Then the following problem*

$$\begin{array}{rcll} (\phi(u''(t)))'+v(t) & = & 0, \quad t>0, & t\neq t_k, \\ u(0) & = & A, & \\ u'(0) & = & B, & \\ u''(+\infty) & = & C, & \\ \triangle u(t_k) & = & I_{1k}(u(t_k),u'(t_k)), & k=1,2,3,\ldots \\ \triangle u'(t_k) & = & I_{2k}(u(t_k),u'(t_k),u''(t_k)), & k=1,2,3,\ldots \\ -\triangle\phi(u''(t_k)) & = & I_{3k}(u(t_k),u'(t_k),u''(t_k)), & k=1,2,3,\ldots \end{array} \tag{2.1}$$

has a unique solution:

$$\begin{aligned} u(t) &= A+\sum_{t_k<t} I_{1k}(u(t_k),u'(t_k))+\left(B+\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)t \\ &+\int_0^t (t-s)\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds. \end{aligned}$$

Proof. For $t\in(0,t_1]$, integrating (2.1) from 0 to t yields that

$$\phi(u''(t_1^-))=\phi(u''(0))-\int_0^{t_1} v(s)ds \tag{2.2}$$

and, for $t\in[t_1,t_2]$,

$$\phi(u''(t))=\phi(u''(t_1^+))-\int_{t_1}^t v(s)ds. \tag{2.3}$$

Adding (2.2) and (2.3) gives

$$\phi(u''(t_1^-))+\phi(u''(t))=\phi(u''(0))+\phi(u''(t_1^+))-\int_0^t v(s)ds.$$

Hence

$$\begin{aligned} \phi(u''(t)) &= \phi(u''(t_1^+))-\phi(u''(t_1^-))+\phi(u''(0))-\int_0^t v(s)ds \\ &= -I_{31}(u(t_1),u'(t_1),u''(t_1))+\phi(u''(0))-\int_0^t v(s)ds. \end{aligned}$$

Repeating this process for every positive t, we find

$$\phi(u''(t))=\phi(u''(0))-\int_0^t v(s)ds-\sum_{t_k<t} I_{3k}(u(t_k),u'(t_k),u''(t_k)).$$

Passing to the limit, as $t\to+\infty$, we obtain

$$\phi(C)=\phi(u''(0))-\sum_{k=1}^{\infty} I_{3k}(u(t_k),u'(t_k),u''(t_k))-\int_0^\infty v(s)ds.$$

It follows that

$$\phi(u''(0))=\phi(C)+\int_0^\infty v(s)ds+\sum_{k=1}^{\infty} I_{3k}(u(t_k),u'(t_k),u''(t_k)).$$

Hence

$$\begin{aligned} \phi(u''(t)) &= \phi(C)+\int_t^\infty v(s)ds-\sum_{t_k<t} I_{3k}(u(t_k),u'(t_k),u''(t_k)) \\ &\quad+\sum_{k=1}^{\infty} I_{3k}(u(t_k),u'(t_k),u''(t_k)). \end{aligned}$$

Equivalently, one has

$$\phi(u''(t))=\phi(C)+\int_t^\infty v(s)ds+\sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k)).$$

This implies that

$$u''(t)=\phi^{-1}\left(\phi(C)+\int_t^\infty v(s)ds+\sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right). \tag{2.4}$$

Integrating (2.4) gives

$$u'(t_1^-)=B+\int_0^{t_1}\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds$$

and

$$u'(t)=u'(t_1^+)+\int_{t_1}^{t}\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds.$$

Hence

$$\begin{aligned} u'(t) &= u'(t_1^-)+I_{21}(u(t_1),u'(t_1),u''(t_1)) \\ &\quad +\int_{t_1}^t\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds. \end{aligned}$$

For all positive t, one has

$$\begin{aligned} u'(t) &= B+I_{21}(u(t_1),u'(t_1),u''(t_1)) \\ &\quad +\int_0^t\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds \\ &= B+\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k)) \\ &\quad +\int_0^t\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds. \end{aligned} \tag{2.5}$$

Integrating (2.5), we get successively

$$\begin{aligned} u(t_1^-) &= A+(B+\sum_{t_k<t_1} I_{2k}(u(t_k),u'(t_k),u''(t_k)))t_1 \\ &\quad +\int_0^{t_1}\int_0^r\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)dsdr \end{aligned}$$

and

$$\begin{aligned} u(t) &= u(t_1^+)+\left(B+\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)(t-t_1) \\ &\quad +\int_{t_1}^t\int_0^r\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)dsdr. \end{aligned}$$

Since $u(t_1^+)-u(t_1^-)=I_{11}(u(t_1),u'(t_1))$, one has

$$\begin{aligned} u(t) &= A+\left(B+\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)t \\ &\quad +\int_0^t\int_0^r\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)dsdr \\ &\quad +I_{11}(u(t_1),u'(t_1)). \end{aligned}$$

By repeating the same process as above, we arrive at

$$\begin{aligned} u(t) &= A+\sum_{t_k<t} I_{1k}(u(t_k),u'(t_k))+\left(B+\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)t \\ &\quad +\int_0^t(t-s)\phi^{-1}\left(\phi(C)+\int_s^\infty v(\tau)d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds. \end{aligned}$$

□

3. EXISTENCE RESULTS

Define the operator $T : P \to E$ by

$$
\begin{aligned}
Tu(t) \;=\; & A+\sum_{t_k<t} I_{1k}(u(t_k),u'(t_k)) + \left(B+\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)t \\
& + \int_0^t (t-s)\phi^{-1}\Big(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau \\
& + \sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\Big)\, ds
\end{aligned}
\tag{3.1}
$$

and let $D \subset E$ be a bounded set. We start with a technical lemma and an important compactness result.

Lemma 3.1. *Suppose that Hypothesis ($\mathscr{H}_0$) holds. Then, for all $u \in P\cap D$, the integrals*

$$
\int_t^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau,\ \sum_{t_k<t} I_{1k}(u(t_k),u'(t_k)),
$$
$$
\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k)),\ \text{and}\ \sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))
$$

converge.

Proof. For $u \in P\cap D$, there exists $r_0 > 0$ such that $\|u\| \le r_0$. By ($\mathscr{H}_0$), we have the estimates:

$$
\begin{aligned}
\sum_{t_k<t} I_{1k}(u(t_k),u'(t_k)) &\le \sum_{k=1}^{+\infty} I_{1k}(u(t_k),u'(t_k)) \\
&\le \sum_{k=1}^{+\infty} (a_{1k}u(t_k)+b_{1k}u'(t_k)+c_{1k}) \\
&\le \sum_{k=1}^{+\infty} \big(a_{1k}(1+t_k^2)+b_{1k}(1+t_k)\big)\|u\|+c_{1k} \\
&\le \sum_{k=1}^{+\infty} \big(a_{1k}(1+t_k^2)+b_{1k}(1+t_k)\big)r_0+c_{1k} \\
&< \infty.
\end{aligned}
\tag{3.2}
$$

By the same way, we can prove that

$$
\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k)) < \infty \ \text{and}\ \sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k)) < \infty.
$$

Also

$$
\begin{aligned}
& \int_t^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau \\
=\;& \int_t^{+\infty} q(\tau)f\Big(\tau,(1+\tau^2)\tfrac{u(\tau)}{1+\tau^2},(1+\tau)\tfrac{u'(\tau)}{1+\tau},u''(\tau)\Big)d\tau \\
\le\;& M_{r_0}\int_0^{+\infty} q(\tau)d\tau \\
<\;& \infty,
\end{aligned}
\tag{3.3}
$$

where M_{r_0} is defined by (1.3). □

Lemma 3.2. *Operator T is completely continuous.*

Proof.

Claim 1. *$T : P\cap D \to P$ is well defined.* Indeed, for $u \in P\cap D$,

$$\begin{aligned}(Tu)''(t) &= \phi^{-1}\left(\phi(C)+\int_t^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right.\\ &\quad\left.+\sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\\ &\leq \phi^{-1}\left(\phi(C)+M_{r_0}\int_t^{+\infty} q(\tau)d\tau+\sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\\ &\leq \phi^{-1}\left(\phi(C)+M_{r_0}\int_t^{+\infty} q(\tau)d\tau\right.\\ &\quad\left.+\sum_{t_k>t}((a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k})r_0+d_{3k})\right),\end{aligned}$$

which converges to C, as $t\to+\infty$. One can easily verify that $(Tu)(t)\geq 0$ and

$$\begin{aligned}(Tu)'(t) &= \left(B+\sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)\\ &\quad+\int_0^t \phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right.\\ &\quad\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds\geq 0.\end{aligned}$$

This shows that $T(P\cap D)\subset P$.

Claim 2. *T is continuous.* Let (u_n) be a sequence in $P\cap D$ and $u\in P\cap D$ be such that $\|u_n-u\|_E\to 0$, as $n\to+\infty$. Then there exists $\bar{r}>0$ such that $\|u_n\|<\bar{r}$. We have

$$\begin{aligned}&|\phi((Tu_n)'')(t)-\phi((Tu)'')(t)|\\ =&\left|\int_t^{+\infty} q(s)f(s,u_n(s),u_n'(s),u_n''(s))ds+\sum_{t_k>t} I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))\right.\\ &\left.-\int_t^{+\infty} q(s)f(s,u(s),u'(s),u''(s))ds-\sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right|\\ \leq&\int_t^{+\infty} q(s)\left|f(s,u_n(s),u_n'(s),u_n''(s))-f(s,u(s),u'(s),u''(s))\right|ds\\ &+\sum_{t_k>t}\left|I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))-I_{3k}(u(t_k),u'(t_k),u''(t_k))\right|\\ \leq&\int_0^{+\infty} q(s)\left|f(s,u_n(s),u_n'(s),u_n''(s))-f(s,u(s),u'(s),u''(s))\right|ds\\ &+\sum_{k=0}^{\infty}\left|I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))-I_{3k}(u(t_k),u'(t_k),u''(t_k))\right|.\end{aligned}$$

Since

$$\left|f(t,u_n(t),u_n'(t),u_n''(t))-f(t,u(t),u'(t),u''(t))\right|\leq 2M_{\bar{r}}$$

one has

$$\begin{aligned}&\int_0^{+\infty} q(s)\left|f(s,u_n(s),u_n'(s),u_n''(s))-f(s,u(s),u'(s),u''(s))\right|ds\\ \leq&\ 2M_{\bar{r}}\int_0^{+\infty} q(\tau)d\tau<\infty,\end{aligned}$$

Lebesgue's dominated convergence theorem guarantees that

$$\int_0^{+\infty} q(s)\left|f(s,u_n(s),u_n'(s),u_n''(s))-f(s,u(s),u'(s),u''(s))\right|ds$$

converges to 0, as $n\to+\infty$. Moreover, since

$$\begin{aligned}&\left|I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))-I_{3k}(u(t_k),u'(t_k),u''(t_k))\right|\\ \leq&\ 2\bar{r}(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k})+2d_{3k}=m(t_k)\end{aligned}$$

and

$$\sum_{k=0}^{+\infty} m(t_k)<\infty,$$

the dominated convergence theorem for series implies that

$$\left[\sum_{k=0}^{+\infty} I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k)) - I_{3k}(u(t_k),u'(t_k),u''(t_k))\right]$$

converges to 0, as $n\to+\infty$. Also

$$\begin{aligned}
&\left|\frac{(Tu_n)'(t)}{1+t}-\frac{(Tu)'(t)}{1+t}\right|\\
= &\left|\frac{1}{1+t}\left(\sum_{t_k<t} I_{2k}(u_n(t_k),u_n'(t_k),u_n''(t_k)) - \sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)\right.\\
&+\frac{1}{1+t}\left(\int_0^t \phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u_n(\tau),u_n'(\tau),u_n''(\tau))d\tau\right.\right.\\
&+\left.\sum_{t_k>s} I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))\right)\\
&-\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right.\\
&\left.\left.\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\right) ds\right)\Bigg|\\
\leq &\frac{1}{1+t}\left|\left(\sum_{t_k<t} I_{2k}(u_n(t_k),u_n'(t_k),u_n''(t_k)) - \sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)\right|\\
&+\frac{1}{1+t}\int_0^t \left|\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u_n(\tau),u_n'(\tau),u_n''(\tau))d\tau\right.\right.\\
&+\left.\sum_{t_k>s} I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))\right)\\
&-\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right.\\
&\left.\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\right| ds.
\end{aligned}$$

Since ϕ^{-1} is continuous, we find that

$$\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u_n(\tau),u_n'(\tau),u_n''(\tau))d\tau+\sum_{t_k>s} I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))\right)$$

converges to

$$\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right),$$

as $n\to+\infty$. Moreover

$$\begin{aligned}
&\left|\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u_n(\tau),u_n'(\tau),u_n''(\tau))d\tau\right.\right.\\
&+\left.\sum_{t_k>s} I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))\right)\\
&-\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right.\\
&\left.\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\right|\\
\leq\ & 2\phi^{-1}\left(\phi(C)+M_{r_0}\int_0^{+\infty} q(\tau)d\tau+(a_3+b_3+c_3)r_0+d_3\right)
\end{aligned}$$

for

$$\int_0^t 2\phi^{-1}\left(\phi(C)+M_{r_0}\int_0^{+\infty} q(\tau)d\tau+(a_3+b_3+c_3)r_0+d_3\right)ds<+\infty.$$

Again Lebesgue's dominated convergence theorem implies that

$$\begin{array}{l}\int_0^t \Big|\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u_n(\tau),u_n'(\tau),u_n''(\tau))d\tau\right.\\ \left.+\sum\limits_{t_k>s} I_{3k}(u_n(t_k),u_n'(t_k),u_n''(t_k))\right)\\ -\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right.\\ \left.+\sum\limits_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\Big| ds\end{array}$$

converges to 0, as $n \to +\infty$. Hence $\|Tu_n - Tu\|_1 \to 0$. By the same way, we can prove that $\|Tu_n - Tu\|_0 \to 0$. Hence $\|Tu_n - Tu\|_E \to 0$, showing that T is continuous on $P \cap D$.

Claim 3. *$T(D)$ is uniformly bounded*. For $u \in P \cap D$, arguing as in (3.3), we have

$$\begin{array}{rcl}\frac{(Tu)(t)}{1+t^2} & = & \frac{A}{1+t^2}+\frac{1}{1+t^2}\sum\limits_{t_k<t} I_{1k}(u(t_k),u'(t_k))\\ & & +\frac{1}{1+t^2}\left(B+\sum\limits_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)t\\ & & +\frac{1}{1+t^2}\left(\int_0^t (t-s)\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right.\right.\\ & & \left.\left.+\sum\limits_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds\right)\\ & \leq & A+(a_1+b_1)r_0+c_1+(B+(a_2+b_2+c_2)r_0+d_2)\\ & & +\phi^{-1}\left(\phi(C)+M_{r_0}\int_0^{+\infty} q(\tau)d\tau+(a_3+b_3+c_3)r_0+d_3\right).\end{array}$$

Then $\sup\limits_{t\in\mathbb{R}^+}\left|\frac{(Tu)(t)}{1+t^2}\right| < +\infty$. Likewise $\sup\limits_{t\in\mathbb{R}^+}\left|\frac{(Tu)'(t)}{1+t}\right| < +\infty$. Indeed

$$\begin{array}{rl}\frac{(Tu)'(t)}{1+t} = & \frac{1}{1+t}\left(B+\sum\limits_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)\\ & +\frac{1}{1+t}\left(\int_0^t \phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right.\right.\\ & \left.\left.+\sum\limits_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds\right)\\ \leq & (B+(a_2+b_2+c_2)r_0+d_2)\\ & +\phi^{-1}\left(\phi(C)+M_{r_0}\int_0^{+\infty} q(\tau)d\tau+(a_3+b_3+c_3)r_0+d_3\right).\end{array}$$

Also $\sup\limits_{t\in\mathbb{R}^+}|(Tu)''(t)| < +\infty$ for

$$\begin{array}{rcl}|(Tu)''(t)| & \leq & \phi^{-1}\left(\phi(C)+M_{r_0}\int_t^{+\infty} q(\tau)d\tau\right.\\ & & \left.+\sum\limits_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\\ & \leq & \phi^{-1}\left(\phi(C)+M_{r_0}\int_0^{+\infty} q(\tau)d\tau\right.\\ & & \left.+\sum\limits_{k=0}^{+\infty}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right)r_0+d_{3k}\right).\\ & < & \infty.\end{array} \tag{3.4}$$

Claim 4. *T is equicontinuous*. For some positive $T_1 \in (0,+\infty)$, let $t_1, t_2 \in J_k \cap [0,T_1]$ be such that $t_1 < t_2$. The following estimates hold:

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^2}-\frac{Tu(t_1)}{1+t_1^2}\right| \\
\leq\ & \left|A\left(\frac{1}{1+t_2^2}-\frac{1}{1+t_1^2}\right)\right| + \left|\frac{1}{1+t_2^2}-\frac{1}{1+t_1^2}\right| \sum_{t_1<t_k} I_{1k}(u(t_k),u'(t_k)) \\
&+\left(B+\sum_{t_1<t_k} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)\left|\frac{t_2}{1+t_2^2}-\frac{t_1}{1+t_1^2}\right| \\
&+\int_0^{t_1}\left|\frac{t_2-s}{1+t_2^2}-\frac{t_1-s}{1+t_1^2}\right| \phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right. \\
&\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right) ds \\
&+\int_{t_1}^{t_2}\left|\frac{t_2-s}{1+t_2^2}\right| \phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right. \\
&\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right) ds \\
\leq\ & \left|A\left(\frac{1}{1+t_2^2}-\frac{1}{1+t_1^2}\right)\right| + \left|\frac{1}{1+t_2^2}-\frac{1}{1+t_1^2}\right| \sum_{k=1}^{+\infty}\left(a_{1k}(1+t_k^2)+b_{1k}(1+t_k)\right) r_0 + \\
& c_{1k} + \left|\frac{t_2}{1+t_2^2}-\frac{t_1}{1+t_1^2}\right| \left(B+\sum_{k=1}^{+\infty}\left(a_{2k}(1+t_k^2)+b_{2k}(1+t_k)+c_{2k}\right) r_0 + d_{2k}\right) \\
&+\int_0^{t_1}\left|\frac{t_2-s}{1+t_2^2}-\frac{t_1-s}{1+t_1^2}\right| \phi^{-1}\left(\phi(C)+M_{r_0}\int_0^{+\infty} q(\tau))d\tau\right. \\
&\left.+\sum_{k=1}^{+\infty}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right) r_0 + d_{3k})\right) ds \\
&+\frac{t_2-t_1}{1+t_2^2}\int_{t_1}^{t_2}\phi^{-1}\Big(\phi(C)+M_{r_0}\int_0^{+\infty} q(\tau)d\tau+\sum_{k=1}^{+\infty}\left(a_{3k}(1+t_k^2)\right) \\
&+b_{3k}(1+t_k)+c_{3k}) r_0 + d_{3k})\Big) ds
\end{aligned}$$

which tends to 0, as $|t_1-t_2| \to 0$. Regarding the first derivatives, we have

$$\begin{aligned}
&\left|\frac{(Tu)'(t_2)}{1+t_2}-\frac{(Tu)'(t_1)}{1+t_1}\right| \\
=\ & \left|\frac{1}{1+t_2}\left(B+\sum_{t_2<t_k} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)\right. \\
&+\frac{1}{1+t_2}\int_0^{t_2}\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right. \\
&\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right) ds \\
&-\frac{1}{1+t_1}\left(B+\sum_{t_1<t_k} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right) \\
&-\frac{1}{1+t_1}\int_0^{t_1}\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right. \\
&\left.\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right) ds\right| \\
\leq\ & \left|\frac{1}{1+t_2}-\frac{1}{1+t_1}\right|\left(B+\sum_{t_1<t_k} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right) \\
&+\left|\frac{1}{1+t_2}-\frac{1}{1+t_1}\right|\int_0^{t_1}\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right. \\
&\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right) ds \\
&+\frac{1}{1+t_2}\int_{t_1}^{t_2}\phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right. \\
&\left.+\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))ds\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\frac{(Tu)'(t_2)}{1+t_2}-\frac{(Tu)'(t_1)}{1+t_1}\right| \\
\leq\ & \left|\frac{1}{1+t_2}-\frac{1}{1+t_1}\right|\left(B+\sum_{k=1}^{+\infty}\left(a_{2k}(1+t_k^2)+b_{2k}(1+t_k)+c_{2k}\right)r_0+d_{2k}\right) \\
&+\left|\frac{1}{1+t_2}-\frac{1}{1+t_1}\right|\int_0^{t_1}\phi^{-1}\left(\phi(C)+M_{r_0}\int_0^{+\infty}q(\tau)d\tau\right. \\
&\left.+\sum_{k=1}^{+\infty}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right)r_0+d_{3k}\right)ds \\
&+\frac{1}{1+t_2}\int_{t_1}^{t_2}\phi^{-1}\left(\phi(C)+M_{r_0}\int_0^{+\infty}q(\tau)d\tau\right. \\
&\left.+\sum_{k=1}^{+\infty}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right)r_0+d_{3k}\right)ds,
\end{aligned}$$

which tends to 0, as $|t_1-t_2|\to 0$. Also, for all $u\in P\cap D$,

$$\begin{aligned}
&|\phi((Tu)'')(t_2)-\phi((Tu)'')(t_1))| \\
=\ & \left|\int_{t_2}^{+\infty}q(s)f(s,u(s),u'(s),u''(s))ds+\sum_{t_k>t_2}I_{3k}(u(t_k),u'(t_k),u''(t_k))\right. \\
&\left.-\int_{t_1}^{+\infty}q(s)f(s,u(s),u'(s),u''(s))ds-\sum_{t_k>t_1}I_{3k}(u(t_k),u'(t_k),u''(t_k))\right| \\
\leq\ & \int_{t_1}^{t_2}q(s)f(s,u(s),u'(s),u''(s))ds+\sum_{t_1<t_k<t_2}I_{3k}(u(t_k),u'(t_k),u''(t_k)),
\end{aligned}$$

which tends to 0, as $|t_1-t_2|\to 0$. So $|(Tu)''(t_2)-(Tu)''(t_1)|\to 0$, as $|t_1-t_2|\to 0$.

Claim 5. *T is equiconvergent at infinity*. For all $u\in P\cap D$, we have $\lim\limits_{t\to+\infty}(Tu)''(t)=C$. By Lemma 1.1, $\lim\limits_{t\to+\infty}\frac{(Tu)'(t)}{1+t}=C$ and $\lim\limits_{t\to+\infty}\frac{Tu(t)}{1+t^2}=\frac{C}{2}$. In addition,

$$\begin{aligned}
|(Tu)''(t)-C| &= \left|\phi^{-1}\left(\phi(C)+\int_t^{+\infty}q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right.\right. \\
&\quad\left.\left.+\sum_{t_k>t}I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)-C\right| \\
&\leq \left|\phi^{-1}\left(\phi(C)+M_{r_0}\int_t^{+\infty}q(\tau)d\tau\right.\right. \\
&\quad\left.\left.+\sum_{t_k>t}I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)-C\right| \\
&\leq \left|\phi^{-1}\left(\phi(C)+M_{r_0}\int_t^{+\infty}q(\tau)d\tau\right.\right. \\
&\quad\left.\left.+\sum_{t_k>t}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right)r_0+d_{3k}\right)-C\right|.
\end{aligned}\tag{3.5}$$

It follows that

$$\begin{aligned}
\sup_{u\in D}|(Tu)''(t)-C| &\leq \left|\phi^{-1}\left(\phi(C)+M_{r_0}\int_t^{+\infty}q(\tau)d\tau\right.\right. \\
&\quad\left.\left.+\sum_{t_k>t}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right)r_0+d_{3k}\right)-C\right|.
\end{aligned}$$

Hence $\sup_{u\in D}|(Tu)''(t)-C|\longrightarrow 0$, as $t\to+\infty$. Similarly

$$\begin{aligned}\left|\frac{(Tu)'(t)}{1+t}-C\right| &= \Big|\tfrac{1}{1+t}\Big(B+\sum_{t<t_k} I_{2k}(u(t_k),u'(t_k),u''(t_k))\Big)\\ &\quad +\tfrac{1}{1+t}\int_0^t \phi^{-1}\Big(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\\ &\quad +\sum_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\Big)\,ds-C\Big|\\ &\le \Big|\tfrac{1}{1+t}\Big(B+\sum_{k=1}^{+\infty}\left(a_{2k}(1+t_k^2)+b_{2k}(1+t_k)+c_{2k}\right)r_0+d_{2k}\Big)\\ &\quad +\tfrac{1}{1+t}\int_0^t \phi^{-1}\Big(\phi(C)+M_{r_0}\int_s^{+\infty} q(\tau)d\tau\\ &\quad +\sum_{t_k>s}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right)r_0+d_{3k}\Big)-C\Big|.\end{aligned}$$

Since

$$\begin{aligned}&\lim_{t\to+\infty}\tfrac{1}{1+t}\int_0^t \phi^{-1}\Big(\phi(C)+M_{r_0}\int_s^{+\infty} q(\tau)d\tau\\ &\quad +\sum_{t_k>s}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right)r_0+d_{3k}\Big)\\ =\;&\lim_{t\to+\infty}\phi^{-1}\Big(\phi(C)+M_{r_0}\int_t^{+\infty} q(\tau)d\tau\\ &\quad +\sum_{t_k>t}\left(a_{3k}(1+t_k^2)+b_{3k}(1+t_k)+c_{3k}\right)r_0+d_{3k}\Big),\end{aligned}$$

we have

$$\sup_{u\in D}\left|\frac{(Tu)'(t)}{1+t}-C\right|\to 0,$$

as $t\to+\infty$. By the same way, for all $u\in P\cap D$,

$$\sup_{u\in D}\left|\frac{(Tu)(t)}{1+t^2}-\frac{C}{2}\right|\to 0,$$

as $t\to+\infty$. This completes the proof of Lemma 3.2. □

We are now in position to prove our main existence result. For the sake of convenience, we use the following notation. For $b>a>0$, let

$$f_0=\liminf_{(u,v,w)\to 0}\min_{t\in[a,b]}\frac{f(t,(1+t^2)u,(1+t)v,w)}{\phi(u)}.$$

Theorem 3.3. *Further to Hypothesis ($\mathscr{H}_0$), assume that, for some $b>a>0$,*

$$(\mathscr{H}_1)\qquad f_0>\phi\left(\frac{2(1+b^2)}{a^2\phi^{-1}\left(\frac{1}{2}\int_a^b q(\tau)d\tau\right)}\right).$$

There exists $R>0$ such that

$$\begin{aligned}(\mathscr{H}_2)\qquad R &\ge A+(a_1+b_1)R+c_1+\tfrac{1}{2}(B+(a_2+b_2+c_2)R+d_2)\\ &\quad+\phi^{-1}\left(\phi(C)+M_R\|q\|_{L^1}+(a_3+b_3+c_3)R+d_3\right),\end{aligned}$$

where M_R is defined by (1.3). Then problem (1.1) has at least one positive solution.

Proof. By the definition of f_0, for every $\varepsilon>0$, there exists $r_1>0$ such that

$$f(t,(1+t^2)u,(1+t)v,w)\ge(1-\varepsilon)f_0\phi(u),\ \ \forall u,v,w\le r_1,\ \forall t\in[a,b]. \tag{3.6}$$

Let $0 < \varepsilon < \frac{1}{2}$ and define the open sets $\Omega_R = \mathscr{B}_E(O,R)$ and $\Omega_r = \mathscr{B}_E(O,r)$, where $0 < r < \min(R, r_1)$. We check that

$$u \neq Tu + \lambda\psi,\ \forall u \in P \cap \partial\Omega_r,\ \forall \lambda > 0.$$

If not, then there would exist $u_0 \in P \cap \partial\Omega_r$ and $\lambda_0 > 0$ such that $u_0 = Tu_0 + \lambda_0\psi$. Let $\mu = \min\limits_{t\in[a,b]} u_0(t)$. For any $t \in [a,b]$, we have the estimates

$$\begin{aligned}
u_0(t) &= (Tu_0)(t) + \lambda_0 \\
&= A + \sum_{t_k<t} I_{1k}(u_0(t_k), u_0'(t_k)) + (B + \sum_{t_k<t} I_{2k}(u_0(t_k), u_0'(t_k), u_0''(t_k)))t \\
&\quad + \int_0^t (t-s)\phi^{-1}\Big(\phi(C) + \int_s^\infty q(\tau) f(\tau, u_0(\tau), u_0'(\tau), u_0''(\tau))d\tau \\
&\quad + \sum_{t_k>s} I_{3k}(u_0(t_k), u_0'(t_k), u_0''(t_k))\Big) ds + \lambda_0 \\
&\geq \int_0^a (a-s)\phi^{-1}\left(\int_a^b q(\tau) f(\tau, u_0(\tau), u_0'(\tau), u_0''(\tau))d\tau\right) ds + \lambda_0 \\
&\geq \int_0^a (a-s)\phi^{-1}\left(\int_a^b q(\tau)(1-\varepsilon) f_0 \phi(\tfrac{u_0(\tau)}{1+b^2})d\tau\right) ds + \lambda_0.
\end{aligned}$$

Indeed, taking into account (3.6), we have

$$\begin{aligned}
f(\tau, u_0(\tau), u_0'(\tau), u_0''(\tau)) &= f(\tau, (1+\tau^2)\tfrac{u_0(\tau)}{1+\tau^2}, (1+\tau)\tfrac{u_0'(\tau)}{1+\tau}, u_0''(\tau)) \\
&\geq (1-\varepsilon) f_0 \phi\left(\tfrac{u_0(\tau)}{1+\tau^2}\right) \geq (1-\varepsilon) f_0 \phi\left(\tfrac{u_0}{1+b^2}\right),
\end{aligned}$$

for $\left|\frac{u_0(\tau)}{1+\tau^2}\right| \leq \|u_0\| \leq r \leq r_1$, $\left|\frac{u_0(\tau)}{1+\tau}\right| \leq \|u_0\| \leq r \leq r_1$, and $|u_0''(\tau)| \leq \|u_0\| \leq r \leq r_1$. Hence

$$\begin{aligned}
u_0(t) &\geq \int_0^a (a-s)\phi^{-1}\left(\int_a^b q(\tau)(1-\varepsilon) f_0 \phi\left(\tfrac{\mu}{1+b^2}\right) d\tau\right) ds + \lambda_0 \\
&\geq \int_0^a (a-s)\phi^{-1}\left(\phi\left(\tfrac{\mu}{1+b^2}\right)(1-\varepsilon) f_0 \int_a^b q(\tau)d\tau\right) ds + \lambda_0 \\
&\geq \tfrac{\mu}{1+b^2} \int_0^a (a-s)\phi^{-1}\left((1-\varepsilon) f_0 \int_a^b q(\tau)d\tau\right) ds + \lambda_0 \\
&\geq \tfrac{\mu}{1+b^2} \phi^{-1}(f_0) \int_0^a (a-s)\phi^{-1}\left(\tfrac{1}{2}\int_a^b q(\tau)d\tau\right) ds + \lambda_0 \\
&\geq \tfrac{\mu}{1+b^2} \phi^{-1}(f_0)\phi^{-1}\left(\tfrac{1}{2}\int_a^b q(\tau)d\tau\right) \tfrac{1}{2}a^2 + \lambda_0 \\
&\geq \mu + \lambda_0.
\end{aligned}$$

Passing to the infimum over $t \in [a,b]$, we get $\mu \geq \mu + \lambda_0$, which is a contradiction. For all $u \in P \cap \partial\Omega_R$, we deduce the estimates:

$$\begin{aligned}
\tfrac{Tu(t)}{1+t^2} &= \tfrac{A}{1+t^2} + \tfrac{1}{1+t^2}\sum_{t_k<t} I_{1k}(u(t_k), u'(t_k)) \\
&\quad + \tfrac{t}{1+t^2}\left(B + \sum_{t_k<t} I_{2k}(u(t_k), u'(t_k), u''(t_k))\right) \\
&\quad + \tfrac{1}{1+t^2}\int_0^t (t-s)\phi^{-1}\Big(\phi(C) + \int_s^{+\infty} q(\tau) f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau \\
&\quad + \sum_{t_k>s} I_{3k}(u(t_k), u'(t_k), u''(t_k))\Big) ds \\
&\leq A + \sum_{k=1}^{+\infty} I_{1k}(u(t_k), u'(t_k)) + \tfrac{1}{2}\left(B + \sum_{k=1}^{+\infty} I_{2k}(u(t_k), u'(t_k), u''(t_k))\right) \\
&\quad + \tfrac{t}{1+t^2}\int_0^t \phi^{-1}\left(\phi(C) + M_R\int_0^{+\infty} q(\tau)d\tau + ((a_3+b_3+c_3)R + d_3)\right) ds \\
&\leq A + (a_1+b_1)R + c_1 + \tfrac{1}{2}(B + (a_2+b_2+c_2)R + d_2) \\
&\quad + \tfrac{t^2}{1+t^2}\phi^{-1}\left(\phi(C) + M_R\int_0^{+\infty} q(\tau)d\tau + (a_3+b_3+c_3)R + d_3\right) \\
&\leq R = \|u\|.
\end{aligned} \tag{3.7}$$

So $\|Tu\|_0 \leq R, \forall u \in P\cap\partial\Omega_R$. Arguing in a similar way, we get

$$\begin{array}{rcl} \frac{(Tu)'(t))}{1+t} & = & \frac{1}{1+t}\left(B+\sum\limits_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right) \\ & & +\frac{1}{1+t}\int_0^t \phi^{-1}\left(\phi(C)+\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau\right. \\ & & \left.+\sum\limits_{t_k>s} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)ds \\ & \leq & \frac{1}{2}(B+(a_2+b_2+c_2)R+d_2) \\ & & +\frac{t}{1+t}\phi^{-1}\left(\phi(C)+M_R\int_0^{+\infty} q(\tau)d\tau+(a_3+b_3+c_3)R+d_3\right) \\ & \leq & R=\|u\|, \end{array} \tag{3.8}$$

that is, $\|Tu\|_1 \leq R, \forall u \in P\cap\partial\Omega_R$. We can also prove that $\|Tu\|_2 \leq R$. Thus $\|Tu\| \leq \|u\| = R$. Finally $\psi = 1 \in P$. Using Lemma 1.2 and Lemma 3.2, we conclude that problem (1.1) has at least one solution in $P\cap(\overline{\Omega_R}\setminus\Omega_r)$. □

Example 3.4. Consider the boundary value problem:

$$\left\{\begin{array}{rcl} ((u''(t))^3)' & = & -e^{-t}\left(|\sin t|+\ln(1+\frac{u}{1+t^2})+\frac{u'}{1+t}+(u''+1)^{\frac{1}{3}}\right),\ t>0,\ t\neq 2^k \\ u(0) & = & 0, \\ u'(0) & = & 0, \\ u''(+\infty) & = & 0, \\ \triangle u(t_k) & = & \frac{1}{2^k}(1+\frac{u(t_k)}{2^k}+u'(t_k))^{\frac{10^{-4}}{2^k}}, \quad k=1,2,3,\ldots \\ \triangle u'(t_k) & = & \frac{1}{3^k}(1+u(t_k)+u'(t_k))^{\frac{10^{-4}}{3^k}}, \quad k=1,2,3,\ldots \\ -\triangle\phi(u''(t_k)) & = & \frac{1}{5^k}(1+u(t_k)+u'(t_k)+u''(t_k))^{\frac{1}{5^k}}, \quad k=1,2,3,\ldots \end{array}\right. \tag{3.9}$$

We have

$$t_k=2^k, q(t)=e^{-t}, I_{1k}(u,v)=\frac{1}{2^k}(1+\frac{u}{2^k}+v)^{\frac{10^{-4}}{2^k}},$$
$$I_{2k}(u,v,w)=\frac{1}{3^k}(1+u+v)^{\frac{10^{-4}}{3^k}}, I_{3k}(u,v,w)=\frac{1}{5^k}(1+u+v+w)^{\frac{1}{5^k}},$$

and

$$f(t,u,v,w)=|\sin t|+\ln\left(1+\frac{u}{1+t^2}\right)+\frac{v}{1+t}+(w+1)^{\frac{1}{3}},$$
$$\phi(u)=u^3, \text{ so that } \phi^{-1}(uv)=(u)^{\frac{1}{3}}v^{\frac{1}{3}}=\phi^{-1}(u)\phi^{-1}(v).$$

Clearly

$$\int_0^{+\infty} q(t)dt=1 \text{ and } \int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} q(\tau)d\tau\right)ds=3.$$

Using the inequality

$$(1+u)^\alpha \leq 1+\alpha u, \text{ for all } u\geq 0, \text{ and } 0<\alpha<1,$$

we get

$$I_{1k}=\frac{1}{2^k}(1+\frac{u}{2^k}+v)^{\frac{10^{-4}}{2^k}} \leq \frac{1}{2^k}+\frac{10^{-4}}{2^{3k}}u+\frac{10^{-4}}{2^{2k}}v.$$

Hence

$$I_{1k} \leq a_{1k}u+b_{1k}v+c_{1k}$$

with

$$a_{1k}=\frac{10^{-4}}{2^{3k}}, b_{1k}=\frac{10^{-4}}{2^{2k}}, c_{1k}=\frac{1}{2^k}.$$

Then

$$a_1 = 10^{-4}\sum a_{1k}(1+t_k^2) = \tfrac{8.10^{-4}}{7},$$
$$b_1 = 10^{-4}\sum b_{1k}(1+t_k) = \tfrac{4.10^{-4}}{3},$$
$$c_1 = \sum c_{1k} = 1,$$
$$I_{2k} = \frac{1}{3^k}(1+u+v)^{\frac{10^{-4}}{3^k}} \leq \frac{1}{3^k} + \frac{10^{-4}}{3^{2k}}u + \frac{10^{-4}}{3^{2k}}v.$$

It follows that

$$I_{2k} \leq a_{2k}u + b_{2k}v + d_{2k}$$

with

$$a_{2k} = b_{2k} = \frac{10^{-4}}{3^{2k}},\, d_{2k} = \frac{1}{3^k}.$$
$$a_2 = 10^{-4}\sum a_{2k}(1+t_k^2) = \tfrac{17.10^{-4}}{8},$$
$$b_2 = 10^{-4}\sum b_{2k}(1+t_k) = \tfrac{23.10^{-4}}{56},\, c_2 = 0,\, d_2 = \sum d_{1k} = \tfrac{1}{2},$$
$$I_{3k} = \frac{1}{5^k}(1+u+v+w)^{\frac{1}{5^k}} \leq \frac{1}{5^k} + \frac{1}{5^{2k}}(u+v+w).$$

It follows that

$$I_{3k} \leq a_{3k}u + b_{3k}v + c_{3k}w + d_{3k}$$

with

$$a_{3k} = b_{3k} = c_{3k} = \frac{1}{5^{2k}},\, d_{3k} = \frac{1}{5^k}.$$

Hence

$$a_3 = \sum a_{3k}(1+t_k^2) = \tfrac{13}{56},\, b_3 = \sum b_{3k}(1+t_k) = \tfrac{71}{652},$$
$$c_3 = \sum c_{1k} = \tfrac{1}{24},\, d_3 = \tfrac{1}{4}.$$

Finally

$$f(t,(1+t^2)u,(1+t)v,w) = |\sin t| + \ln(1+u) + v + (w+1)^{\frac{1}{3}} \leq 2+u+v+\frac{1}{3}w.$$

Hence $f(t,(1+t^2)u,(1+t)v,w)$ is bounded whenever u,v,w are bounded, so that $(\mathscr{H}_0)$ holds. Also $\frac{f}{u^3} \geq \frac{\ln(1+u)}{u^3} \longrightarrow +\infty$, as $u \to 0$ implies that $f_0 = +\infty$. Thus $(\mathscr{H}_1)$ is satisfied in the super-linear case. Moreover we get by computations the values $R = 3.33$, $M_R = 1 + ln(1+R) + R + (R+1)^{1/3} = 7.4255$, and $3.33 \geq 3.3275$, which make validation of $(\mathscr{H}_2)$. Therefore all hypotheses of Theorem 3.3 are fulfilled. We conclude that problem (3.9) admits at least one positive solution u in P.

Next, we construct iterative solutions for the particular case of (1.1) when $u''(+\infty) = 0$ and $\phi^{-1}(uv) = \phi^{-1}(u)\phi^{-1}(v)$, $\forall u,v \in \mathbb{R}$, for example, the usual p-Laplacian operator $\phi(s) = |s|^{p-1}s$.

Theorem 3.5. *In addition to Hypothesis $(\mathscr{H}_0)$, assume that there exists $a > 3\Lambda$ such that*

$$\begin{aligned}
&(\mathscr{H}_1')\quad f(t,x_1,y_1,z_1) \leq f(t,x_2,y_2,z_2),\\
&0 \leq t < \infty,\ 0 \leq x_1 \leq x_2 \leq a,\ y_1 \leq y_2 \leq a,\ z_1 \leq z_2 \leq a.\\
&(\mathscr{H}_2')\quad f(t,(1+t^2)a,(1+t)a,a) \leq \phi(\tfrac{a}{3M}),\ f(t,0,0,0) \neq 0,\ 0 \leq t < \infty.\\
&(\mathscr{H}_3)\quad I_{1k}(x_1,y_1) \leq I_{1k}(x_2,y_2),\ 0 \leq x_1 \leq x_2 \leq a,\ y_1 \leq y_2 \leq a,\\
&\qquad I_{ik}(x_1,y_1,z_1) \leq I_{ik}(x_2,y_2,z_2),\\
&0 \leq x_1 \leq x_2 \leq a,\ y_1 \leq y_2 \leq a,\ z_1 \leq z_2 \leq a,\ i = 2,3.\\
&(\mathscr{H}_4)\quad I_{3k}((1+t^2)a,(1+t)a,a) \leq \phi(\tfrac{a}{3M})e_{3k},\ 0 \leq t < \infty,\\
&\qquad e_3 = \sum_{k=1}^{\infty} e_{3k} < \infty,
\end{aligned} \tag{3.10}$$

where

$$\Lambda = \max\left\{\frac{B+d_2}{\frac{1}{3}-a_2-b_2-c_2}, \frac{A+c_1}{\frac{1}{3}-a_1-b_1}\right\}, \quad M = \phi^{-1}\left(\int_0^{+\infty} q(s)ds + e_3\right).$$

Then there exist two iterate sequences (v_n) *and* (w_n) *such that* $v^* = \lim_{n\to\infty} v_n$ *and* $w^* = \lim_{n\to\infty} w_n$ *are positive nondecreasing solutions of problem* (1.1) *with* $0 < \|v^*\|, \|w^*\| \le a$.

Proof. By Lemma 3.2, operator $T : P \to P$ is completely continuous. From (3.1), we can easily check that $Tx_1 \le Tx_2$ for any $x_1, x_2 \in P$ with $x_1 \le x_2, x_1' \le x_2', x_1'' \le x_2''$. Let $P_a = P \cap \mathscr{B}_E(0,a)$. Then $\overline{P}_a = P \cap \overline{\mathscr{B}}(0,a)$. Next, we prove $T : \overline{P}_a \to \overline{P}_a$. If $x \in \overline{P}_a$, then $\|x\| \le a$ and

$$\begin{aligned}
(Tu)''(t) &= \phi^{-1}\left(\int_t^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau \right.\\
&\quad \left. + \sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\\
&\le \phi^{-1}\left(\phi(\tfrac{a}{3M})\int_t^{+\infty} q(\tau)d\tau + \sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\\
&\le \phi^{-1}\left(\phi(\tfrac{a}{3M})\int_t^{+\infty} q(\tau)d\tau + \phi(\tfrac{a}{3M})\sum_{k=1}^{+\infty} e_{3k}\right)\\
&\le \phi^{-1}\left(\phi(\tfrac{a}{3M})\left(\int_t^{+\infty} q(\tau)d\tau + \sum_{k=1}^{+\infty} e_{3k}\right)\right)\\
&\le \tfrac{a}{3M}\phi^{-1}\left(\int_0^{+\infty} q(\tau)d\tau + \sum_{k=1}^{+\infty} e_{3k}\right)\\
&\le \tfrac{a}{3M}\phi^{-1}\left(\int_0^{+\infty} q(\tau)d\tau + e_3\right)\\
&\le \tfrac{a}{3} < a.
\end{aligned}$$

Also

$$\begin{aligned}
\tfrac{(Tu)'(t)}{1+t} &= \tfrac{1}{1+t}\left(B + \sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)\\
&\quad + \tfrac{1}{1+t}\left(\int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau \right.\right.\\
&\quad \left.\left. + \sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\right) ds\\
&\le (B + (a_2+b_2+c_2)a + d_2)\\
&\quad + \tfrac{a}{3M}\phi^{-1}\left(\int_0^{+\infty} q(\tau)d\tau + \sum_{k=1}^{+\infty} e_{3k}\right)\\
&\le \tfrac{2a}{3}
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{(Tu)(t)}{1+t^2} &= \tfrac{A}{1+t^2} + \tfrac{1}{1+t^2}\sum_{t_k<t} I_{1k}(u(t_k),u'(t_k))\\
&\quad + \tfrac{1}{1+t^2}\left(B + \sum_{t_k<t} I_{2k}(u(t_k),u'(t_k),u''(t_k))\right)t\\
&\quad + \tfrac{1}{1+t^2}\left(\int_0^t (t-s)\phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau \right.\right.\\
&\quad \left.\left. + \sum_{t_k>t} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right)\right) ds\\
&\le A + (a_1+b_1)a + c_1 + (B + (a_2+b_2+c_2)a + d_2)\\
&\quad + \tfrac{a}{3M}\phi^{-1}\left(\int_0^{+\infty} q(\tau)d\tau + \sum_{k=1}^{+\infty} e_{3k}\right)\\
&\le \tfrac{a}{3} + \tfrac{a}{3} + \tfrac{a}{3} = a.
\end{aligned}$$

Thus $\|Tu\| \le a$. Let $w_0(t) = \frac{a}{2}(1+t+\frac{1}{t^2}), t \ge 0$. Then $w_0(t) \in \overline{P}_a$. Let $w_1(t) = Tw_0(t)$. Then $w_1(t) \in \overline{P}_a$. Inductively, we define the sequence

$$w_{n+1} = Tw_n = T^n w_0, n = 1,2,\ldots. \tag{3.11}$$

Since $T : \overline{P}_a \to \overline{P}_a$, we have $w_n(t) \in T(\overline{P}_a) \subset \overline{P}_a$, $n = 1,2,\ldots$. Since T is completely continuous, we find that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in \overline{P}_a$ such that $w_{n_k} \to w^*$, as $k \to +\infty$. By $(\mathscr{H}_1) - (\mathscr{H}_3)$, we have

$$\begin{aligned} w_1(t) &= Tw_0(t) \\ &= A + \sum_{t_k<t} I_{1k}(w_0(t_k), w_0'(t_k)) + \left(B + \sum_{t_k<t} I_{2k}(w_0(t_k), w_0'(t_k), w_0''(t_k))\right)t \\ &\quad + \int_0^t (t-s)\phi^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, w_0(\tau), w_0'(\tau), w_0''(\tau))d\tau\right. \\ &\quad \left. + \sum_{t_k>s} I_{3k}(w_0(t_k), w_0'(t_k), w_0''(t_k))\right) ds \\ &\le w_0(t) \end{aligned}$$

and

$$\begin{aligned} w_1'(t) &= (Tw_0)'(t) \\ &= \left(B + \sum_{t_k<t} I_{2k}(w_0(t_k), w_0'(t_k), w_0''(t_k))\right) \\ &\quad + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, w_0(\tau), w_0'(\tau), w_0''(\tau))d\tau\right. \\ &\quad \left. + \sum_{t_k>s} I_{3k}(w_0(t_k), w_0'(t_k), w_0''(t_k))\right) ds, \\ &\le (w_0)'(t), \end{aligned}$$

$$\begin{aligned} w_1''(t) &= (Tw_0)''(t) \\ &= \phi^{-1}\left(\int_t^{+\infty} q(\tau) f(\tau, w_0(\tau), w_0'(\tau), w_0''(\tau))d\tau\right. \\ &\quad \left. + \sum_{t_k>t} I_{3k}(w_0(t_k), w_0'(t_k), w_0''(t_k))\right) \\ &\le w_0''(t). \end{aligned}$$

Hence, for positive t,

$$\begin{aligned} w_2(t) = Tw_1(t) &\le Tw_0(t) = w_1(t), \\ w_2'(t) = (Tw_1)'(t) &\le (Tw_0)'(t) = (w_1)'(t). \end{aligned}$$

By induction we have for $n = 0,1,2,\ldots$ and positive t

$$\begin{aligned} w_{n+1}(t) &\le w_n(t), \\ w_{n+1}'(t) &\le w_n'(t), \\ w_{n+1}''(t) &\le w_n''(t). \end{aligned}$$

We claim that $w_n \to w^*$ as $n \to \infty$. By the continuity of T and the fact that $w_{n+1}(t) = Tw_n(t)$, we obtain that $Tw^* = w^*$. Let $v_0(t) = 0$, $t \ge 0$. Then $v_0(t) \in \overline{P}_a$. Letting $v_1 = Tv_0, v_2 = T^2 v_0$, we have $v_1 \in \overline{P}_a$ and $v_2 \in \overline{P}_a$. Denote

$$v_{n+1} = Tv_n = T^{n+1}v_0,\ n = 0,1,2,\cdots.$$

Since $T : \overline{P}_a \to \overline{P}_a$, we have $v_n \in \overline{P}_a$, $n = 1,2,3,\ldots$ Since T is completely continuous, we find that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set. So it has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists

$v^* \in \overline{P}_a$ such that $v_{n_k} \to v^*$. Since $v_1 = Tv_0 \in \overline{P}_a$, we have, for positive t,

$$\begin{aligned} v_1(t) &= (Tv_0)(t) = T0(t) \geq 0, \\ v_1'(t) &= (Tv_0)'(t) = (T0)'(t) \geq 0 = v_0'(t), \\ v_1''(t) &= (Tv_0)''(t) = (T0)''(t) \geq 0 = v_0''(t). \end{aligned}$$

By $(\mathscr{H}_1')-(\mathscr{H}_4)$, we have for $t \geq 0$

$$\begin{aligned} v_2(t) = (Tv_1)(t) &\geq T0(t) = v_1(t), \\ v_2'(t) = (Tv_1)'(t) &\geq (Tv_0)'(t) = v_1'(t), \\ v_2''(t) = (Tv_1)''(t) &\geq (T0)''(t) \geq 0 = v_1''(t). \end{aligned}$$

By induction, we finally obtain for $n = 1, 2, 3, \ldots$ and positive t

$$\begin{aligned} v_{n+1} &\geq v_n, \\ v_{n+1}' &\geq v_n', \\ v_{n+1}'' &\geq v_n''. \end{aligned}$$

Hence $v_n \to v^*$ as $n \to \infty$. By the continuity of T and the fact that $v_{n+1} = Tv_n$, we get $v^* = Tv^*$. Thus v^* is a positive solution. □

Remark 3.6. Theorem 3.3 extends the existence results in [5, 10, 12], where problem (1.1) was studied without impulse points and in [11], where the ϕ-Laplacian problem was investigated on a bounded interval. In [14], the existence results for an impulsive ϕ-Laplacian problem were obtained for a second-order differential operator set on a bounded interval of the real line. Theorem 3.5 provides a concrete construction of approximate solutions to a third-order impulsive ϕ-Laplacian BVP. We refer to [19], where iterative solutions were obtained for a third-order ϕ-Laplacian BVP without impulses and set on $[0,1]$.

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